EXTREMAL PROBLEM OF A QUADRATICALLY HYPERSONAL WEIGHTED SHIFT

Hee Yul Lee

Department of Mathematics, College of Natural Sciences,
Kyoungpook National University,
Daegu 702-701, Korea

Abstract

Let \( \hat{\alpha}(x, y) : = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge \) be a weight sequence with \( 1 \leq x \leq y \) and \( 0 < a < 1 \) and let \( \mathcal{R} = \{(x, y) : W_{\hat{\alpha}(x, y)} \) is quadratically hyponormal and \( \|W_{\hat{\alpha}(x, y)}\| = 1 \} \). In this note we obtain concrete expressions of extremal values of \( \mathcal{R} \) with respect to \( x \) and \( y \).

1. Introduction and Preliminaries. Let \( \mathcal{H} \) be a separable, infinite dimensional, complex Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) be the algebra of all bounded linear operators on \( \mathcal{H} \). For \( A, B \in \mathcal{L}(\mathcal{H}) \) let \( [A, B] := AB - BA \). We say that an \( n \)-tuple \( T = (T_1, \ldots, T_n) \) of operators in \( \mathcal{L}(\mathcal{H}) \) is hyponormal if the operator matrix \( ([T_{ij}], \mathbb{J})_{i,j=1}^n \) is positive on the direct sum of \( n \) copies of \( \mathcal{H} \). For \( k \geq 1 \) and \( T \in \mathcal{L}(\mathcal{H}) \), \( T \) is \( k \)-hyponormal if \( (I, T, \ldots, T^k) \) is hyponormal. Recall that \( T = (T_1, \ldots, T_n) \) is weakly-hyponormal if \( \lambda_1 T_1 + \cdots + \lambda_n T_n \) is hyponormal for every \( \lambda_i \in \mathbb{C}, \) \( i = 1, \ldots, n \), where \( \mathbb{C} \) is the set of complex numbers. An operator \( T \) is weakly \( k \)-hyponormal if \( (T, \ldots, T^k) \) is weakly hyponormal. In particular, weak 2-hyponormality, often referred to as quadratic hyponormality, was discussed in [Cu], [CuF1], and [CuF2]. To detect the quadratic hyponormality of weighted shifts, Fialkow-Curto introduced the concept of positively quadratically hyponormal weighted shifts whose definition appears in [CuF2]. Also it was shown in [JP1] that two notions of quadratical hyponormality and positively quadratical hyponormality are equivalent in the one-step extended weighted shifts \( W_{\hat{\alpha}} \) with a tail induced recursively by three numbers \( 0 < b < c < d \), where \( \hat{\alpha} : = (\sqrt{b}, \sqrt{c}, \sqrt{d})^\wedge \). Furthermore, the flatness of weighted shifts makes an important role to study the quadratical hyponormality. As one of such models for studying its flatness, in [CuJ] they considered the recursively weighted shift \( \hat{\alpha}(x, y) : = (1, 1, \sqrt{z}, \sqrt{y})^\wedge \) with \( 1 \leq x \leq y \) and obtain that the set \( \mathcal{R} = \{(x, y) : W_{\hat{\alpha}(x, y)} \) is quadratically hyponormal\} is a convex set with nonempty interior and there exist unique maximum values \( x_M \) and \( y_M \) of \( x \) and \( y \) such that \( \mathcal{R} \cap \{x_M \times \mathbb{R} \} \) and \( \mathcal{R} \cap (\mathbb{R} \times \{y_M \}) \) are singletons. And they suggested the following external value problem.

Problem 1.1 ([CuJ, Problem 5.1]). Find a concrete expression for \( x_M \) and \( y_M \).

According to Corollary 2.2 below, it is worthwhile to consider only the case of weighted shift \( W_{\hat{\alpha}} \) with \( \|W_{\hat{\alpha}}\| = 1 \) to detect the quadratical hyponormality. For a given \( a \in (0, 1), \)

\footnote{2000 Mathematics subject classification: Primary 05C38, 15A15; Secondary 05A15, 15A18.}

\footnote{Key words and phrases: quadratically hyponormal weighted shifts, extremal values.}
let $\alpha(x, y) : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$ be a weight sequence with $1 \leq x \leq y$. In this note we solve Problem 1.1 for the weighted shift $W_\alpha(x, y)$.

We now recall [CuF1] that a weighted shift $W_\alpha$ is said to be recursively generated if there exist $i \geq 1$ and $\Psi = (\Psi_0, \cdots, \Psi_{i-1}) \in C^i$ such that

$$\gamma_n = \Psi_{i-1} \gamma_{n-1} + \cdots + \Psi_0 \gamma_{n-i} \quad (n \geq i),$$

where $\gamma_n (n \geq 0)$ is the moment sequence of $W_\alpha$, i.e., $\gamma_0 := 1$, $\gamma_{n+1} := \alpha_n^2 \gamma_n \quad (n \geq 0)$. Furthermore, (2) is equivalent to

$$\alpha_n^2 = \Psi_{i-1} + \frac{\Psi_{i-2}}{\alpha_{n-1}^2} + \cdots + \frac{\Psi_0}{\alpha_{n-i+1}^2} \quad (n \geq i).$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_{2k} \quad (k \geq 0)$, there is a canonical procedure to generate a sequence (denote $\hat{\alpha}$) in such a way that $W_\hat{\alpha}$ is a recursively generated shift having $\alpha$ as an initial segment of weights (cf. [CuF1]). We now review this procedure in a special case of $k = 1$. Given $\alpha : \alpha_0, \alpha_1, \alpha_2 \quad (0 < \alpha_0 < \alpha_1 < \alpha_2)$, let

$$v_0 := \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}, \quad v_1 := \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \quad v_2 := \begin{bmatrix} \gamma_2 \end{bmatrix}.$$

