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Recent Development in Subfactor Theoretic Approach to (2+1)-dimensional Topological Quantum Field Theory

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§0 Backgrounds

Before stating the main results, we shall review some backgrounds from various viewpoints.

We begin with physics of *quantization of the classical Chern-Simons theory* for a (2+1)-dimensional quantum field theory. Certainly, E. Witten made new trends for theories of knots, links and 3-manifolds. He constructed a Chern-Simons quantum field theory, which does not depend on the metric of three manifolds. This kind of quantum field theories is called topological quantum field theory(TQFT). However, the above connstruction uses mathematically undefined path integration.

And it is M. Atiyah who axiomatized topological quantum field theory in the mathematical language [1].

To make the quantized Chern-Simons theory mathematically rigorous, basically the following two mathods had been developped by making use of the tensor category of the representations of the quantum group $SU_q(2)$, where q is a root of unity.

- Turaev-Viro TQFT (using a triangulation of a 3-manifold.) [22]

- Reshetikhin-Turaev TQFT (using a Dehn surgery description of a 3-manifold.) [19]

Here come in *subfactors*. The first method was extended to the *tensor categories obtained from subfactors* by A. Ocneanu and nowadays, it is called *Turaev-Viro-Ocneanu* TQFT.

A. Ocneanu has claimed that a Turaev-Viro-Ocneanu invariant of closed 3-manifolds is equal to a Reshetikhin-Turaev invariant constructed out of the categorical quantum double of an original data(bimodules and intertwiners obtained from a subfactor) was proved by Kawahigashi-Sato-Wakui [11]. (See [10] for the definition of the categorical quantum double, in that book, which is called the center construction.)

Moreover, Ocneanu has claimed (without a proof) a formula for the Turaev-Viro-Ocneanu invariant of closed 3-manifolds constructed out of a degenerate *braided system of bimodules* arising from a subfactor.

There are type II_1 subfactors which give rise to the same tensor cateogry as $SU(N)_k$ Wess-Zumino-Witten model [7]. In the case of $N = 2, 3$, Evans and Kawahigashi succeeded to describe the categorical quantum double of an original braided (but not non-degenerate in general) system Δ of bimodules arising from subfactors in terms of the full system of $\hat{\Delta}$ [8].

By using *sector theory* arising from infinte subfactors, M. Izumi obtained the categorical quantum double of Δ [9] and this construction was nothing but the center construction of V. Drinfel'd [10], which was pointed out by M. Müger. Izumi further investigated some examples of his construction in particular in the case of $SU(N)_k$ WZW model for general N. For the author, the categorical quantum double of this tensor category looks quite close to Müger's crossed product in category theory , namely dividing out the double category $\hat{\Delta} \otimes \hat{\Delta}^{op}$ by the group symmetry $\mathbb{Z}_N$.

Müger's theory was inspired by a problem in *algebraic quantum field theory*. K.-H. Rehren conjectured that the extending endomorphisms on the observable algebra to the ones on the field algebra removes the degeneracy of the braiding [17, 18]. Müger solved this conjecture [13] and he noticed that it could be possible to formulate the whole theory in terms of tensor category. His formulation crucially depends on Doplicher-Roberts duality theory. (Almost at the same time, A. Bruguières had a similar result in a more algebraic way by using duality theorem of Deligne[5].)

This note is an exposition of the published paper [21] and we will overview what is written in this paper.

Main results

- In the case that we have Longo-Rehren inclusions $A \supset B_\Delta \supset B_{\hat{\Delta}}$ for a minimal non-degenerate extension $\hat{\Delta} \supset \Delta$, we have a simple explicit description of the quantum double of Δ (Theorem 1).
- As an application of an orbifold aspect of the inclusions $A \supset B_\Delta \supset B_{\hat{\Delta}}$, we have an explicit description of the Reshetikhin-Turaev invariant of closed 3-manifolds constructed from the quantum double of Δ by using the framed link invariants of $\hat{\Delta}$ (Theorem 2).

§1 Preliminaries

We explain the terms mentioned in the previoius section.

1.1 Braided system of endomorphisms

Braided system of endomorphisms.

Let M be an infinite factor, and Δ_0 be the set of irreducible normal $*$-endomorphisms of M closed under the following sector operations:

(i) Different elements in Δ_0 are unitarily inequivalent.
(ii) $id_M \in \Delta_0$.
(iii) For every $\xi \in \Delta_0$ there exists $\bar{\xi} \in \Delta_0$ such that $\overline{[\xi]} = [\bar{\xi}]$.
(iv) There exists a non-negative integer $N_{\xi\eta}^{\zeta}$ such that $[\xi][\eta] = \oplus_{\zeta \in \Delta_0} N_{\xi\eta}^{\zeta}[\zeta]$.

