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Orbital approach to free entropy and free entropy dimension

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INTRODUCTION

The (microstate) free entropy (as well as the free entropy dimension) is a highlight in free probability theory and its definition is based on the idea to regard matrices as microstates which approximate noncommutative random variables. In fact, the free entropy of several noncommutative random variables is the asymptotic growth rate of the volume of the set of matrices approximating those random variables in moments.

In this report we propose a somewhat new approach to microstate free entropy and free entropy dimension based on the joint work [6] with T. Miyamoto and Y. Ueda. §1 is a short survey on the microstate free entropy χ mostly developed by Voiculescu [15]–[18] and [20]. (Also, Voiculescu developed the non-microstate free entropy χ^* in [19].) In §2 we introduce the orbital free entropy which is defined in terms of the unitary orbital microstates of given noncommutative random variables. We establish the relation between χ and χ_{orb}. The quantity $i := -\chi_{\mathrm{orb}}$ is the free probabilistic analog of the classical mutual information, which is also considered as the microstate counterpart of the mutual free information i^* introduced in [21]. §3 is a brief survey on the free entropy dimension δ and its modified version δ_0 developed in [16] and [17]. An important fact due to Jung [11] is that δ_0 is equal to the fractal free entropy dimension δ_1 defined via the packing number of the set of approximating microstates. In §4 we introduce the orbital versions $\delta_{0,\mathrm{orb}}$ of δ_0 and $\delta_{1,\mathrm{orb}}$ of δ_1. We discuss the relations among δ_0, $\delta_{0,\mathrm{orb}}$ and $\delta_{1,\mathrm{orb}}$.

1. MICROSTATE FREE ENTROPY

Let us start with the classical result providing the microstate formulation for the Boltzmann-Gibbs entropy. Let $\vec{X} = (X_1, \dots, X_n)$ be an n-tuple of classical random variables, whose Boltzmann-Gibbs entropy $H(\vec{X})$ is defined by

$$H(\vec{X}) := \begin{cases} -\int_{\mathbb{R}^n} p(\vec{x}) \log p(\vec{x})\, d\vec{x} & \text{if } \mu_{\vec{X}} \ll d\vec{x} \text{ and } p := d\mu_{\vec{X}}/d\vec{x}, \\ -\infty & \text{otherwise,} \end{cases}$$

where $\mu_{\vec{X}}$ is the distribution measure of $\vec{X}$ and $d\vec{x}$ the Lebesgue measure on $\mathbb{R}^n$. Here, assume that all X_i are bounded, and choose $R \geq \max_{1\leq i\leq n} \|X_i\|_\infty$. We consider n-tuples of $\mathbb{R}^N$-vectors as microstates, which are conveniently written in the matrix form

$$\vec{x} = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N) = \begin{bmatrix} x_{11} & x_{22} & \cdots & x_{1N} \\ x_{21} & x_{22} & \cdots & x_{2N} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nN} \end{bmatrix}.$$

For each $N, m \in \mathbb{N}$ and $\delta > 0$ define the sets of microstates approximating $\vec{X}$ as follows:

$$\Delta(\vec{X}; N, m, \delta) := \left\{ \vec{x} \in (\mathbb{R}^N)^n : \left| \frac{1}{N} \sum_{k=1}^{N} x_{i_1 k} x_{i_2 k} \cdots x_{i_r k} - \mathbb{E}(X_{i_1} X_{i_2} \cdots X_{i_r}) \right| \leq \delta \right.$$
$$\left. \text{for all } 1 \leq i_1, \ldots, i_r \leq n \text{ and } 1 \leq r \leq m \right\}, \quad (1.1)$$
$$\Delta_R(\vec{X}; N, m, \delta) := \Delta(\vec{X}; N, m, \delta) \cap ([-R, R]^N)^n.$$

Proposition 1.1. *With the above definitions,*

$$H(\vec{X}) = \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \lim_{N\to\infty} \frac{1}{N} \log \lambda_N^{\otimes n}(\Delta_R(\vec{X}; N, m, \delta)) \quad (1.2)$$

independently of the choice of $R \geq \max_{1\leq i\leq n} \|X_i\|_\infty$, *where* λ_N *is the Lebesgue measure on* $\mathbb{R}^N$.

The definition of Voiculescu's microstate free entropy of an n-tuple of noncommutative random variables is the matricial microstate version of the above formula for $H(\vec{X})$.

Definition 1.2. Let M_N^{sa} denote the space of all Hermitian matrices in $M_N(\mathbb{C})$. Let $\vec{X} = (X_1, \ldots, X_n)$ be an n-tuple of noncommutative self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M}, \tau)$. For each $N, m \in \mathbb{N}$ and $\delta > 0$ define the set of microstates approximating $\vec{X}$ by

$$\Gamma(\vec{X}; N, m, \delta)$$
$$:= \left\{ \vec{A} = (A_1, \ldots, A_n) \in (M_N^{sa})^n : \left| \mathrm{tr}_N(A_{i_1} A_{i_2} \cdots A_{i_r}) - \tau(X_{i_1} X_{i_2} \cdots X_{i_r}) \right| \leq \delta \right.$$
$$\left. \text{for all } 1 \leq i_1, \ldots, i_r \leq n \text{ and } 1 \leq r \leq m \right\}, \quad (1.3)$$
$$\Gamma_R(\vec{X}; N, m, \delta) := \Gamma(\vec{X}; N, m, \delta) \cap (M_N^{sa})_R,$$

where $(M_N^{sa})_R := \{A \in M_N^{sa} : \|A\|_\infty \le R\}$. Furthermore, with the "Lebesgue" measure Λ_N on M_N^{sa} (the measure induced via the isometric isomorphism $M_N^{sa} \cong \mathbb{R}^{N^2}$) define

$$\chi_R(\vec{X}; m, \delta) := \limsup_{N\to\infty} \left(\frac{1}{N^2} \log \Lambda_N^{\otimes n}(\Gamma_R(\vec{X}; N, m, \delta)) + \frac{n}{2} \log N \right), \tag{1.4}$$

$$\chi_R(\vec{X}) := \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \chi_R(\vec{X}; m, \delta),$$

$$\chi(\vec{X}) := \sup_{R>0} \chi_R(\vec{X}).$$

Then $\chi(\vec{X})$ is called the (*microstate*) *free entropy* of $\vec{X}$.

The definition itself justifies that the free entropy $\chi(\vec{X})$ is the free probabilistic analog of the Boltzmann-Gibbs entropy. The analogy between (1.1) and (1.3) becomes clearer when we write

$$\frac{1}{N} \sum_{k=1}^{N} x_{i_1 k} x_{i_2 k} \cdots x_{i_r k} = \mathrm{tr}_N(A_{i_1} A_{i_2} \cdots A_{i_r}) \quad \text{for } A_i := \mathrm{Diag}(x_{i1}, x_{i2}, \ldots, x_{iN}).$$

Obvious differences of (1.4) from (1.2) are the scaling $1/N^2$, the term $\frac{n}{2}\log N$ and the lim sup instead of lim. The $1/N^2$-scaling is quite natural since microstates are matrices in $M_N^{sa} \cong \mathbb{R}^{N^2}$ and the $\frac{n}{2}\log N$-term is an appropriate renormalization from the choice of the volume Λ_N. We must take lim sup because the existence of limit is not at all guaranteed in (1.4), which makes the microstate free entropy quite difficult to handle.

The following are basic properties of $\chi(\vec{X})$ ([16, 18]; also [7, Chapter 6]).