The vectors $v_0$ and $v_1$ are linearly independent in $R^2$, so there exists a unique $\Psi = (\Psi_0, \Psi_1) \in R^2$ such that $v_2 = \Psi_0 v_0 + \Psi_1 v_1$. In fact,

$$\Psi_0 = -\frac{\alpha_0 \alpha_2 (\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_1^2 (\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

Let $\hat{\gamma} := \gamma_n \quad (0 \leq n \leq 1)$ and let $\hat{\gamma}_n := \Psi_1 \gamma_{n-1} + \Psi_0 \gamma_{n-2} \quad (n \geq 2)$. Then $\hat{\alpha}_n := \sqrt{\hat{\gamma}_{n+1}} / \sqrt{\hat{\gamma}}_n \quad (n \geq 0)$ (so that $\hat{\alpha}_n = \alpha_n$ for $0 \leq n \leq 2$) and the coefficients of a recursively generated weighted shift is $\hat{\alpha}_2 \wedge = \Psi_1 + \Psi_0 / \alpha_{n-1}^2 \quad (n \geq 1)$. Such a recursively weight sequence is written by $(\alpha_0, \alpha_1, \alpha_2)^\wedge$.

This note will be appeared in some other journal as a full version.

2. Striving extremal values. We consider recursively generated weighted shifts of the general form $W_\alpha$ with a weight sequence $\alpha : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$ and $0 < \alpha < x < y$. In special case, we focus on the weighted shift $W_\alpha$ having the norm one which, however, involves without loss of generality.

We begin with the following elementary lemma.

**Lemma 2.1.** Let $0 < a \leq b \leq c$. Then $\sqrt{s} \cdot W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge} = W_{(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge}$ for any $s \in (0, \infty)$.

The following corollary follows immediately from Lemma 2.1.

**Corollary 2.2.** Let $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \cdots, \sqrt{\alpha_n}, (\sqrt{\alpha_n}, \sqrt{\alpha_{n+1}}, \sqrt{\alpha_{n+2}})^\wedge$ with $0 < \alpha_{i-1} \leq \alpha_i$ for all $i \geq 1$. Then the unilateral weighted shift $W_\alpha$ has norm $\sqrt{s}$ if and only if the shift $W_\alpha$ with $\alpha' : \sqrt{s}, \sqrt{s}, \cdots, \sqrt{s}, (\sqrt{s}, \sqrt{s}, \sqrt{s})^\wedge$ has norm 1.
Theorem 2.3 Let $W_{\alpha}$ be a recursively generated weighted shift with $\alpha : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$, $0 < a < x < y \leq 1$, and $\|W_{\alpha}\| = 1$. Then $W_{\alpha}$ is quadratically hyponormal if and only if $x \in (a, r_{a}]$ where $r_{a}$ is the root of $f(x) = 0$, where $f(x) = \sum_{i=0}^{4} c_i x^i$ with

\begin{align*}
c_0 &= a > 0, \\
c_1 &= -(a^5 - a^4 - a^3 + 3a^2 + 1) < 0, \\
c_2 &= a(2a^4 - 3a^3 + a^2 + 3) > 0, \\
c_3 &= -a^2(a^3 - 2a^2 - a + 3) < 0, \\
c_4 &= a^3(1 - a) > 0.
\end{align*}

(Note that $0 < r_{a} < 1$.)

Remark 2.4. By a simple computation we have that

$$r_{a} = -\frac{c_3}{4c_4} - \frac{1}{2} G - \frac{1}{2} \sqrt{\frac{c_3^2}{2c_4} - \frac{4c_2}{3c_4} - A - B - \frac{t}{4G}},$$

where

$$A = \frac{2^\frac{1}{3}q}{3c_4(p + \sqrt{-4q^3 + p^2})^\frac{1}{3}},$$

$$B = \frac{(p + \sqrt{-4q^3 + p^2})^\frac{1}{3}}{32^\frac{1}{3}c_4},$$

$$G = \sqrt{\frac{c_3^2}{4c_4^2} - \frac{2c_2}{3c_4} + A + B},$$

$$t = -\frac{c_3^2}{c_4^2} + \frac{4c_2c_3}{c_4^2} - \frac{8c_1}{c_4},$$

$$p = 2c_2^2 - 9c_1c_2c_3 + 27c_1c_4c_2 + 27c_4c_3^2 - 72c_0c_2c_4,$$

$$q = c_2^2 - 3c_1c_3 + 12c_0c_4.$$


