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NORMAL COVERS OF RECTANGULAR PRODUCTS WITH ORDINAL FACTORS

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1. NORMAL COVERS

All spaces considered here are assume to be *Hausdorff*.

Let X be a space and $\mathcal{U}$ a cover of it. A cover $\mathcal{A}$ of X is an *refinement* of $\mathcal{U}$ if each $A \in \mathcal{A}$ is contained in some $U_A \in \mathcal{U}$. A cover $\mathcal{V}$ of X is a *star-refinement* of $\mathcal{U}$ if $\{\mathrm{St}(V, \mathcal{W}) \colon V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}$, where $\mathrm{St}(V, \mathcal{V}) = \bigcup\{W \in \mathcal{V} \colon W \cap V \neq \emptyset\}$.

An open cover $\mathcal{O}$ of a space X is *normal* if there is a sequence $\{\mathcal{U}_n\}$ of open covers of X such that $\mathcal{U}_{n+1}$ is a star-refinement of $\mathcal{U}_n$ for each $n \in \omega$, where $\mathcal{U}_0 = \mathcal{O}$.

The study of normal covers was begun by Tukey [T]. The concept of normal covers is very important because of the following result proved by Stone [S].

Stone's Theorem. *A space X is paracompact if and only if every open cover of X is normal. Hence every metric space is paracompact.*

An open set U in a space X is a *cozero-set* if $U = \{x \in X \colon f(x) > 0\}$ for some continuous function $f \colon X \to [0,1]$. A closed set Z in X is a *zero-set* if it is the complement of a cozero-set in X. An open cover $\mathcal{U}$ of X has a *shrinking* $\{F_U : U \in \mathcal{U}\}$ if it is a closed cover of X such that $F_U \subset U$ for each $U \in \mathcal{U}$.

Subsequently, normal covers were characterized in terms of cozero refinements in various forms. This was summarized in [M] as follows.

Stone-Michael-Morita's Theorem. *Let X be a space and $\mathcal{O}$ an open cover of X. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite cozero refinement.*
(c) *$\mathcal{O}$ has a σ-discrete cozero refinement.*
(d) *$\mathcal{O}$ has a locally finite cozero refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, cozero refinement $\mathcal{U}$ which has a shrinking $\{Z_U : U \in \mathcal{U}\}$ consisting of zero-sets.*

2. NORMAL COVERS OF RECTANGULAR PRODUCTS

Let $X \times Y$ be a product space. A subset of the form $A \times B$ in $X \times Y$ is called a *rectangle.* A rectangle $U \times V$ in $X \times Y$ is called a *cozero rectangle (zero-set rectangle)* if U and V is cozero-sets (zero-sets) in X and Y, respectively, A cover $\mathcal{G}$ of $X \times Y$ is called *rectangular cozero* (respectively, *rectangular zero-set*) if each member of $\mathcal{G}$ is a cozero rectangle (respectively, zero-set rectangle) in $X \times Y$.

A product space $X \times Y$ is said to be *rectangular* if every normal (equivalently, finite cozero cover or binary cozero) cover of $X \times Y$ has a σ-locally finite rectangular cozero refinement.

This concept was introduced by Pasynkov [P] to prove the product theorem in dimension theory.

Pasynkov's Theorem. *Let X and Y be a Tychonoff spaces. If $X \times Y$ is rectangular, then* $\dim X \times Y \leq \dim X + \dim Y$.

Now, we have raised the following question concerning normal covers.

Question 1. Is there a characterization analogous to Stone-Michael-Morita's Theorem for normal covers of rectangular products in terms of rectangular cozero refinements?

A space X is called a *σ-space* if it has a σ-locally finite closed net. Making use of it, the second author has given an answer to Question 1 as follows.

Theorem 2.1 [Y2]. *Let $X \times Y$ be a rectangular product with a paracompact σ-space factor X and $\mathcal{O}$ an open cover of $X \times Y$. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite rectangular cozero refinement.*
(c) *$\mathcal{O}$ has a σ-discrete rectangular cozero refinement.*
(d) *$\mathcal{O}$ has a locally finite rectangular cozero refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.*

3. Normal covers of infinite products

Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be an infinite product. For each $\lambda \in \Lambda$, π_λ denotes the projection of X onto X_λ. A subset of the form $\bigcap_{\lambda \in \theta} \pi_\lambda^{-1}(U_\lambda)$ in X is called a *rectangle* if θ is a finite subset of Λ, where $U_\lambda \subset X_\lambda$ for each $\lambda \in \theta$. A rectangle $\bigcap_{\lambda \in \theta} \pi_\lambda^{-1}(U_\lambda)$ in X is called a *cozero rectangle* (respectively, *zero-set rectangle*) if U_λ is a cozero-set (respectgively, zero-set) for each $\lambda \in \theta$. A cover $\mathcal{G}$ of the product X is said to be *rectangular* (respectgively, *rectangular cozero, rectangular zero-set*) if each member of $\mathcal{G}$ is a rectangle (respectgively, cozero rectangle, zero-set rectangle) in X.