1° $\chi(\vec{X}) = \chi_R(\vec{X})$ for any $R \ge \|\vec{X}\|_\infty := \max_{1\le i\le n} \|X_i\|_\infty$.

2° **(Single variable case)** For every single X with the distribution measure μ, $\chi(X)$ is equal to $\Sigma(\mu) := \iint_{\mathbb{R}^2} \log|x-y|\, d\mu(x)\, d\mu(y)$ up to an additive constant, i.e.,

$$\chi(X) = \Sigma(\mu) + \frac{3}{4} + \frac{1}{2}\log 2\pi.$$

Moreover, the lim sup in (1.4) can be replaced by lim for the single variable case.

3° **(Upper bound)**

$$\chi(\vec{X}) \le \frac{n}{2} \log\left(\frac{2\pi e}{n} \tau(X_1^2 + \cdots + X_n^2) \right).$$

4° **(Subadditivity)** $\chi(\vec{X}, \vec{Y}) \le \chi(\vec{X}) + \chi(\vec{Y})$ for all $\vec{X} = (X_1, \ldots, X_n)$ and $\vec{Y} = (Y_1, \ldots, Y_m)$.

5° **(Upper semicontinuity)** If $\vec{X}^{(k)} = (X_1^{(k)}, \ldots, X_n^{(k)})$, $k \in \mathbb{N}$, are n-tuples of self-adjoint random variables in $(\mathcal{M}, \tau)$ such that $\vec{X}^{(k)} \to \vec{X}$ in the distribution sense (i.e., in the sense of moment convergence) and $\sup_k \|\vec{X}^{(k)}\|_\infty < +\infty$, then

$$\chi(\vec{X}) \ge \limsup_{k\to\infty} \chi(\vec{X}^{(k)}).$$

6° **(Change of variable formula by noncommutative power series)** See [16] for details.

7° **(Separate change of variable formula)** Assume that $\chi(X_i) > -\infty$ for $1 \le i \le n$. If $f_1, \dots, f_n$ are real increasing continuous functions on $\mathbb{R}$, then

$$\chi(f_1(X_1), \dots, f_n(X_n)) \ge \chi(\vec{X}) + \sum_{i=1}^{n} (\chi(f_i(X_i)) - \chi(X_i)).$$

Moreover, if $f_1, \dots, f_n$ are strictly increasing, then

$$\chi(f_1(X_1), \dots, f_n(X_n)) = \chi(\vec{X}) + \sum_{i=1}^{n} (\chi(f_i(X_i)) - \chi(X_i)).$$

8° **(Infinitesimal change of variable formula)** If $P_1, \dots, P_n \in \mathbb{C}\langle t_1, \dots, t_n \rangle$ are noncommutative polynomials such that $P_i^* = P_i$, then the differential formula

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \chi(\vec{X} + \varepsilon P(\vec{X})) = \sum_{i=1}^{n} (\tau \otimes \tau)(\partial_i P_i(\vec{X}))$$

holds, where ∂_i is the free partial derivative with respect to X_i.

9° **(Additivity and freeness)** If $X_1, \dots, X_n$ are freely independent, then

$$\chi(\vec{X}) = \chi(X_1) + \cdots + \chi(X_n).$$

Moreover, the converse of the above holds true whenever $\chi(X_i) > -\infty$ for $1 \le i \le n$.

2. Orbital free entropy (or microstate mutual free information)

For $N \in \mathbb{N}$ let $\gamma_{\mathrm{U}(N)}$ denote the Haar probability measure on the unitary group $\mathrm{U}(N)$ of order N. For each $\alpha \in M_N^{sa}$ its distribution of $\alpha \in M_N^{sa}$ with respect to tr_N is denoted by μ_α, which is given by $\mu_\alpha = \frac{1}{N}\sum_{j=1}^{N} \delta_{\alpha_j}$ with the eigenvalues $\alpha_1, \dots, \alpha_N$ of α with counting multiplicities. We also define the map $\xi_{N,\alpha} : \mathrm{U}(N) \to M_N^{sa}$ by

$$\xi_{N,\alpha}(U) := U\alpha U^* \quad \text{for } U \in \mathrm{U}(N).$$

Let $\vec{X} = (X_1, \dots, X_n)$ be an n-tuple of self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M}, \tau)$. For each $1 \le i \le n$ we choose and fix a sequence $\alpha_i(N) \in M_N^{sa}$, $N \in \mathbb{N}$, such that $\mu_{\alpha_i(N)}$ converges to μ_{X_i} in moments as $N \to \infty$, i.e., $\mathrm{tr}_N(\alpha_i(N)^m) \to \tau(X_i^m)$ as $N \to \infty$ for all $m \in \mathbb{N}$. Of course, one can choose $\alpha_i(N)$ so that $\|\alpha_i(N)\|_\infty \le \|X_i\|_\infty$ for all N and $\mu_{\alpha_i(N)} \to \mu_{X_i}$ weakly as $N \to \infty$. For $\vec{\alpha}(N) := (\alpha_1(N), \dots, \alpha_n(N))$ chosen above, we write $\xi_{\vec{\alpha}(N)}$ in short for the map $\prod_{i=1}^{n} \xi_{N,\alpha_i(N)} : \mathrm{U}(N)^n \longrightarrow (M_N^{sa})^n$, i.e.,

$$\xi_{\vec{\alpha}(N)}(\vec{U}) := (U_1\alpha_1(N)U_1^*, \dots, U_n\alpha_n(N)U_n^*) \quad \text{for } \vec{U} = (U_1, \dots, U_n) \in \mathrm{U}(N)^n.$$

Definition 2.1. With the above notations, for each $N, m \in \mathbb{N}$ and $\delta > 0$, define

$$\Gamma_{\mathrm{orb}}(\vec{X}|\vec{\alpha}(N); N, m, \delta) := \xi_{\vec{\alpha}(N)}^{-1}(\Gamma(\vec{X}; N, m, \delta))$$

with $\Gamma(\vec{X}; N, m, \delta)$ given in (1.3). We then define

$$\chi_{\mathrm{orb}}(X_1; \ldots; X_n) := \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2} \log \gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_{\mathrm{orb}}(\vec{X}|\vec{\alpha}(N); N, m, \delta)\big),$$

$$i(X_1; \ldots; X_n) := -\chi_{\mathrm{orb}}(X_1; \ldots; X_n).$$

We call $\chi_{\mathrm{orb}}(X_1; \ldots; X_n)$ the *orbital free entropy* of $\vec{X}$ since it is defined in terms of the volume of some unitary orbital microstates. On the other hand, we call $i(X_1; \ldots; X_n)$ the *microstate mutual free information* of $\vec{X}$.

The next proposition says that the above definition of χ_{orb} is well defined independently of the choices of $\vec{\alpha}(N)$.

Proposition 2.2. *$\chi_{\mathrm{orb}}(X_1; \ldots; X_n)$ is independent of the choices of $\alpha_i(N) \in M_N^{sa}$, $N \in \mathbb{N}$, with $\mu_{\alpha_i(N)} \to \mu_{X_i}$ in moments as $N \to \infty$ for $1 \le i \le n$.*

The following are basic properties of the orbital free entropy χ_{orb}. The corresponding properties of i are obvious.

Proposition 2.3.

1° **(Single variable case)** $\chi_{\mathrm{orb}}(X) = 0$ *for a single variable* X.

2° **(Negativity)** $\chi_{\mathrm{orb}}(X_1; \ldots; X_n) \le 0$.