We denote by Δ the subset of $\mathrm{End}(M)_0$ whose element is decomposed into finite direct sums of the elements in Δ_0 as sectors.

A system of endomorphisms Δ_0 is called *braided* if for any $\lambda, \mu \in \Delta_0$ there exists a unitary intertwiner $\varepsilon(\lambda, \mu) \in \mathrm{Hom}(\lambda \cdot \mu, \mu \cdot \lambda)$ with $\varepsilon(id, \mu) = \varepsilon(\lambda, id) = 1$ satisfying the following (the Braiding-Fusion equations):
pause For any $\lambda, \mu, \nu \in \Delta_0$, $t \in \mathrm{Hom}(\lambda, \mu \cdot \nu)$,

$$
\begin{aligned}
\sigma(t)\varepsilon(\lambda, \sigma) &= \varepsilon(\mu, \sigma)\mu(\varepsilon(\nu, \sigma))t \\
t\varepsilon(\sigma, \lambda) &= \mu(\varepsilon(\sigma, \nu))\varepsilon(\sigma, \mu)\sigma(t) \\
\sigma(t)^*\varepsilon(\mu, \sigma)\mu(\varepsilon(\nu, \sigma)) &= \varepsilon(\lambda, \sigma)t^* \\
t^*\mu(\varepsilon(\sigma, \nu))\varepsilon(\sigma, \mu) &= \varepsilon(\sigma, \lambda)\rho(t)^*.
\end{aligned}
$$

We call above ε a *braiding* on Δ_0. For a given braiding $\varepsilon(\lambda, \mu)$ on Δ_0, unitary intertwiners $\varepsilon(\mu, \lambda)^*$ also satisfies the above conditions of the braiding. We will use the notations $\varepsilon^+(\lambda, \mu) = \varepsilon(\lambda, \mu)$ and $\varepsilon^-(\lambda, \mu) = \varepsilon(\mu, \lambda)^*$ to emphasize the difference.

Degenerate sectors.

A sector $\xi \in \Delta$ is said to be *degenerate* if $\varepsilon^+(\xi, \eta) = \varepsilon^-(\xi, \eta)$ for every $\eta \in \Delta_0$. Δ is said to be *non-degenerate* if id_M is the only degenerate sector. We denote the set of all of degenerate sectors in Δ by Δ^d and the set of all of irreducible sectors in Δ^d by Δ^d_0. Note that Δ^d is a symmetric C^*-tensor subcategory of Δ with direct sums, subobjects and conjugates.

For $\xi \in \Delta^d_0$, $\phi_\xi(\varepsilon(\xi, \xi)) = \lambda_\xi \in \mathbb{C}$, where ϕ_ξ is the standard left inverse of ξ. The polar decomposition of λ_ξ is given by $\frac{\omega_\xi}{d(\xi)}$. It is easy to show that $\omega_\xi = \pm 1$ for $\xi \in \Delta^d$ (more generally, for an object in a symmetric C^*-tensor category). Δ^d is said to be *even* if $\omega_\xi = 1$ for every irreducible $\xi \in \Delta^d$. We assume Δ^d is even in the sequel. Then, by Doplicher-Roberts duality theory [6], there exists a finite group G up to isomorphism such that $\Delta^d \cong U(G)$, where $U(G)$ is a category of finite dimensional unitary representations of G.

α-induction.

Let $M \supset N$ be an inclusion of infinite factors with finite index and γ be its canonical endomorphism. Let $\Delta_0 \subset \mathrm{End}(N)_0$ be a braided system of endomorphisms with a braiding ε. We define the α-induced endomorphism of $\lambda \in \Delta_0$ $\alpha_\lambda \in \mathrm{End}(M)$ by

$$\alpha_\lambda = \gamma^{-1} \cdot Ad(\varepsilon(\lambda, \theta)) \cdot \lambda \cdot \gamma,$$

where $\theta = \gamma|_N$. This definition of the α-induction may look awful, but not much as we will see in the case of inclusions of crossed product types.