Question 2. Is there a similar version to our Theorem 2.1 above for normal covers of infinite products?

Recall that a space X is an *M-space* if there is a quasi-perfect map f (that is, f is a closed map such that each $f^{-1}(y)$ is countably compact) from X onto some metric space M. Note that the class of paracompact M-spaces is coincided with that of paracompact p-spaces and is countably productive.

Filippov's Theorem. *Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be an infinite product of paracompact M-spaces. Then every normal cover of X has a σ-locally finite rectangular cozero refinement.*

In order to solve Question 2, the Filippov's proof for the above in [F] was very useful. Making use of his idea, the second author has given an answer to Question 2 extending his result.

Theorem 3.1 [Y3]. *Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be an infinite product of paracompact M-spaces. Let $\mathcal{O}$ be an open cover of X. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite rectangular cozero refinement.*
(c) *$\mathcal{O}$ has a σ-discrete rectangular cozero refinement.*
(d) *$\mathcal{O}$ has a locally finite rectangular cozero refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.*

Recall that a regular space S is a *strong Σ-space* if there are a σ-locally finite closed cover $\mathcal{F}$ of S and a cover $\mathcal{K}$ of S by compact sets such that, whenever $K \in \mathcal{K}$ and U is open in S with $K \subset U$, one can find $F \in \mathcal{F}$ with $K \subset F \subset U$.

Note that a paracompact Σ-space is a strong Σ-space and the class of all paracompact Σ-spaces is countably productive.

Moreover, we have also given another answer to Question 2, which strengthens [Y1, Proposition 3].

Theorem 3.2 [Y3]. *Let $X = \prod_{\alpha\in\Lambda} X_\alpha$ be an infinite product of paracompact Σ-spaces, and each finite subproduct of which has countable tightness. Let $\mathcal{O}$ be an open cover of X. Then the following are equivalent.*

(a) *$\mathcal{O}$ is normal.*
(b) *$\mathcal{O}$ has a σ-locally finite rectangular cozero refinement.*
(c) *$\mathcal{O}$ has a σ-discrete rectangular cozero refinement.*
(d) *$\mathcal{O}$ has a locally finite rectangular cozero refinement.*
(e) *$\mathcal{O}$ has a locally finite, σ-discrete, rectangular cozero refinement which has a rectangular zero-set shrinking.*

In [Y3], Theorems 3.1 and 3.2 have been simultaneously derived from a more generalized form in terms of perfect maps and β-spaces instead of M-spaces and Σ-spaces.

4. Rectangular products with ordinal factors

As stated the previous three sections, the study of the characterization of "Normal Covers" has been stepped by the following order:

(1) a topological space X,
(2) a product space $X \times Y$,
(3) an infinite product $\prod_{\alpha\in\Lambda} X_\alpha$,

So it is natural to ask

Question 4.1 What kind of products is the next subject for the above study?

Let λ be an ordinal. Then λ is considered as a space with the usual order topology. The study of the products of two subspaces of an ordinal was stimulated with the following result in [KOT].

Kemoto-Ohta-Tamano's Theorem. *Let A and B be subspaces of an ordinal.*
(1) *Then the following are equivalent.*

(a) *$A \times B$ is normal.*
(b) *$A \times B$ is collectionwise normal.*
(c) *$A \times B$ is shrinking.*

(2) *If $A \times B$ is normal, then $A \times B$ is countably paracompact.*

Note that $\dim A = 0$ holds for each subspace A of an ordinal. Moreover, the following was given in [FKT].

Fleissner-Kemoto-Terasawa's Theorem. *Let A and B be subspaces of an ordinal. Then* $\dim A \times B = 0$ *holds.*

The world of such products of two ordinal spaces is rather special. However, such a speciality has sometimes brought quite unexpected results. So, as an answer to Question 4.1, we investigate normal covers of the products of two subspaces of an ordinal.

First, from Pasynkov's Theorem and FKT's Theorem above, we have to ask

Question 4.2. For two subspaces A and B of an ordinal, is the product $A \times B$ rectangular?

However, it is rather easy to find a negative answer to Question 4.2. In fact, take two disjoint stationary subsets A and B of ω_1. Then it is not so difficult to show that $A \times B$ is not rectangular. Hence we see that it is necessary to assume the rectangularity of such products.

Now, we can raise many questions for rectangular products of two subspaces of an ordinal from KOT's and FKT's Theorems above as follows.

Question 4.3. Let A and B be subspaces of an ordinal.