3° **(Subadditivity)** $\chi_{\mathrm{orb}}(X_1; \ldots; X_n) \le \chi_{\mathrm{orb}}(X_1; \ldots; X_k) + \chi_{\mathrm{orb}}(X_{k+1}; \ldots; X_n)$ *for every* $1 \le k < n$.

4° **(Upper semicontinuity)** *If $\vec{X}^{(k)} = (X_1^{(k)}, \ldots, X_n^{(k)})$, $k \in \mathbb{N}$, are n-tuples of self-adjoint random variables and $\vec{X}^{(k)} \to \vec{X}$ in distribution, then*

$$\chi_{\mathrm{orb}}(X_1; \ldots; X_n) \ge \limsup_{k\to\infty} \chi_{\mathrm{orb}}(X_1^{(k)}; \ldots; X_n^{(k)}).$$

Theorem 2.4. ([6])

$$\chi(\vec{X}) = \chi_{\mathrm{orb}}(X_1; \ldots; X_n) + \sum_{i=1}^{n} \chi(X_i).$$

The theorem says that $\chi_{\mathrm{orb}}(X_1; \ldots; X_n)$ is the free entropy for mutual relation among X_i's disregarding $\chi(X_i)$ for each separate X_i. To justify the terminology of $i = -\chi_{\mathrm{orb}}$, let us consider its analogy to the classical mutual information.

For two n-tuples $\vec{X} = (X_1, \ldots, X_n)$ and $\vec{Y} = (Y_1, \ldots, Y_n)$ of classical random variables, the (classical) *mutual information* $I(\vec{X}, \vec{Y})$ of $\vec{X}, \vec{Y}$ is normally defined by

$$I(\vec{X}; \vec{Y}) := S(\mu_{(\vec{X},\vec{Y})}, \mu_{\vec{X}} \otimes \mu_{\vec{Y}}) \left(= \int \log \frac{d\mu_{(\vec{X},\vec{Y})}}{d(\mu_{\vec{X}} \otimes \mu_{\vec{Y}})}\, d\mu_{(\vec{X},\vec{Y})} \right),$$

which is also expressed as

$$I(\vec{X}; \vec{Y}) = -H(\vec{X}, \vec{Y}) + H(\vec{X}) + H(\vec{Y})$$

in terms of Boltzmann-Gibbs entropies whenever the latter expression is meaningful. For two self-adjoint random variables X, Y Theorem 2.4 says that

$$i(X; Y) = -\chi_{\mathrm{orb}}(X; Y) = -\chi(X, Y) + \chi(X) + \chi(Y) \tag{2.5}$$

as long as both $\chi(X)$ and $\chi(Y)$ are finite. An advantage of the orbital free entropy χ_{orb} (or i) is that it can be defined (and often finite) for any self-adjoint random variables X, Y while the right-hand side of (2.5) makes sense only when both $\chi(X)$ and $\chi(Y)$ are finite. For example, the original χ is meaningless for projections since χ always takes $-\infty$ for them. In this connection, the exact formula of $\chi_{\mathrm{orb}}(p;q)$ for two projections p, q was obtained in [8] (see Example 4.9 in the last).

The expression (2.5) itself suggests that $i(X;Y)$ is the free analog of the classical mutual information. The analogy can be more strongly justified as follows.

Remark 2.5. For $N \in \mathbb{N}$ let γ_{S_N} be the uniform probability measure on the symmetric group S_N. For each $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ we define the map $\xi_{N,\alpha} : S_N \to \mathbb{R}^N$ by

$$\xi_{N,\alpha}(\sigma) := (\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \dots, \alpha_{\sigma(N)}) \quad \text{for } \sigma \in S_N.$$

Let $\vec{X} = (X_1, \dots, X_n)$ be an n-tuple of classical real bounded random variables and for $1 \le i \le n$ choose a sequence $\alpha_i(N) \in \mathbb{R}^N$, $N \in \mathbb{N}$, such that $\mu_{\alpha_i(N)} \to \mu_{X_i}$ weakly as $N \to \infty$ (here $\mu_\alpha := N^{-1} \sum_{j=1}^N \delta_{\alpha_j}$ for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$). We denote by $\xi_{\vec{\alpha}(N)}$ the map $\prod_{I=1}^n \xi_{N,\alpha_i(N)} : (S_N)^n \to (\mathbb{R}^N)^n$. For $N, m \in \mathbb{N}$ and $\delta > 0$ define

$$\Delta_{\mathrm{sym}}(\vec{X} | \vec{\alpha}(N); N, m, \delta) := \xi_{\vec{\alpha}(N)}^{-1}(\Delta(\vec{X}; N, m, \delta))$$

with $\Delta(\vec{X}; N, m, \delta)$ given in (1.1). We then define

$$\overline{H}_{\mathrm{sym}}(X_1; \dots; X_n) := \lim_{\substack{m\to\infty \\ \delta \searrow 0}} \limsup_{N\to\infty} \frac{1}{N} \log \gamma_{S_N}^{\otimes n}(\Delta_{\mathrm{sym}}(\vec{X} | \vec{\alpha}(N); N, m, \delta)),$$

$$\underline{H}_{\mathrm{sym}}(X_1; \dots; X_n) := \lim_{\substack{m\to\infty \\ \delta \searrow 0}} \liminf_{N\to\infty} \frac{1}{N} \log \gamma_{S_N}^{\otimes n}(\Delta_{\mathrm{sym}}(\vec{X} | \vec{\alpha}(N); N, m, \delta)).$$

As Proposition 2.2 it is easy to check that $\overline{H}_{\mathrm{sym}}(X_1; \dots; X_n)$ as well as $\underline{H}_{\mathrm{sym}}(X_1; \dots; X_n)$ is independent of the choices of $\alpha_i(N) \in \mathbb{R}^N$ with $\mu_{\alpha_i(N)} \to \mu_{X_i}$. Moreover one can show that

$$H(\vec{X}) = \overline{H}_{\mathrm{sym}}(X_1; \dots; X_n) + \sum_{i=1}^n H(X_i) = \underline{H}_{\mathrm{sym}}(X_1; \dots; X_n) + \sum_{i=1}^n H(X_i).$$

In particular, when X and Y are real bounded random variables with $H(X) > -\infty$ and $H(Y) > -\infty$, we have

$$I(X;Y) = -\overline{H}_{\mathrm{sym}}(X;Y) = -\underline{H}_{\mathrm{sym}}(X;Y).$$

In this way, the "classical analog" of $i(X;Y) = -\chi_{\mathrm{orb}}(X;Y)$ provides a new definition (a kind of "discretization") of the classical mutual information $I(X;Y)$.