The systematic use of α-induction was first made by Feng Xu [23], and further studied in a series of papers by Böckenhauer and Evans [2, 3, 4]. We list some properties of the α-induction:

(i) $d(\alpha_\lambda) = d(\lambda)$
(ii) $\alpha_\lambda \cdot \alpha_\mu = \alpha_{\lambda \cdot \mu}$ for any $\lambda, \mu \in \Delta_0$
(iii) $\alpha_\mu \cdot \alpha_\lambda = Ad(\varepsilon(\lambda, \mu)) \cdot \alpha_\lambda \cdot \alpha_\mu$ for any $\lambda, \mu \in \Delta_0$
(iv) If $[\lambda] = [\lambda_1] \oplus [\lambda_2]$, $\lambda, \lambda_1, \lambda_2 \in \Delta$, then $[\alpha_\lambda] = [\alpha_{\lambda_1}] \oplus [\alpha_{\lambda_2}]$ and
(v) $[\alpha_{\bar\lambda}] = [\overline{\alpha_\lambda}], \lambda \in \Delta_0$.

1.2 Premodular categories

Assumption

We assume that $\mathcal{C}$ is a C^*-tensor category with conjugate, direct sums, subobjects, irreducible unit object ι and a unitary braiding ε.

We use the following notations which are popular in the context of the algebraic quantum field theory:
We use small Greek letters ρ, σ etc for objects of $\mathcal{C}$, and the tensor product is denoted by $\rho\sigma$ instead of $\rho \otimes \sigma$.

For operations of arrows, we denote the composition of arrows $S \in \mathrm{Hom}(\rho, \sigma)$, $T \in \mathrm{Hom}(\sigma, \tau)$ by $T \circ S \in \mathrm{Hom}(\rho, \tau)$, the tensor product of $S \in \mathrm{Hom}(\rho_1, \sigma_1)$, $T \in \mathrm{Hom}(\rho_2, \sigma_2)$ by $S \times T \in \mathrm{Hom}(\rho_1\rho_2, \sigma_1\sigma_2)$. We denote by $\mathcal{C}_0$ the set of isomorphism classes of irreducible objects.

We remark that under Assumption $\mathcal{C}$ is a ribbon category and we denote a twist for each irreducible object $\rho \in \mathcal{C}$ by ω_ρ.

Since we assume that $\mathcal{C}$ has a conjugate $\bar{\rho}$ for each object ρ, there are $R_\rho \in \mathrm{Hom}(\iota, \bar{\rho}\rho)$ and $\bar{R}_\rho \in \mathrm{Hom}(\iota, \rho\bar{\rho})$ satisfying

$$(\bar{R}_\rho^* \times id_\rho) \circ (id_\rho \times R_\rho) = id_\rho, \ (id_\rho \times R_\rho{}^*) \circ (\bar{R}_\rho \times id_\rho) = id_\rho.$$

Then, the dimension of an irreducible object ρ is defined by $d(\rho) = R_\rho{}^* \circ R_\rho$, which takes its value in $[1, \infty)$.

If the set $\mathcal{C}_0$ is finite, the category is called *rational.* Then, its dimension is defined by $\dim \mathcal{C} = \sum_{\xi \in \mathcal{C}_0} d(\xi)^2$. In subfactor context, this is called the *global index.*

When $\mathcal{C}$ is rational, then we set the complex number

$$S'(\xi, \eta) id_\iota = (R_\xi{}^* \times \bar{R}_\eta^*) \circ (id_{\bar{\xi}} \times (\varepsilon(\eta, \xi) \circ \varepsilon(\xi, \eta)) \times id_{\bar{\eta}}) \circ (R_\xi \times \bar{R}_\eta)$$

for $\xi, \eta \in \mathcal{C}_0$.

If S' is invertible, $\mathcal{C}$ is called *modular.* When $\mathcal{C}$ is modular, the matrices

$$S = \dim \mathcal{C}^{-\frac{1}{2}} S', \ T = \left(\frac{\Delta_\mathcal{C}}{|\Delta_\mathcal{C}|}\right)^{\frac{1}{3}} \mathrm{Diag}(\omega_\xi)$$

are unitaries and satisfy the relations

$$S^2 = (ST)^3 = C,\ TC = CT,$$

where $\Delta_{\mathcal{C}} = \sum_{\xi\in\mathcal{C}_0} d(\xi)^2\omega(\xi)^{-1}$ and $C = \delta_{\xi,\bar{\eta}}$.

Definition. If $\mathcal{C}$ satisfies Assumption and is rational, we say $\mathcal{C}$ is C^*-premodular.