(1) If $A \times B$ is normal, is it rectangular?
(2) If $A \times B$ is rectangular, is it countably paracompact?
(3) If $A \times B$ is rectangular, every normal cover of $A \times B$ has a discrete rectangular clopen refinement?
(4) Is there a nice equivalent condition for the rectangularity of $A \times B$?

The plenty of Question 4.3 suggests us without any doubt that the rectangularity of $A \times B$ deserves to investigate.

Now, the results below in this section will be published in our recent paper [KY3].

It is quite unexpected that we can obtain the following result, which answers all questions (1)-(4) in Question 4.3 simultaneously.

Theorem 4.1. *Let A and B be subspaces of an ordinal. Then the following are equivalent.*

(a) *$A \times B$ is rectangular.*
(b) *$A \times B$ is countably paracompact.*
(c) *Every binary cozero cover of $A \times B$ has a discrete rectangular clopen refinement.*
(d) *Every normal cover of $A \times B$ has a discrete rectangular clopen refinement.*

As an immediate consequence of Theorem 4.1, we have

Corollary 4.2. *Let A and B be subspaces of an ordinal. Then the following are equivalent.*

(a) *$A \times B$ is normal.*
(b) *Every binary open cover of $A \times B$ has a σ-locally finite rectangular open refinement.*
(c) *Every binary open cover of $A \times B$ has a discrete rectangular clopen refinement.*

Remark 1. For a countably paracompact (and normal) space X which is not paracompact, there is a paracompact space Y such that $X \times Y$ is countably paracompact (and normal) but not rectangular (see [O]). So we generally have

$$\text{Countable paracompactness of } X \times Y \underset{\Longleftarrow}{\not\Longrightarrow} \text{Rectangularity of } X \times Y.$$

Remark 2. Note that $\omega_1 \times (\omega_1 + 1)$ is countably compact but not normal. So we have

$$\text{Rectangularity of } A \times B \not\Longrightarrow \text{Normality of } A \times B.$$

Remark 3. Notice that Theorem 4.1 and Corollary 4.2 give topological characterizations for countable paracompactness and normality of $A \times B$, respectively. From these characterizations, we can easily see a quite delicate difference between countable paracompactness and normality of the products of ordinals. Moreover, from the difference, we can immediately see that no $A \times B$ is a Dowker space (see [KOT]).

5. Products with one ordinal factor

As the last section, we look into an intermediate world between the products of two general spaces and the products of two ordinal spaces. The intermediate one is the class of products of a general space X and an ordinal space A. Here we discuss some covering properties of such products. The new results stated in this section will be also published in the paper [KY3].

Proposition 5.1 [KY1]. *Let A and B be subspaces of an ordinal. The following are equivalent.*

(1) *A and B are paracompact.*
(2) *A and B are weakly submetacompact.*
(3) *$A \times B$ is paracompact.*
(4) *$A \times B$ is weakly submetacompact.*

This is generalized as follows:

Theorem 5.2. *If X is a paracompact space and A is a paracompact subspace of an ordinal, then $X \times A$ is paracompact and rectangular.*

Moreover, we obtain

Theorem 5.3. *Let X be a collectionwise normal and countably paracompact space with* $\dim X = 0$. *Let A be a subspace of an ordinal. Then every σ-point-finite rectangular open cover of $X \times A$ has a discrete rectangular clopen refinement.*

Immdiately, we have

Corollary 5.4. *Let X be a collectionwise normal and countably paracompact space with* $\dim X = 0$. *Let A be a subspace of an ordinal. If $X \times A$ is rectangular, then every normal cover of $X \times A$ has a discrete rectangular clopen refinement.*

Corollary 5.4 immediately yields that (1) implies (4) in Theorem 4.1.

Recall that a space X is *orthocompact* if every open cover $\mathcal{U}$ of X has an interior-preserving open refinement $\mathcal{V}$ (that is, $\bigcap \mathcal{W}$ is open in X for each $\mathcal{W} \subset \mathcal{V}$).

This concept gives another equivalent condition to the normality of $A \times B$ as follows.

Theorem 5.5 [KY1]. *Let A and B be subspaces of an ordinal. Then $A \times B$ is normal if and only if it is orthocompact.*

Kunen pointed out that a space X is paracompact if and only if $X \times (\kappa + 1)$ is normal, where κ is a sufficiently large cardinal. On the other hand, taking off the last point from the $\kappa + 1$, we could obtain an opposite implication which is quite unexpected as follows.

Theorem 4.5 [KY2]. *Let X be a paracompact space and κ an uncountable regular cardinal. If $X \times \kappa$ is orthocompact, then it is normal.*

This result is also generalized as follows.

Theorem 4.6. *Let X be a paracompact space and A a subspace of an ordinal. If $X \times A$ is orthocompact, then it is normal and rectangular.*

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