Next, let us generalize the quantity $i = -\chi_{\mathrm{orb}}$ to n-blocks $(\vec{X}^{(1)}, \dots, \vec{X}^{(n)})$ of non-commutative random variables. Now, let $\vec{X}^{(i)} = (X_1^{(i)}, \dots, X_{k_i}^{(i)})$ be a k_i-tuple of non-commutative random variables in a tracial W^*-probability space $(\mathcal{M}, \tau)$ for $1 \le i \le n$. Throughout the rest of this section we assume that, for each $1 \le i \le n$, the von Neumann subalgebra $W^*(\vec{X}^{(i)})$ generated by $\vec{X}^{(i)}$ is hyperfinite. Then one can choose sequences $\vec{\alpha}^{(i)}(N) = (\alpha_1^{(i)}(N), \dots, \alpha_{k_i}^{(i)}(N))$ of microstates in $(M_N^{sa})^{k_i}$, $1 \le i \le n$, such that $\vec{\alpha}^{(i)}(N)$ converges to $\vec{X}^{(i)}$ in the distribution sense. (Such sequences of microstates can be chosen whenever $W^*(\vec{X}^{(i)})$, $1 \le i \le n$, are embeddable into the ultraproduct R^ω of the hyperfinite II_1 factor R; however, the hyperfiniteness of $W^*(\vec{X}^{(i)})$ will be essential in our discussions below.) Define

$$\xi_{\vec{\alpha}^{(1)}(N),\dots,\vec{\alpha}^{(n)}(N)} : \mathrm{U}(N)^n \longrightarrow \prod_{i=1}^{n} (M_N^{sa})^{k_i}$$

by

$$\xi_{\vec{\alpha}^{(1)}(N),\dots,\vec{\alpha}^{(n)}(N)}(\vec{U}) := \left(U_i \vec{\alpha}^{(i)}(N) U_i^*\right)_{i=1}^{n} \quad \text{for } \vec{U} = (U_1,\dots,U_n) \in \mathrm{U}(N)^n,$$

where

$$U_i \vec{\alpha}^{(i)}(N) U_i^* := \left(U_i \alpha_1^{(i)}(N) U_i^*, \dots, U_i \alpha_{k_i}^{(i)}(N) U_i^*\right), \quad 1 \le i \le n.$$

Definition 2.6. With the above notations, for each $N, m \in \mathbb{N}$ and $\delta > 0$, define

$$\begin{aligned}
&\Gamma_{\mathrm{orb}}^{\mathrm{block}}(\vec{X}^{(1)},\dots,\vec{X}^{(n)} \,|\, \vec{\alpha}^{(1)}(N),\dots,\vec{\alpha}^{(n)}(N); N, m, \delta) \\
&\quad := \xi^{-1}_{\vec{\alpha}^{(1)}(N),\dots,\vec{\alpha}^{(n)}(N)}(\Gamma(\vec{X}^{(1)},\dots,\vec{X}^{(n)}; N, m, \delta)).
\end{aligned}$$

We then define

$$\begin{aligned}
\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) &:= \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \limsup_{N\to\infty} \\
&\quad \frac{1}{N^2} \log \gamma_{\mathrm{U}(N)}^{\otimes n}\left(\Gamma_{\mathrm{orb}}^{\mathrm{block}}(\vec{X}^{(1)},\dots,\vec{X}^{(n)} \,|\, \vec{\alpha}^{(1)}(N),\dots,\vec{\alpha}^{(n)}(N); N, m, \delta)\right), \\
i(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) &:= -\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}).
\end{aligned}$$

The block-wise *orbital free entropy* $\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$ is well defined independently of the choices of $\vec{\alpha}^{(i)}(N)$, $1 \le i \le n$, as Proposition 2.2, and it has the same basic properties as those of $\chi_{\mathrm{orb}}(X_1;\dots;X_n)$ given in Proposition 2.3. In particular, note that $\chi_{\mathrm{orb}}(\vec{X}) = 0$ for a single block $\vec{X}$. In fact, this is obvious because $\Gamma_{\mathrm{orb}}^{\mathrm{block}}(\vec{X}|\vec{\alpha}; N, m, \delta)$ is the whole $\mathrm{U}(N)$ whenever N is large.

The following theorem tells us that $i(\vec{X}_1;\dots;\vec{X}_n)$ can be called the *microstate mutual free information* of the n-tuple of hyperfinite subalgebras $(W^*(\vec{X}_1),\dots,W^*(\vec{X}_n))$.

Theorem 2.7. ([6]) *Let $\vec{X}^{(i)} = (X_1^{(i)},\dots,X_{k_i}^{(i)})$ and $\vec{Y}^{(i)} = (Y_1^{(i)},\dots,Y_{l_i}^{(i)})$ be self-adjoint random variables in $(\mathcal{M},\tau)$ for $1 \le i \le n$. If $W^*(\vec{X}^{(i)}) = W^*(\vec{Y}^{(i)})$ and it is hyperfinite for each $1 \le i \le n$, then*

$$\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{n)}) = \chi_{\mathrm{orb}}(\vec{Y}^{(1)};\dots;\vec{Y}^{n)}).$$

The "additivity theorem" for χ_{orb} is presented as follows.

Theorem 2.8. ([6]) *Let $\vec{X}^{(i)} = (X_1^{(i)},\dots,X_{k_i}^{(i)})$, $1 \le i \le n$, be self-adjoint random variables in $(\mathcal{M},\tau)$ such that $W^*(\vec{X}^{(i)})$ is hyperfinite for each $1 \le i \le n$. Then $\vec{X}^{(1)},\dots,\vec{X}^{(n)}$ are free if and only if $\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) = 0$ (or $\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) = \sum_{i=1}^{n} \chi_{\mathrm{orb}}(\vec{X}^{(i)})$ since χ_{orb} is zero for a single block).*

In particular, when each $\vec{X}^{(i)}$ is a single variable, the additivity theorem for χ (i.e., property 9° in §1) directly follows from Theorems 2.4 and 2.8. Incidentally, the formula

$$\chi(\vec{X}^{(1)},\dots,\vec{X}^{(n)}) = \chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) + \sum_{i=1}^{n}\chi(\vec{X}^{(i)})$$

is meaningless because both sides are $-\infty$ as long as $W^*(\vec{X}^{(i)})$, $1 \le i \le n$, are hyperfinite and some $\vec{X}^{(i)}$ is not single. Although Theorem 2.8 is an additivity theorem in some sense, we should note that it has no contribution to the block-additivity problem for χ: if $\vec{X}$ and $\vec{Y}$ are free, then $\chi(\vec{X},\vec{Y}) = \chi(\vec{X}) + \chi(\vec{Y})$?

Remark 2.9. By restricting only to projections and by applying a change of variable formula specialized to projections, the following pair block-wise additivity theorem was shown in [9]: Let $p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}$ be projections in $(\mathcal{M},\tau)$. Then we have:

(a) If $\{p_1,q_1\}$, …, $\{p_n,q_n\}$, $\{r_1\}$, …, $\{r_{n'}\}$ are free, then

$$\chi_{\mathrm{orb}}(p_1;q_1;\dots;p_n;q_n;r_1;\dots;r_{n'}) = \chi_{\mathrm{orb}}(p_1;q_1) + \dots + \chi_{\mathrm{orb}}(p_n;q_n).$$

(b) Conversely, if $\chi_{\mathrm{orb}}(p_i;q_i) > -\infty$ for $1 \le i \le n$ and equality in (a) holds, then $\{p_1,q_1\}$, …, $\{p_n,q_n\}$, $\{r_1\}$, …, $\{r_{n'}\}$ are free.

(c) In particular, $\chi_{\mathrm{orb}}(p_1;\dots;p_n) = 0$ if and only if $p_1,\dots,p_n$ are free.

The above (c) is of course a particular case of Theorem 2.8; however, (a) and (b) are not covered by Theorem 2.8 since $\chi_{\mathrm{orb}}(p_i;q_i)$ is not the orbital free entropy of a single pair block (p_i,q_i).

3. Free entropy dimension

First, recall the definition of the modified version of free entropy.