For a C^*-premodular category $\mathcal{C}$ and its full subcategory $\mathcal{S}$, we define $\mathcal{C}\cap\mathcal{S}'$, a full subcategory of $\mathcal{C}$, by

$$\mathrm{Obj}\ \mathcal{C}\cap\mathcal{S}' = \{\rho\in\mathcal{C}|\varepsilon(\sigma,\rho)\circ\varepsilon(\rho,\sigma) = id_{\rho\sigma} \text{ for all } \sigma\in\mathcal{S}\}.$$

We remark that if $\mathcal{C}$ is modular we have

$$\dim\ \mathcal{C}\cap\mathcal{S}' = \frac{\dim\ \mathcal{C}}{\dim\ \mathcal{S}}$$

due to a Theorem of Müger.

Let $\mathcal{C}$ be a C^*-premodular category and we set $\mathcal{D}_{\mathcal{C}} = \mathcal{C}\cap\mathcal{C}'$. We assume that $\mathcal{D}_{\mathcal{C}}$ is even, i.e., twist $\omega_\xi = 1$ for each irreducible object ξ. Then, by Doplicher-Roberts duality theory [6], there is a finite group such that $\mathcal{D}_{\mathcal{C}}$ is equivalent to $U(G)$ as symmetric tensor $*$-categories with conjugates, where $U(G)$ is a category of finite dimensional unitary representations of G.

Let F be an invertible functor from $\mathcal{D}_{\mathcal{C}}$ to $U(G)$ which gives the equivalence, $\hat{G}$ be the set of all isomorphism classes of irreducible objects in $\mathcal{D}_{\mathcal{C}}$, $\{\gamma_k|k\in\hat{G}\}$ be a section of objects in $\mathcal{D}_{\mathcal{C}}$ such that $\gamma_0 = \iota$ and $\mathcal{H}_k = F(\gamma_k)$. We choose an orthonormal basis $\{V_{k,l}^{m,\alpha}\}_{\alpha=1}^{N_{kl}^m}$ of $\mathrm{Hom}(\gamma_m, \gamma_k\gamma_l)$.

Müger's crossed product.

M. Müger has defined a new tensor category $\mathcal{C}\rtimes_0\mathcal{D}_{\mathcal{C}}$ out of $\mathcal{C}$. The objects and morphisms are defined in the following manner [14].

- $\mathrm{Obj}\ \mathcal{C}\rtimes_0\mathcal{D}_{\mathcal{C}} = \mathrm{Obj}\ \mathcal{C}$ with the same tensor product as $\mathcal{C}$
- $\mathrm{Hom}_{\mathcal{C}\rtimes_0\mathcal{D}_{\mathcal{C}}}(\rho,\sigma) = \bigoplus_{k\in\hat{G}}\mathrm{Hom}_{\mathcal{C}}(\gamma_k\rho,\sigma)\otimes\mathcal{H}_k$.

With additional conditions on the morphisms such as the compositions, tensor products and $*$-operations.

§2 Müger's crossed product versus α-induction for subfactors

Let M, Δ and Δ^d be as in Subsection 1.1, and we assume that Δ_0 is a finite set. We further assume that Δ^d is even and $\Delta^d \cong U(G)$, where G is a finite group. Then, by Doplicher-Roberts duality theory [20] there exists a factor, denoted by $M \rtimes \hat{G}$, which contains M as a subfactor with index $|G|$.

We may assume that $M \rtimes \hat{G}$ is generated by M and isometries $\{\psi_i^{(\sigma)}$, $i = 1, \cdots, d(\sigma), \sigma \in \Delta_0^d\}$ satisfying :

$$\psi^{(\iota)} := \psi_1^{(\iota)} = 1 \tag{1}$$

$$\psi_i^{(\sigma)*}\psi_j^{(\sigma')} = \delta_{i,j}\delta_{\sigma,\sigma'} \tag{2}$$

$$\sum_{i=1}^{d(\sigma)} \psi_i^{(\sigma)}\psi_i^{(\sigma)*} = 1 \tag{3}$$

$$\psi_i^{(\sigma)}x = \sigma(x)\psi_i^{(\sigma)}, \ x \in M \tag{4}$$

$$\psi_i^{(\rho)}\psi_j^{(\sigma)} = \sum_{\tau\in\Delta_0^d}\sum_{k=1}^{d(\tau)} V_{(\rho,i)(\sigma,j)}^{(\tau,k)}\psi_k^{(\tau)} \tag{5}$$

$$\psi_i^{(\sigma)*} = R_\sigma^*\psi_i^{(\bar{\sigma})} \tag{6}$$

$$\sum_{i=1}^{d(\sigma_1)}\sum_{j=1}^{d(\sigma_2)} \psi_j^{(\sigma_2)}\psi_i^{(\sigma_1)}\psi_j^{(\sigma_2)*}\psi_i^{(\sigma_1)*} = \varepsilon(\sigma_1,\sigma_2), \tag{7}$$

where $V_{(\rho,i)(\sigma,j)}^{(\tau,k)} \in \mathrm{Hom}(\tau, \rho\cdot\sigma)$ and $R_\sigma \in \mathrm{Hom}(\iota, \bar{\sigma}\cdot\sigma)$.

