数理解析研究所講究録
1535巻 2007年 112-118



# Matrix functions and unitarily invariant norms (行列関数とユニタリ不変ノルム)

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## 1 Introduction

The eigenvalues of an $n\times n$ Hermitian matrix $H$ are denoted by $\lambda_i(H)\quad(i=1,2,\cdots,n)$ and arranged in increasing order, that is,
$\lambda_1(H)\leqq\lambda_2(H)\leqq\cdots\leqq\lambda_n(H)$.

$$\sigma_{(k)}(H):=\sum_{i=1}^{k}\lambda_i(H),$$

$$\sigma^{(k)}(H):=\sum_{i=n-k+1}^{n}\lambda_i(H).$$

A norm $|||\cdot|||$ on the $n\times n$ matrices is called a *unitarily invariant norm* if

$$|||UXV|||=|||X|||$$

for all $X$ and for all unitary matrices $U$ and $V$.

**Example 1.1** The following are tipical unitarily invariant norm:

The operator norm $\|X\|$,

Schatten p-norms $\|X\|_p:=\{\sum_{i=1}^n\lambda_i(|X|)^p\}^{1/p}$, $(p\geqq 1)$,

Ky Fan k-norms $\|X\|_{(k)}:=\sigma^{(k)}(|X|)$, $(k=1,2,\cdots,n)$.

The following useful result is due to **Ky Fan**:
$\|X\|_{(k)}\leqq\|Y\|_{(k)}(\forall k)$ impies $|||X|||\leqq|||Y|||$ for every unitarily invariant norm $|||\cdot|||$.

We begin with a simple fact:

**Proposition 1.1** Let $f(t)$ be a concave function on an interval $I$, and let $A, B$ be $n \times n$ Hermitian matrices with the spectra in $I$. Then for $R, S$ such that $R^*R + S^*S = 1$ and for $k = 1, 2, \cdots, n$

$$\sigma_{(k)}(f(R^*AR + S^*BS)) \geqq \sigma_{(k)}(R^*f(A)R + S^*f(B)S).$$

Moreover, if $f(t)$ is monotone, then

$$\lambda_k(f(R^*AR + S^*BS)) \geqq \lambda_k(R^*f(A)R + S^*f(B)S).$$

*Proof.* Let $\{\lambda_i\}_{i=1}^n$ be the eigenvalues of $X^*AX + Y^*BY$ so that $f(\lambda_1) \leqq f(\lambda_2) \leqq \cdots \leqq f(\lambda_n)$, and let $\{\mathbf{e}_i\}$ be the corresponding eigenvectors. Then the left side of (??) equals $f(\lambda_1) + \cdots + f(\lambda_k)$. By the concavity of $f$, we have

$$\begin{aligned} & \sum_{i=1}^k \langle (X^*f(A)X + Y^*f(B)Y)\mathbf{e}_i, \mathbf{e}_i\rangle \\ = & \sum_{i=1}^k \{\|X\mathbf{e}_i\|^2 \langle f(A)\frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|}, \frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|}\rangle + \|Y\mathbf{e}_i\|^2 \langle f(B)\frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|}, \frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|}\rangle\} \\ \leqq & \sum_{i=1}^k \{\|X\mathbf{e}_i\|^2 f(\langle A\frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|}, \frac{X\mathbf{e}_i}{\|X\mathbf{e}_i\|}\rangle) + \|Y\mathbf{e}_i\|^2 f(\langle B\frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|}, \frac{Y\mathbf{e}_i}{\|Y\mathbf{e}_i\|}\rangle)\} \\ \leqq & \sum_{i=1}^k f(\langle (X^*AX + Y^*BY)\mathbf{e}_i, \mathbf{e}_i\rangle) = \sum_{i=1}^k f(\lambda_i). \end{aligned}$$

Thus, by the min-max theorem, we get the first inequality.

If $f(t)$ is increasing, we can arrange eigenvalues $\{\lambda_i\}_{i=1}^n$ as $\lambda_i \leqq \lambda_{i+1}$ and $f(\lambda_i) \leqq f(\lambda_{i+1})$. For any unit vector $\mathbf{x}$ that is a linear combination of $\mathbf{e}_1, \cdots, \mathbf{e}_k$

$$\begin{aligned} \langle (X^*f(A)X + Y^*f(B)Y)\mathbf{x}, \mathbf{x}\rangle & \leqq f(\langle (X^*AX + Y^*BY)\mathbf{x}, \mathbf{x}\rangle) \\ & \leqq f(\lambda_k), \end{aligned}$$

for $\langle (X^*AX + Y^*BY)\mathbf{x}, \mathbf{x}\rangle \leqq \lambda_k$. From this, the second inequality follows. It can be similarly shown even if $f(t)$ is decreasing. □

**Corollary 1.2** Let $g(t)$ be a convex function on $I$. Then for $1 \leqq k \leqq n$ and for all $R, S$ such that $R^*R + S^*S = 1$

$$\sigma^{(k)}(g(R^*AR + S^*BS)) \leqq \sigma^{(k)}(R^*g(A)R + S^*g(B)S).$$

Moreover, if $g(t)$ is monotone,

$$\lambda_k(g(R^*AR + S^*BS)) \leqq \lambda_k(R^*g(A)R + S^*g(B)S).$$

**Remark**: The case $k = n$ of the first inequality in the corollary had been shown by Brown-Kosaki, Hansen-Pedersen.
The second inequality was shown by Bourin.
The case where $R, S$ are scalars are due to Aujla- Silva.

# 2 Essential results

It is well known that $trBA^2B = trAB^2A$. But it is difficult to estimate

$$trCBA^2BC - trCAB^2AC.$$

J. C. Bourin [5] got a nice result to do it. The next special case follows from it, however we can give a simple and direct proof.