Definition 3.1. Let $\vec{X} = (X_1,\dots,X_n)$ and $\vec{Y} = (Y_1,\dots,Y_l)$ be self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M},\tau)$. For $N,m \in \mathbb{N}$, $\delta > 0$ and $R > 0$ define

$$\begin{aligned}&\Gamma_R(\vec{X}:\vec{Y};N,m,\delta)\\ &\qquad := \left\{\vec{A} \in (M_N^{sa})^n : (\vec{A},\vec{B}) \in \Gamma_R(\vec{X},\vec{Y};N,m,\delta) \text{ for some } \vec{B} \in (M_N^{sa})^l\right\}\end{aligned}$$

(i.e., the projection of $\Gamma_R(\vec{X},\vec{Y};N,m,\delta) \subset (M_N^{sa})^n \times (M_N^{sa})^l$ to the first n-components) and

$$\chi_R(\vec{X}:\vec{Y}) := \lim_{\substack{m\to\infty\\ \delta\searrow 0}} \limsup_{N\to\infty}\left(\frac{1}{N^2}\log\Lambda_N^{\otimes n}(\Gamma_R(\vec{X}:\vec{Y};N,m,\delta)) + \frac{n}{2}\log N\right).$$

Then the *modified free entropy* of $\vec{X}$ in the presence of $\vec{Y}$ is

$$\chi(\vec{X}:\vec{Y}) := \sup_{R>0}\chi_R(\vec{X}:\vec{Y}).$$

Definition 3.2. Let $\vec{X} = (X_1,\dots,X_n)$ and $\vec{S} = (S_1,\dots,S_n)$ be n-tuples of self-adjoint random variables in $(\mathcal{M},\tau)$ such that $\vec{S}$ is a standard semicircular system free from $\vec{X}$ (i.e., a free family of self-adjoint variables S_i with the standard semicircular distribution). Write $\vec{X}+\varepsilon\vec{S} := (X_1+\varepsilon S_1,\dots,X_n+\varepsilon S_n)$ for $\varepsilon>0$. Then, the *free entropy dimension* $\delta(\vec{X})$ and the *modified free entropy dimension* $\delta_0(\vec{X})$ are defined by

$$\delta(\vec{X}) := n + \limsup_{\varepsilon\searrow 0} \frac{\chi(\vec{X}+\varepsilon\vec{S})}{|\log\varepsilon|},$$

$$\delta_0(\vec{X}) := n + \limsup_{\varepsilon\searrow 0} \frac{\chi(\vec{X}+\varepsilon\vec{S} : \vec{S})}{|\log\varepsilon|}.$$

It seems that the modified version δ_0 is technically more convenient than δ. The following are some basic properties of δ and δ_0 ([16, 17]; also [7, §7.3]).

1° **(Trivial inequalities)** $\delta_0(\vec{X}) \le \delta(\vec{X}) \le n$ if $\vec{X}$ consists of n-variables.

2° **(Subadditivity)** $\delta(\vec{X},\vec{Y}) \le \delta(\vec{X})+\delta(\vec{Y})$ and $\delta_0(\vec{X},\vec{Y}) \le \delta_0(\vec{X})+\delta_0(\vec{Y})$.

3° **(Single variable case)** Let X, S be self-adjoint random variables in $(\mathcal{M},\tau)$ such that S is a standard semicircular free from X. If μ is the distribution measure of X, then

$$\lim_{\varepsilon\searrow 0} \frac{\chi(X+\varepsilon S)}{|\log\varepsilon|} = -\sum_{t\in\mathbb{R}} \mu(\{t\})^2$$

and $\delta_0(X)=\delta(X)=1-\sum_{t\in\mathbb{R}}\mu(\{t\})^2$.

4° **(Lower semicontinuity in the single variable case)** If $X_k \to X$ in distribution with $\sup_k \|X_k\|_\infty < +\infty$, then

$$\delta(X) \le \liminf_{k\to\infty} \delta(X_k).$$

5° **(Additivity in the free case)** If $X_1,\dots,X_n$ are free, then

$$\delta_0(\vec{X}) = \delta(\vec{X}) = \delta(X_1)+\cdots+\delta(X_n).$$

Indeed, a slightly more stronger result hold: If $\vec{X}$ and a single Y are free, then

$$\delta(\vec{X},Y) = \delta(\vec{X})+\delta(Y), \quad \delta_0(\vec{X},Y) = \delta_0(\vec{X})+\delta(Y).$$

The following properties from [16, 20] are useful to compute/estimate δ and δ_0. Let $\vec{X}=(X_1,\dots,X_n)$ and $\vec{Y}=(Y_1,\dots,Y_l)$ be in $(\mathcal{M},\tau)$. (For (a) and (b), see also Proposition 3.5 below.)

(a) If $\vec{Y}\subset W^*(\vec{X})$ and $\chi(\vec{X})>-\infty$, then $\delta(\vec{X},\vec{Y}) \ge \delta(\vec{X}) = n$.

(b) If $\vec{Y}\subset \mathrm{Alg}(\vec{X})$ (in fact, a weaker assumption is in [16]) and $\chi(\vec{X})>-\infty$, then $\delta(\vec{X},\vec{Y}) = \delta(\vec{X}) = n$.

(c) If $\vec{Y}\subset W^*(\vec{X})$, then $\delta_0(\vec{X},\vec{Y}) \ge \delta_0(\vec{X})$.

(d) If $\mathrm{Alg}(\vec{X}) = \mathrm{Alg}(\vec{Y})$, then $\delta_0(\vec{X}) = \delta_0(\vec{Y})$, that is, δ_0 is an algebraic invariant.

In [16] Voiculescu posed the question of whether δ has the lower semicontinuity property or not; namely, if $\vec{X}^{(k)} \to \vec{X}$ strongly in $(\mathcal{M},\tau)$, then

$$\delta(\vec{X}) \le \liminf_{k\to\infty} \delta(\vec{X}^{(k)})\,?$$

Thanks to the above (a) and (b), the positive answer to this question implies the non-isomorphism of free group factors: $\mathcal{L}(\mathbb{F}_n) \not\cong \mathcal{L}(\mathbb{F}_m)$ if $n\ne m$. Moreover, thanks to (c) and (d), the positive answer of the same question for δ_0 implies that $\delta_0(\vec{X}) = \delta_0(\vec{Y})$ if $W^*(\vec{X}) = W^*(\vec{Y})$. Recently, Shlyakhtenko [14] gave a counter-example to the lower semicontinuity question for δ (also for δ_0). But, he posed some weaker versions of the question, which are still sufficient to settle the non-isomorphism of free group factors. For example, if $\vec{X}^{(k)} \to \vec{X}$ strongly in $(\mathcal{M}, \tau)$ and $W^*(\vec{X}^{(k)}) = W^*(\vec{X}) = \mathcal{M}$, then $\delta(\vec{X}) \le \liminf_{k\to\infty} \delta(\vec{X}^{(k)})$?

Next, let us recall the notions of covering/packing numbers. Let $(\mathcal{X}, d)$ be a Polish space and $\Gamma \subset \mathcal{X}$. Consider Γ as a metric space with the restriction of d on Γ. For each $\varepsilon > 0$ we denote by $K_\varepsilon(\Gamma)$ the minimum number of open ε-balls covering Γ, and by $P_\varepsilon(\Gamma)$ the maximum number of elements in a family of mutually disjoint open ε-balls in Γ, where ε-balls in Γ are taken as subsets of Γ.