Remark.
(1) It is known that $\{\psi_i^{(\sigma)}$, $i = 1, \cdots, d(\sigma), \sigma \in \Delta_0^d\}$ is a left M-module basis.
(2) When $x = \sum_{\sigma,i} t_i^{(\sigma)}\psi_i^{(\sigma)} \in M \rtimes \hat{G}$, the conditional expectation $E : M \rtimes \hat{G} \longrightarrow M$ is given by $E(x) = t^{(\iota)}$. By computations, one has $E(\psi_i^{(\sigma)}\psi_j^{(\rho)*}) = \delta_{\sigma,\rho}\delta_{i,j}\frac{1}{d(\sigma)}$, where $\lambda = [M \rtimes \hat{G} : M]$.

Lemma.
Let $v = \sum_{\sigma,i} t_i^{(\sigma)}\psi_i^{(\sigma)} \in \mathrm{Hom}(id, \gamma)$. Then, we have the relations $t_i^{(\sigma)} = d(\sigma)E(v\psi_i^{(\sigma)*}) \in \mathrm{Hom}(\sigma, \theta)$ and $\psi_i^{(\sigma)} = \frac{\lambda}{d(\sigma)}t_i^{(\sigma)*}v$. Furthermore, $t_i^{(\sigma)}$, $i = 1, \cdots, d(\sigma)$ satisfy $t_i^{(\sigma)*}t_j^{(\rho)} = \delta_{\sigma,\rho}\delta_{i,j}\frac{d(\sigma)}{\lambda}$ and $\sum_{\sigma,i}\frac{\lambda}{d(\sigma)}t_i^{(\sigma)}t_i^{(\sigma)*} = 1$.

Proposition. The equation (7) is equivalent to the identity $\varepsilon(\theta,\theta)v^2 = v^2$ for $v \in \mathrm{Hom}(id, \gamma)$.

Remark. The identity $\varepsilon(\theta,\theta)v^2 = v^2$ is called the *chiral locality* condition. Chiral locality naturally appears in the context of the algebraic quantum field theory in the approach using subfactors. But, for general subfactors, not appearing in algeraic quantum field theory, this chiral locality does not hold in general.

Lemma. For $\lambda \in \Delta$, we have

$$\alpha_\lambda^\pm(\psi_i^{(\sigma)}) = \varepsilon^\pm(\lambda,\sigma)^*\psi_i^{(\sigma)}, \tag{8}$$

where $\sigma \in {\Delta^d}_0, i = 1,\cdots,d(\sigma)$. In particular, $\alpha_\lambda^+ = \alpha_\lambda^-$ for $\lambda \in \Delta \cap \Delta^{d'} = \{\rho_\xi \in \Delta | \varepsilon(\xi,\sigma)\varepsilon(\sigma,\xi) = 1, \ \forall\sigma \in \Delta_0^d\}$.

Lemma. For $\lambda, \mu \in \Delta$,

$$\mathrm{Hom}(\alpha_\lambda,\alpha_\mu) = \{\sum_{\sigma\in\Delta_0^d}\sum_{i=1}^{d(\sigma)} t_i^{(\sigma)}\psi_i^{(\sigma)}; t_i^{(\sigma)} \in \mathrm{Hom}(\sigma\cdot\lambda,\mu),\ i = 1,\cdots,d(\sigma)\}.$$

Remark. By the above lemma, we have

$$\mathrm{Hom}(id,\alpha_\rho) = \{\sum_{i=1}^{d(\rho)} t_i^{(\rho)}\psi_i^{(\rho)}; t_i^{(\rho)}\rho(x) = \rho(x)t_i^{(\rho)},\ \forall x \in M, i = 1,\cdots,d(\rho)\}$$

for $\rho \in \Delta_0^d$, which is a Hilbert space with dimension $d(\rho)$. Since $d(\alpha_\rho) = d(\rho)$, we conclude that $\alpha_\rho \cong \oplus_{i=1}^{d(\rho)} id$. This can be read that α-induction trivializes degenerate sectors.

Let $\lambda \in \Delta \cap \Delta^{d'}$ and we use the notation α_λ instead of $\alpha_\lambda^+ = \alpha_\lambda^-$. We denote by $(\Delta \cap \Delta^{d'})^\alpha$ the subset of $\mathrm{End}(M \rtimes \hat{G})_0$ consisting of subsectors of α_λ, when λ varies in $\Delta \cap \Delta^{d'}$.