**Lemma 2.1** Let $A \geqq 0$ and $B \geqq 0$, and let $Q$ be an orthogonal projection such that $QB = BQ$. If

$$\begin{aligned} &\inf\{\|B\mathbf{x}\| : Q\mathbf{x} = \mathbf{x}, \|\mathbf{x}\| = 1\} \\ \geqq\ &\sup\{\|B\mathbf{x}\| : (1-Q)\mathbf{x} = \mathbf{x}, \|\mathbf{x}\| = 1\}, \end{aligned}$$

then

$$\begin{aligned} &trQBA^2BQ \geqq tr\, QAB^2AQ, \\ &tr(1-Q)AB^2A(1-Q) \\ \geqq\ &tr(1-Q)BA^2B(1-Q). \end{aligned}$$

**Corollary 2.2** Let $A \geqq 0$ and $B \geqq 0$, and let $Q$ be an orthogonal projection such that $QB = BQ$.

Suppose the strict inequality:

$$\inf\{\|B\mathbf{x}\| : Q\mathbf{x} = \mathbf{x}, \|\mathbf{x}\| = 1\} > \sup\{\|B\mathbf{x}\| : (1-Q)\mathbf{x} = \mathbf{x}. \|\mathbf{x}\| = 1\}.$$

Then

$$tr\, QBA^2BQ = tr\, QAB^2AQ \Leftrightarrow QA = AQ.$$

**Proposition 2.3** Let $h(t)$ be a continuous function on $[0, \infty)$.

If $h(t)$ is decreasing and $th(t)$ is increasing,

or

if $h(t)$ is increasing and $th(t)$ is decreasing,

then for $A, B \geqq 0$ and for every unitarily invariant norm $|\!|\!| \cdot |\!|\!|$

$$|\!|\!| A^{1/2}h(A+B)A^{1/2} + B^{1/2}h(A+B)B^{\frac{1}{2}} |\!|\!| \geqq |\!|\!| (A+B)h(A+B) |\!|\!|.$$

**Corollary 2.4** Let $A$ and $B$ be non-negative Hermitian matrices such that $A + B$ is invertible. Then the following are equivalent:

(i) $H := A^{1/2}(A+B)^{-1}A^{1/2} + B^{1/2}(A+B)^{-1}B^{1/2} \leqq 1$,

(ii) $H = 1$,

(iii) $AB = BA$.

**Remark**: We give one fact relevant to the above (i).

$$(A+B)^{-1/2}A^{1/2}A^{1/2}(A+B)^{-1/2} + +(A+B)^{-1/2}B^{1/2}B^{1/2}(A+B)^{-1/2} = 1.$$

# 3 Applications

We can easily give another proof of

**Theorem A.** (Ando and Zhan)
Let $f(t) \geqq 0$ be an operator monotone function on $[0, \infty)$ such that $f(t)$ is continuous at $t = 0$. Then

$$|||f(A+B)||| \leqq |||f(A)+f(B)||| \tag{1}$$

for every unitarily invariant norm $|||\cdot|||$ and for all $A, B \geqq 0$.

*Proof* We may assume that $A+B$ is invertible. Then, since $(A+B)^{-1/2}A^{1/2}$ is contractive, by Hansen-Pedersen's inequality [7] we have

$$\begin{aligned} \varphi(A) &= \varphi(A^{1/2}(A+B)^{-1/2}(A+B)(A+B)^{-1/2}A^{1/2}) \\ &\geq A^{1/2}(A+B)^{-1/2}\varphi(A+B)(A+B)^{-1/2}A^{1/2}, \\ \varphi(B) &\geq B^{1/2}(A+B)^{-1/2}\varphi(A+B)(A+B)^{-1/2}B^{1/2}. \end{aligned}$$

Since $\varphi(t)$ is increasing and $\varphi(t)/t$ is decreasing, by Proposition 2.3 we get

$$\sigma^{(k)}(\varphi(A)+\varphi(B)) \geq \sigma^{(k)}(A^{1/2}(\varphi/t)(A+B)A^{1/2} + B^{1/2}(\varphi/t)(A+B)B^{1/2})$$
$$\geq \sigma^{(k)}(\varphi(A+B)) \quad (1 \leqq k \leqq n).$$

□

Also we can get the following generalization of (1),
For $A_i \geqq 0 \quad (1 \leqq i \leqq k)$

$$|||f(\sum_{i=1}^{k} A_i)||| \leqq |||\sum_{i=1}^{k} f(A_i)|||$$

Let $f(t)$ be a non-negative concave function on $0 \leqq t < \infty$. The following is known:

**Theorem B** (Rotfel'd [9]) (see also [10, 6, 4, 5])
For $A, B \geqq 0$

$$\|f(A+B)\|_1 \leqq \|f(A)\|_1 + \|f(B)\|_1.$$

We will extend this to every unitarily invariant norm.

**Theorem 3.1** ([12])

$$|||f(|X+Y|)||| \leqq |||f(|X|)||| + |||f(|Y|)||| \quad (\forall X, Y).$$

for every unitarily invariant norm $|||\cdot|||$ .

Now we can slightly improve this as follows:

**Theorem 3.2** Let $f$ be a non-negative (not necessarily continuous) concave function defined on $[0, \infty)$, and let $\{X_i\}(i = 0, 1, \cdots, k)$ be a finite set of matrices. Then there are unitary matrices $U_i(i = 1, \cdots, k)$ such that the inequality

$$\begin{aligned} &|||\ f(|X_0 + X_1 + \cdots + X_k|)||| \leqq \\ &|||\ f(|X_0|) + U_1^* f(|X_1|)U_1 + \cdots + U_k^* f(|X_k|)U_k||| \end{aligned}$$

holds for every unitarily invariant norm.

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