On the space $(M_N^{sa})^n$ ($\cong \mathbb{R}^{nN^2}$) we consider the metric d_2 induced from the Hilbert-Schmidt norm with respect to $\mathrm{tr}_N = N^{-1}\mathrm{Tr}_N$:

$$d_2(\vec{A}, \vec{B}) := \|\vec{A} - \vec{B}\|_{2,\mathrm{tr}_N} = \left(\mathrm{tr}_N\left(\sum_{i=1}^n (A_i - B_i)^2\right)\right)^{1/2}.$$

In [11] Jung introduced another definition of free entropy dimension via the notions of covering/packing numbers and proved its coincidence with the modified free entropy dimension δ_0.

Definition 3.3. Let $\vec{X} = (X_1, \dots, X_n)$ be an n-tuple of self-adjoint random variables in a tracial W^*-probability space $(\mathcal{M}, \tau)$, and choose $R \ge \|\vec{X}\|_\infty$. Define the *fractal* (or *packing*) *free entropy dimension* of $\vec{X}$ to be

$$\delta_1(\vec{X}) := \limsup_{\varepsilon \searrow 0} \frac{\mathbb{K}_\varepsilon(\vec{X})}{|\log \varepsilon|} = \limsup_{\varepsilon \searrow 0} \frac{\mathbb{P}_\varepsilon(\vec{X})}{|\log \varepsilon|},$$

where

$$\mathbb{K}_\varepsilon(\vec{X}) := \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2} \log K_\varepsilon(\Gamma_R(\vec{X}; N, m, \delta)),$$

$$\mathbb{P}_\varepsilon(\vec{X}) := \lim_{\substack{m\to\infty \\ \delta\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2} \log P_\varepsilon(\Gamma_R(\vec{X}; N, m, \delta)).$$

In the above definition, $\mathbb{K}_\varepsilon(\vec{X})$ and $\mathbb{P}_\varepsilon(\vec{X})$ should be written as $\mathbb{K}_{\varepsilon,R}(\vec{X})$ and $\mathbb{P}_{\varepsilon,R}(\vec{X})$ to be precise. But, note ([3], [12]) that the definition of $\delta_1(\vec{X})$ above is independent of the choice of R with $R \ge \|\vec{X}\|_\infty$ permitting $R = \infty$ (i.e., no cut-off).

Theorem 3.4. (Jung [11]) *For every $\vec{X}$ in $(\mathcal{M}, \tau)$,*

$$\delta_0(\vec{X}) = \delta_1(\vec{X}).$$

In the following, we present a few more basic properties of δ_0 based on the equality $\delta_0 = \delta_1$.

Proposition 3.5. ([17, Proposition 6.10]) *If $\chi(\vec{X}) > -\infty$, then $\delta_0(\vec{X}) = n$ and hence $\delta(\vec{X}) = n$.*

Theorem 3.6. *If $\vec{X} = (X_1, \dots, X_n)$ and $\delta_0(\vec{X}) = n > 1$, then $W^*(\vec{X})$ is a factor. Hence, this is the case if $\chi(\vec{X}) > -\infty$ (see* [17, Corollary 4.2]).

Remark 3.7.

(1) Let $\vec{X} = (X_1, \dots, X_n)$ be a free family of non-atomic variables X_i. Then $W^*(\vec{X})$ is isomorphic to the free group factor $\mathcal{L}(\mathbb{F}_n)$ (Voiculescu's free Gaussian functor theorem) and $\delta_0(\vec{X}) = \delta(\vec{X}) = n$ by property 5°. But, $\chi(\vec{X}) = \sum_{i=1}^n \chi(X_i)$ can easily be $-\infty$ so that the converse of Proposition 3.5 is not true.

(2) The first assertion of Theorem 3.6 seems new though it might be a folklore for specialists. It does not seem that there is a known example of $\vec{X}$ such that $\delta_0(\vec{X}) > 1$ but $W^*(\vec{X})$ is not a factor.

(3) It might be natural to expect that the generated factor $W^*(\vec{X})$ is similar to free group factors when $\vec{X} = (X_1, \dots, X_n)$ and $\delta_0(\vec{X}) = n > 1$ (or more strongly $\chi(\vec{X}) > -\infty$). However, Brown [2] proved the existence of $\vec{X} = (X_1, \dots, X_n)$ such that $\chi(\vec{X}) > -\infty$ but $W^*(\vec{X})$ is not isomorphic to any (not necessarily unital) subalgebra of a free group factor.

In [10] Jung computed the modified free entropy dimension $\delta_0(\vec{X}) = \delta_1(\vec{X})$ in the case where $W^*(\vec{X})$ is hyperfinite. Let $\vec{X} = (X_1, \dots, X_n)$ be an n-tuple of self-adjoint random variables in $(\mathcal{M}, \tau)$. The generated von Neumann algebra $W^*(\vec{X})$ is decomposed as

$$W^*(\vec{X}) = \mathcal{M}_0 \oplus \bigoplus_{j=1}^{s} M_{k_j}(\mathbb{C}),$$

$$\tau|_{W^*(\vec{X})} = \alpha_0 \tau_0 \oplus \bigoplus_{j=1}^{s} \alpha_j \mathrm{tr}_{k_j},$$

where $\mathcal{M}_0$ is a diffuse von Neumann algebra (possibly $\mathcal{M}_0 = \{0\}$), $s \in \{0, 1, \dots, \infty\}$, $\alpha_0 \geq 0$ ($\alpha_0 = 0$ if $\mathcal{M}_0 = \{0\}$) and $\alpha_j > 0$ with $\sum_{j=0}^{s} \alpha_j = 1$. Then, the conclusion is:

Theorem 3.8. ([10]) *If $W^*(\vec{X})$ is hyperfinite, then*

$$\delta_0(\vec{X}) = 1 - \sum_{j=1}^{s} \frac{\alpha_j^2}{k_j^2}.$$

Remark 3.9. Obviously, Theorem 3.8 says that if $(\mathcal{M}, \tau)$ is a hyperfinite tracial W^*-probability space, then $\delta_0(\vec{X}) = \delta_0(\vec{Y})$ of any two finite sets $\vec{X}$ and $\vec{Y}$ of self-adjoint generators for $\mathcal{M}$. In [4] Dykema introduced the notion of the *free dimension* $\mathrm{fdim}(\mathcal{M})$ for a certain class of finite von Neumann algebras, including finite-dimensional algebras, hyperfinite algebras and interpolated free group factors. It is worthwhile to note that if $W^*(\vec{X})$ is hyperfinite, then the two notions of the modified free entropy dimension and the free dimension coincide:

$$\delta_0(\vec{X}) = \mathrm{fdim}(W^*(\vec{X})).$$

In [22] Voiculescu proved that if $X_1, \ldots, X_n$ are non-atomic self-adjoint random variables in $(\mathcal{M}, \tau)$ satisfying the consecutive commuting conditions $X_i X_{i+1} = X_{i+1} X_i$ for $1 \le i < n$, then $\delta_0(\vec{X}) \le 1$. For example, when $n \ge 3$, there is a finite set $\vec{X} = (X_1, \ldots, X_p)$ of self-adjoint generators of the group algebra $\mathcal{L}(SL(n, \mathbb{Z}))$ with the above property. ($\mathcal{L}(SL(n, \mathbb{Z}))$, $n \ge 3$, are typical examples of property T factors.) Later, Ge and Shen [5] obtained a considerably stronger result that $\delta_0(\vec{X}) \le 1$ for every $\vec{X}$ in $(\mathcal{M}, \tau)$ if $\mathcal{M}$ is generated by a sequence of Haar unitaries with some weakened consecutive conditions. But, the problem on δ_0 in the general case where $W^*(\vec{X})$ is a property T von Neumann algebra is recently settled by Jung and Shlyakhtenko as follows.