Under these preliminaries, we have the following

Proposition. $(\Delta \cap \Delta^{d'})^\alpha$ is a modular category.

So far, we have discussed the similarities to Müger's theory of crossed product. In fact, we have the following

Proposition.
For the inclusion $M \rtimes \hat{G} \supset M$, $(\Delta \cap \Delta^{d'})^{\alpha}$ is naturally identified with Müger's crossed product $(\Delta \cap \Delta^{d'}) \rtimes \Delta^{d}$.

§3 Longo-Rehren inclusions $A \supset B_{\Delta} \supset B_{\hat{\Delta}}$

Let Δ be a subset of $\mathrm{End}(M)_0$ with a finite braided system Δ_0, $\hat{\Delta} \supset \Delta$ its non-degenerate extension. The following definition was first introduced by Ocneanu [16].

Definition. The non-degenerate extension $\hat{\Delta} \supset \Delta$ is called minimal if $\hat{\Delta} \cap \Delta' = \Delta^{d}$.

Remark that we have $\dim \hat{\Delta} = \dim \Delta \dim \Delta^{d}$ if the extension is minimal.

We assume the minimality of the non-degenerate extension $\hat{\Delta} \supset \Delta$ in the sequel.

Longo-Rehren inclusion.
Let $\{T(\,^{\zeta}_{\xi,\eta})_i\}_{i=1}^{N^{\zeta}_{\xi,\eta}}$ be an orthonormal basis of $\mathrm{Hom}(\zeta, \xi \cdot \eta)$, $\xi, \eta, \zeta \in \Delta_0$. Let M^{op} be the opposite algebra of M and $j : M \longrightarrow M^{op}$ the anti-linear isomorphism. We set $A = M \otimes M^{op}$, $\xi^{op} = j \cdot \xi \cdot j$, and $\hat{\xi} = \xi \otimes \xi^{op}$. For the isometries $\{V_{\xi}\}_{\xi \in \Delta_0} \subset A$ satisfying $\sum_{\xi \in \Delta_0} V_{\xi} V_{\xi}^{*} = 1$, we define

$$\gamma_{\Delta}(x) = \sum_{\xi \in \Delta_0} V_{\xi} \hat{\xi}(x) V_{\xi}^{*}.$$

Let $V_{\Delta} \in \mathrm{Hom}(id, \gamma)$, $W_{\Delta} \in \mathrm{Hom}(\gamma, \gamma^2)$ be isometries defined by

$$V_{\Delta} = V_{id_M},$$
$$W_{\Delta} = \sum_{\xi,\eta,\zeta \in \Delta_0} \sqrt{\frac{d(\xi)d(\eta)}{\dim \Delta d(\zeta)}} V_{\xi} \hat{\xi}(V_{\eta}) T^{\zeta}_{\xi,\eta} V_{\zeta}^{*},$$

where $T^{\zeta}_{\xi,\eta} = \sum_{i=1}^{N^{\zeta}_{\xi,\eta}} T(\,^{\zeta}_{\xi,\eta})_i \otimes j(T(\,^{\zeta}_{\xi,\eta})_i)$.

Then, one can construct a subfactor B_{Δ} of A such that $\gamma_{\Delta} : A \longrightarrow B_{\Delta}$ is the canonical endomorphism of the inclusion $A \supset B_{\Delta}$. We call the inclusion $A \supset B_{\Delta}$ the *Longo-Rehren inclusion* [12].

In a similar manner, we can construct the Longo-Rehren inclusion $A \supset B_{\hat{\Delta}}$. By their constructions, we have the inclusions $A \supset B_\Delta \supset B_{\hat{\Delta}}$.

We define $D(\Delta)$ to be the set of endomorphisms $\rho \in \mathrm{End}(B_\Delta)_0$ such that $[\iota_\Delta][\rho]$ is a finite direct sum of sectors in the decompositions of $\{[\xi \otimes id^{op}][\iota_\Delta]\}_{\xi\in\Delta_0}$, where ι_Δ is the inclusion map $\iota_\Delta : B_\Delta \hookrightarrow A$. We call $D(\Delta)$ the *quantum double* of Δ. Izumi proved that $D(\hat{\Delta})$ is equivalent to $\hat{\Delta} \otimes \hat{\Delta}^{op}$ as modular categories [9]. (The similar thing in the case of an asymptotic inclusion had been proved by Evans-Kawahigashi [8].)