Theorem 3.10. ([13]) *Let $\vec{X} = (X_1, \ldots, X_n)$ be self-adjoint variables in $(\mathcal{M}, \tau)$. If $W^*(\vec{X})$ is a property T von Neumann algebra, then $\delta_0(\vec{X}) \le 1$. Hence, if $W^*(\vec{X})$ is a diffuse and Property T von Neumann algebra which is embeddable into R^ω, then $\delta_0(\vec{X}) = 1$.*

4. Orbital (or mutual) free entropy dimension

In §2 we proposed a somewhat new approach to free entropy theory called the orbital approach. This can be performed also for the free entropy dimension theory as we explain in this section. We adopt the generalized setting of n-blocks of noncommutative random variables under the hyperfiniteness assumption as in the latter half of §2. To introduce the orbital version of the modified free entropy dimension $\delta_0(\vec{X})$, we first need to define the modified orbital free entropy in the presence of some unitary random variables.

Definition 4.1.

(1) Let $\vec{X} = (X_1, \ldots, X_k)$ be a k-tuple of self-adjoint random variables and $\vec{v} = (v_1, \ldots, v_l)$ an l-tuple of unitary random variables in $(\mathcal{M}, \tau)$. For $N, m \in \mathbb{N}$ and $\delta > 0$ we denote by $\Gamma(\vec{X}; \vec{v}; N, m, \delta)$ the set of all $(\vec{A}, \vec{V}) = (A_1, \ldots, A_k, V_1, \ldots, V_l) \in (M_N^{sa})^k \times \mathrm{U}(N)^l$ such that $|\mathrm{tr}_N(h(\vec{A}, \vec{V})) - \tau(h(\vec{X}, \vec{v}))| \le \delta$ for all $*$-monomials h with degree $\le m$, and by $\Gamma(\vec{X} : \vec{v}; N, m, \delta)$ the set of all $\vec{A} \in (M_N^{sa})^k$ such that $(\vec{A}, \vec{V}) \in \Gamma(\vec{X}; \vec{v}; N, m, \delta)$ for some $\vec{V} \in \mathrm{U}(N)^l$.

(2) Moreover, let $(\vec{X}^{(1)}, \ldots, \vec{X}^{(n)})$ be noncommutative self-adjoint random variables in $(\mathcal{M}, \tau)$ as stated before Definition 2.6, that is, for $1 \le i \le n$, $\vec{X}^{(i)} = (X_1^{(i)}, \ldots, X_{k_i}^{(i)})$ is a k_i-tuple of variables such that $W^*(\vec{X}^{(i)})$ is hyperfinite. Let $\vec{\alpha}^{(i)}(N) = (\alpha_1^{(i)}(N), \ldots, \alpha_{k_i}^{(i)}(N))$, $1 \le i \le n$, and $\xi_{\vec{\alpha}^{(1)}(N), \ldots, \vec{\alpha}^{(n)}(N)} : \mathrm{U}(N)^n \to \prod_{i=1}^n (M_N^{sa})^{k_i}$ be also as stated before Definition 2.6. Define

$$
\begin{aligned}
&\Gamma_{\mathrm{orb}}^{\mathrm{block}}(\vec{X}^{(1)}, \ldots, \vec{X}^{(n)} | \vec{\alpha}^{(1)}(N), \ldots, \vec{\alpha}^{(n)}(N) : \vec{v}; N, m, \delta) \\
&\quad := \xi_{\vec{\alpha}^{(1)}(N), \ldots, \vec{\alpha}^{(n)}(N)}^{-1}(\Gamma(\vec{X}^{(1)}, \ldots, \vec{X}^{(n)} : \vec{v}; N, m, \delta)),
\end{aligned}
$$

and define the block-wise *modified orbital free entropy* in the presence of $\vec{v}$ by

$$\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}:\vec{v}) := \lim_{\substack{m\to\infty\\ \delta\searrow 0}} \limsup_{N\to\infty}$$

$$\frac{1}{N^2}\log\gamma_{\mathrm{U}(N)}^{\otimes n}\big(\Gamma_{\mathrm{orb}}^{\mathrm{block}}(\vec{X}^{(1)},\dots,\vec{X}^{(n)}|\vec{a}^{(1)}(N),\dots,\vec{a}^{(n)}(N):\vec{v};N,m,\delta)\big).$$

To define the orbital version of $\delta_0(\vec{X})$, we also need the notion of free unitary Brownian motion introduced by Biane [1]. A *free unitary Brownian motion* is a noncommutative process $v(t)$, $t\ge 0$, of unitary random variables satisfying the following properties:

(i) $v(t)$ has free left multiplicative increments, i.e., if $0\le t_0<t_1<\cdots<t_n$, then $v(t_i)v(t_{i-1})^*$, $1\le i\le n$, are freely independent.

(ii) $v(t)$ is stationary, i.e., the distribution of $v(t)v(s)^*$ for every $0\le s<t$ is determined by $t-s$.

In the following we always assume that $v(0)=1$. The distribution measures $\nu_t := \mu_{v(t)}\in\mathrm{Prob}(\mathbb{T})$, $t\ge 0$, satisfy the semigroup condition: $\nu_0=\delta_1$ and $\nu_s\boxtimes\nu_t=\nu_{s+t}$.

Definition 4.2. Let $(\vec{X}^{(1)},\dots,\vec{X}^{(n)})$ be as in the above definition, and let $\vec{v}(t) = (v_1(t),\dots,v_n(t))$, $t\ge 0$, be an n-tuple of free unitary Brownian motions with $v_i(0)=1$ which are free each other and moreover free from $\vec{X}^{(1)},\dots,\vec{X}^{(n)}$. (We may always assume that such extra variables are taken in $(\mathcal{M},\tau)$). We write $v_i(t)\vec{X}^{(i)}v_i(t)^* := (v_i(t)X_1^{(i)}v_i(t)^*,\dots,v_i(t)X_{k_i}^{(i)}v_i(t)^*)$ and define the block-wise (*modified*) *orbital free entropy dimension* of $(\vec{X}^{(1)},\dots,\vec{X}^{(n)})$ by

$$\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$$

$$:= 2\limsup_{\varepsilon\searrow 0}\frac{\chi_{\mathrm{orb}}(v_1(\varepsilon)\vec{X}^{(1)}v_1(\varepsilon)^*;\cdots;v_n(\varepsilon)\vec{X}^{(n)}v_n(\varepsilon)^*:\vec{v}(\varepsilon))}{|\log\varepsilon|}.$$

Note that the multiplicative perturbation by unitary free Brownian processes is used in the above definition of $\delta_{0,\mathrm{orb}}$ while the additive perturbation by semicircular processes is used for δ_0.

It is easy to show as Proposition 2.2 that the definition of $\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$ is independent of the choices of $\vec{a}^{(i)}(N)$, $1\le i\le n$, such that $\vec{a}^{(i)}(N)\to\vec{X}^{(i)}$ in distribution as $N\to\infty$.

The next proposition gives basic properties of $\delta_{0,\mathrm{orb}}$.

Proposition 4.3.

1° **(Single variable case)** $\delta_{0,\mathrm{orb}}(\vec{X})=0$ *for a single block* $\vec{X}$.

2° **(Negativity)** $\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})\le 0$.