Proposition.
We assume that $\Delta^d \cong U(G)$, where G is an *abelian group.* Then, there exists an outer action α of G on $B_{\hat{\Delta}}$ and the subfactor $B_\Delta \supset B_{\hat{\Delta}}$ is isomorphic to $B_{\hat{\Delta}} \rtimes_\alpha G \supset B_{\hat{\Delta}}$.

Theorem 1. [21]
Let $D(\Delta)$ be the quantum double of Δ. Then, under the assumptions in Proposition in Section 2, $D(\Delta) = (\hat{\Delta} \otimes \hat{\Delta}^{op} \cap \Delta^{d'}) \rtimes \Delta^d$, where the embedding $\iota_{\Delta^d} : \Delta^d \hookrightarrow \hat{\Delta} \otimes \hat{\Delta}^{op}$ is given by $\iota_{\Delta^d}(\sigma) = (\sigma, \sigma^{op})$.

§4 Application to the Reshetikhin-Turaev invariants for 3-manifolds

We apply Theorem 1 to the Reshetikhin-Turaev invariant of 3-manifolds constructed from the quantum double $D(\Delta)$ to get a simpler description of it in this case. See [21] for the details.

Lemma. Let $\mathcal{M}$ be a premodular category, $\mathcal{P}$ the non-degenerate extension of $\mathcal{M}$ and $\mathcal{D}$ be degenerates of $\mathcal{M}$, i.e., $\mathcal{D} = \mathcal{M} \cap \mathcal{M}'$. Then, we have

$$\sum_{\omega\in\mathcal{M}_0} N^{\omega}_{\eta\bar{\zeta}} d(\omega) = d(\eta\bar{\zeta})\chi_{\mathcal{M}}(\eta\bar{\zeta}), \tag{9}$$

where $\chi_{\mathcal{M}}(\xi) = 1$ if $\xi \in \mathcal{M}, 0$ otherwise.

Let $\mathcal{C}$ be a premodular category. Let L be a framed link with n components in the 3-sphere. We denote the invariant of the colored framed link by $F_{\mathcal{C}}(L, \lambda)$, where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathcal{C}_0^n$. Set

$$\{L\}_{\mathcal{C}} = \sum_{\lambda\in\mathcal{C}_0^n} \prod_{i=1}^{n} d(\lambda_i) F_{\mathcal{C}}(L; \lambda).$$

We may assume that a closed 3-manifold M is obtained from surgery along the framed link L in the 3-sphere S^3. We denote the signature of L by $\sigma(L)$.

Let $\mathcal{C}$ be a modular category and we set $\Delta_{\mathcal{C}} = \sum_{\xi \in \mathcal{C}_0} \omega_\xi^{-1} d(\xi)^2$ and $D_{\mathcal{C}} = (\dim \mathcal{C})^{1/2}$. The Reshetikhin-Turaev invariant $\tau_{\mathcal{C}}$ is defined by

$$\tau_{\mathcal{C}}(M) = (\Delta_{\mathcal{C}})^{\sigma(L)} D_{\mathcal{C}}^{-\sigma(L)-n-1} \{L\}_{\mathcal{C}}.$$

Lemma. Let $\mathcal{C}$ be a premodular category with $\mathcal{C} \cap \mathcal{C}' = \mathcal{D}$ and L be a framed link with n components. Then, we have

$$\{L\}_{\mathcal{C}} = (\dim \mathcal{D})^n \{L\}_{\mathcal{C} \rtimes \mathcal{D}}.$$

We now go back in the case of braided C^*-tensor categories $\hat{\Delta}$ and Δ associated with subfactors. Recall that we have assumed the minimality of the non-degenerate extension $\hat{\Delta} \supset \Delta$. For $\lambda, \mu \in \hat{\Delta}$, we put

$$[\lambda, \mu]_\Delta = \frac{1}{\dim \hat{\Delta}} \sum_{\nu \in \Delta_0} N_{\lambda \bar{\mu}}^{\nu} d(\nu).$$

Theorem 2. [21]
Let M be a closed 3-manifold obtained from surgery along the framed link L with n components. Then, the Reshetikhin-Turaev invariant for $D(\Delta)$ is given by

$$\tau_{D(\Delta)}(M) = \frac{1}{\dim \Delta} \sum_{\lambda, \mu \in \hat{\Delta}_0^n} \prod_{i=1}^{n} [\lambda_i, \mu_i]_\Delta F_{\hat{\Delta}}(L; \lambda) \overline{F_{\hat{\Delta}}(L; \mu)}.$$