3° **(Subadditivity)** *For every* $1\le k<n$,

$$\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})\le\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(k)})+\delta_{0,\mathrm{orb}}(\vec{X}^{(k+1)};\dots;\vec{X}^{(n)}).$$

4° **(Zero in the free case)** *If* $\vec{Y}$ *is free from* $\vec{X}^{(1)},\dots,\vec{X}^{(n)}$, *then*

$$\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;X^{(n)};\vec{Y})=\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;X^{(n)}).$$

Hence, if $\vec{X}^{(1)},\dots,\vec{X}^{(n)}$ are free, then $\delta_{0,\mathrm{orb}}(\vec{X}^{(1)},\dots,\vec{X}^{(n)})=0$.

The next theorem says that $\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{n)})$ can be regarded as the (modified) orbital free entropy dimension of the n-tuple of hyperfinite subalgebras $(W^*(\vec{X}^{(1)}),\dots,W^*(\vec{X}^{(n)}))$.

Theorem 4.4. ([6]) *Let $\vec{X}^{(i)}=(X_1^{(i)},\dots,X_{k_i}^{(i)})$ and $\vec{Y}^{(i)}=(Y_1^{(i)},\dots,Y_{l_i}^{(i)})$ be self-adjoint random variables in $(\mathcal{M},\tau)$ for $1\le i\le n$. If $W^*(\vec{X}^{(i)})=W^*(\vec{Y}^{(i)})$ and it is hyperfinite for each $1\le i\le n$, then*

$$\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{n)})=\delta_{0,\mathrm{orb}}(\vec{Y}^{(1)};\dots;\vec{Y}^{n)}).$$

By adapting Proposition 3.5 to the case of unitary microstates, we have the following:

Proposition 4.5. *If $\chi_{\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})>-\infty$, then $\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})=0$.*

Next, we introduce the orbital version of the fractal free entropy dimension $\delta_1(\vec{X})$.

Definition 4.6. Let $(\vec{X}^{(1)},\dots,\vec{X}^{(n)})$ and $\vec{\alpha}^{(i)}(N)$, $1\le i\le n$, be as in Definition 4.1 (2). For each $\varepsilon>0$ define the block-wise *orbital fractal free entropy dimension* of $(\vec{X}^{(1)},\dots,\vec{X}^{(n)})$ by

$$\delta_{1,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}):=\limsup_{\varepsilon\searrow 0}\frac{\mathbb{K}_\varepsilon(\vec{X}^{(1)};\dots;\vec{X}^{(n)})}{|\log\varepsilon|}=\limsup_{\varepsilon\searrow 0}\frac{\mathbb{P}_\varepsilon(\vec{X}^{(1)};\dots;\vec{X}^{(n)})}{|\log\varepsilon|},$$

where

$$\mathbb{K}_\varepsilon(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) := \lim_{\substack{m\to\infty\\ \delta\searrow 0}}\limsup_{N\to\infty}\frac{1}{N^2}\log K_\varepsilon\big(\Gamma_{\mathrm{orb}}^{\mathrm{block}}(\vec{X}^{(1)},\dots,\vec{X}^{(n)}|\vec{\alpha}^{(1)}(N),\dots,\vec{\alpha}^{(n)}(N);N,m,\delta)\big)$$

and $\mathbb{P}_\varepsilon(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$ is similar with P_ε in place of K_ε. Once again, it is easy to check that the definitions of $\mathbb{K}_\varepsilon(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$ and $\mathbb{P}_\varepsilon(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$ (hence that of $\delta_{1,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})$) are independent of the choices of $\vec{\alpha}^{(i)}(N)$, $1\le i\le n$.

The main result of this section is now stated as follows.

Theorem 4.7. ([6]) *For every n-blocks $(\vec{X}^{(1)},\dots,\vec{X}^{(n)})$ of self-adjoint random variables in $(\mathcal{M},\tau)$, the following hold true:*

(1)
$$\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})=\delta_{1,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})-n.$$

(2)
$$\delta_0(\vec{X}^{(1)},\dots,\vec{X}^{(n)})\le\delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)})+\sum_{i=1}^n\delta_0(\vec{X}^{(i)}).$$

(3) *If $W^*(\vec{X}^{(i)})$ is finite-dimensional for each $1\le i\le n$, then equality holds in (2).*

Problem 4.8. On a parallel with Theorem 2.4, it may be strongly expected that the equality

$$\delta_0(\vec{X}^{(1)},\dots,\vec{X}^{(n)}) = \delta_{0,\mathrm{orb}}(\vec{X}^{(1)};\dots;\vec{X}^{(n)}) + \sum_{i=1}^{n}\delta_0(\vec{X}^{(i)}).$$

holds true for general $\vec{X}^{(i)}$ with hyperfinite $W^*(\vec{X}^{(i)})$.

Example 4.9. (Two projections) The simplest example of non-commuting random variables is a pair of projections. Let p, q be two projections in $(\mathcal{M},\tau)$ with $\alpha := \tau(p)$ and $\beta := \tau(q)$. The von Neumann algebra generated by p, q is represented as

$$W^*(p,q) = \big(L^\infty((0,1),\nu)\otimes M_2(\mathbb{C})\big)\oplus\mathbb{C}(p\wedge q)\oplus\mathbb{C}(p\wedge q^\perp)\oplus\mathbb{C}(p^\perp\wedge q)\oplus\mathbb{C}(p^\perp\wedge q^\perp)$$

with $\tau|_{W^*(p,q)} = (\nu\otimes\mathrm{tr}_2)\oplus\alpha_{11}\oplus\alpha_{10}\oplus\alpha_{01}\oplus\alpha_{00}$, where $\alpha_{11} := \tau(p\wedge q)$, $\alpha_{10} := \tau(p\wedge q^\perp)$, $\alpha_{01} := \tau(p^\perp\wedge q)$ and $\alpha_{00} := \tau(p^\perp\wedge q^\perp)$. Then by Theorem 3.8,

$$\delta(p) = 2\alpha(1-\alpha), \quad \delta(q) = 2\beta(1-\beta),$$

$$\delta_0(p,q) = 1 - \sum_{i,j=0}^{1}\alpha_{ij}^2 - \frac{1}{4}\sum_{t\in(0,1)}\nu(\{t\})^2,$$

from which we can explicitly compute $\delta_{0,\mathrm{orb}}(p;q)$ by Theorem 4.7 (3).

On the other hand, as a consequence of the large deviation principle for two random projection matrices in [8], it is known that if $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10} = 0$ or equivalently

$$\begin{cases}\alpha_{11} = \max\{\alpha+\beta-1, 0\},\\ \alpha_{00} = \max\{1-\alpha-\beta, 0\},\\ \alpha_{10} = \max\{\alpha-\beta, 0\},\\ \alpha_{01} = \max\{\beta-\alpha, 0\},\end{cases}$$

then

$$\begin{aligned}\chi_{\mathrm{orb}}(p;q) = \frac{1}{4}\Sigma(\nu) + \frac{|\alpha-\beta|}{2}\int_{(0,1)}\log x\,d\nu(x)\\ + \frac{|\alpha+\beta-1|}{2}\int_{(0,1)}\log(1-x)\,d\nu(x) - C,\end{aligned}$$

where C is a constant depending on α and β only. Otherwise, $\chi_{\mathrm{orb}}(p;q) = -\infty$. Thus, when $\chi_{\mathrm{orb}}(p;q) > -\infty$, ν is non-atomic so that we get

$$\delta_0(p,q) = 1-(\alpha+\beta-1)^2+(\alpha-\beta)^2 = 2\alpha(1-\alpha)+2\beta(1-\beta) = \delta(p)+\delta(q)$$

so that $\delta_{0,\mathrm{orb}}(p;q) = 0$ as Proposition 4.5 generally says.

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