Final Remark. According to the main theorem in [11], the Turaev-Viro-Ocneanu invariant $Z_\Delta(M)$ obtained from Δ satisfying the same condition as in Theorem 2. , we have the following equality

$$Z_\Delta(M) = \frac{1}{\dim \Delta} \sum_{\lambda, \mu \in \hat{\Delta}_0^n} \prod_{i=1}^{n} [\lambda_i, \mu_i]_\Delta F_{\hat{\Delta}}(L; \lambda) \overline{F_{\hat{\Delta}}(L; \mu)}.$$

This gives a proof of Ocneanu's claim in the case of a group G is abelian. So far, we have no idea to extend our theorem in the case of non-abelian groups. Moreover, There are a few examples of minimal non-degenerate extension. Ocneanu has claimed that there always exists a unique minimal non-degenerate extension of Δ. But, now it is known that uniqueness does not hold. To construct minimal non-degenrate extensions is still left over.

References

[1] M. F. Atiyah, *Topological quantum field theories*, Publ. Math. I.H.E.S. 68 (1989) 175–186.

[2] J. Böckenhauer, D. E. Evans, *Modular invariants, graphs, and α-induction for nets of subfactors. I*, Commun. Math. Phys. 197 (1998), 361–386.

[3] J. Böckenhauer, D. E. Evans, *Modular invariants, graphs, and α-induction for nets of subfactors. II*, Commun. Math. Phys. 200 (1999), 57–103.

[4] J. Böckenhauer, D. E. Evans, *Modular invariants, graphs, and α-induction for nets of subfactors. III*, Commun. Math. Phys. 205 (1999), 183–228.

[5] A. Bruguières, *Catégories prémodulaires, modularisations et invariants de variété de dimension 3*, Math. Ann. 316 (2000), 215–236.

[6] S. Doplicher, J.E. Roberts, *Endomorphisms of C^*-algebras, crossed products and duality for compact groups*, Ann. Math. 130 (1989), 75–119.

[7] D. E. Evans, Y. Kawahigashi, *Orbifold subfactors from Hecke algebras*, Commun. Math. Phys. 165 (1994) 445–484.

[8] D. E. Evans, Y. Kawahigashi, *Orbifold subfactors from Hecke algebras II*, Commun. Math. Phys. 196 (1998) 331–361.

[9] M. Izumi, *The structures of sectors associated with the Longo-Rehren inclusions I. General theory*, Commun. Math.Phys. 213 (2000) 127–179.

[10] C. Kassel, *Quantum groups*, Springer-Verlag 1995.

[11] Y. Kawahigashi, N. Sato, M. Wakui, *(2+1)-dimensional topological quantum field theory from subfactors and Dehn surgery formula for 3-manifold invariants*, Adv. Math. 195 (2005), 165–204, math.OA/0208238.

[12] R. Longo, K.-H. Rehren, *Nets of subfactors* , Rev. in Math. Phys. 7 (1995), 567–597.

[13] M. Müger, *On charged fields with group symmetry and degeneracies of Verlinde's matrix S*, Ann. Inst. Henri Poincaré, 71 (1999), 359–394.

[14] M. Müger, *Galois theory for braided tensor categories and the modular closure*, Adv. Math. 150 (2000), 151–201.

[15] M. Müger, *On the structure of modular categories*, Proc. London Math. Soc. 87 (2003), 291–308.

[16] A. Ocneanu, *Chirality for operator algebras*, 39–63, in "Subfactors", ed. by H. Araki, et al., World Scientific, 1994.

[17] K.-H. Rehren, *Braid group statistics and their super selection rules*, in "The Algebraic Theory of Superselection sectors", ed. by D. Kastler, World Scientific, 1990.

[18] K.-H. Rehren, *Markov traces as characters for local algebras*, Nucl. Phys. B(Proc. Suppl.) 18B (1990), 259–268.

[19] N. Reshetikhin and V. G. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. 103 (1991), p. 547–597.

[20] J. E. Roberts, *Crossed products of von Neumann algebras by group duals*, Sympos. Math. XX (1976), 335–363.

[21] N. Sato, *Orbifold aspects of the Longo-Rehren subfactors*, Commun. Math. Phys. 257 (2005), 151–167.

[22] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, Walter de Gruyter 1994.

[23] F. Xu, *New braided endomorphisms from conformal inclusions*, Commun. Math. Phys. 192 (1998) 349–403.

[24] S. Yamagami, *Group symmetry in tensor categories and duality for orbifolds*, J. Pure Appl. Alg. 167 (2002), 83–128.