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EXTREMAL PROBLEM OF A QUADRATICALLY HYPONORMAL WEIGHTED SHIFT

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Abstract

Let $\widehat{\alpha}(x,y):\sqrt{a},(\sqrt{a},\sqrt{x},\sqrt{y})^{\wedge}$ be a weight sequence with $1\leq x\leq y$ and $0<a<1$ and let $\mathcal{R}=\{(x,y):W_{\widehat{\alpha}(x,y)}$ is quadratically hyponormal and $\|W_{\widehat{\alpha}(x,y)}\|=1\}$. In this note we obtain concrete expressions of extremal values of $\mathcal{R}$ with respect to x and y.

1. Introduction and Preliminaries. Let $\mathcal{H}$ be a separble, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For $A,\ B\in\mathcal{L}(\mathcal{H})$ let $[A,B]:=AB-BA$. We say that an n-tuple $T=(T_1,...,T_n)$ of operators in $\mathcal{L}(\mathcal{H})$ is *hyponormal* if the operator matrix $([T_j^*,T_i])_{i,j=1}^n$ is positive on the direct sum of n copies of $\mathcal{H}$. For $k\geq 1$ and $T\in\mathcal{L}(\mathcal{H})$, T is *k-hyponormal* if $(I,T,...,T^k)$ is hyponormal. Recall that $T=(T_1,...,T_n)$ is *weakly-hyponormal* if $\lambda_1T_1+\cdots+\lambda_nT_n$ is hyponormal for every $\lambda_i\in\mathbb{C}$, $i=1,...,n$, where $\mathbb{C}$ is the set of complex numbers. An operator T is *weakly k-hyponormal* if $(T,...,T^k)$ is weakly hyponormal. In particular, weak 2-hyponormality, often referred to as *quadratic hyponormality*, was discussed in [Cu], [CuF1], and [CuF2]. To detect the quadratical hyponormality of weighted shifts, Fialkow-Curto introduced the concept of positively quadratically hyponormal weighted shifts whose definition appears in [CuF2]. Also it was shown in [JP1] that two notions of quadratical hyponormality and positively quadratical hyponormality are equivalent in the one-step extended weighted shifts $W_{\widehat{\alpha}}$ with a tail induced recursively by three numbers $0<b<c<d$, where $\widehat{\alpha}:\sqrt{a},(\sqrt{b},\sqrt{c},\sqrt{d})^{\wedge}$. Furthermore, the flatness of weighted shifts makes an important role to study the quadratic hyponormality. As one of such models for studying its flatness, in [CuJ] they considered the recursively weighted shift $\widehat{\alpha}(x,y):1,(1,\sqrt{x},\sqrt{y})^{\wedge}$ with $1\leq x\leq y$ and obtain that the set $\mathcal{R}=\{(x,y):W_{\widehat{\alpha}(x,y)}$ is quadratically hyponormal$\}$ is a convex set with nonempty enterior and there exist unique maximum values x_M and y_M of x and y such that $\mathcal{R}\cap(\{x_M\}\times\mathbb{R})$ and $\mathcal{R}\cap(\mathbb{R}\times\{y_M\})$ are singletons. And they suggested the following extermal value problem.

Problem 1.1 ([CuJ, Problem 5.1]). Find a concrete expression for x_M and y_M.

According to Corollary 2.2 below, it is worthwile to consider only the case of weighted shift W_α with $\|W_\alpha\|=1$ to detect the quadratical hyponormality. For a given $a\in(0,1)$,

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let $\widehat{\alpha}(x,y) : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$ be a weight sequence with $1 \le x \le y$. In this note we solve Problem 1.1 for the weighted shift $W_{\widehat{\alpha}(x,y)}$.

We now recall [CuF1] that a weighted shift W_α is said to be *recursively generated* if there exist $i \ge 1$ and $\Psi = (\Psi_0, \cdots, \Psi_{i-1}) \in \mathbb{C}^i$ such that

$$\gamma_n = \Psi_{i-1}\gamma_{n-1} + \cdots + \Psi_0\gamma_{n-i} \quad (n \ge i),$$

where $\gamma_n (n \ge 0)$ is the moment sequence of W_α, i.e., $\gamma_0 := 1$, $\gamma_{n+1} := \alpha_n^2\gamma_n$ $(n \ge 0)$. Furthermore, (2) is equivalent to

$$\alpha_n^2 = \Psi_{i-1} + \frac{\Psi_{i-2}}{\alpha_{n-1}^2} + \cdots + \frac{\Psi_0}{\alpha_{n-1}^2 \cdots \alpha_{n-i+1}^2} \quad (n \ge i).$$

Given an initial segment of weights $\alpha : \alpha_0, \cdots, \alpha_{2k}$ $(k \ge 0)$, there is a canonical procedure to generate a sequence (denote $\widehat{\alpha}$) in such a way that $W_{\widehat{\alpha}}$ is a recursively generated shift having α as an initial segment of weights (cf. [CuF1]). We now review this procedure in a special case of $k = 1$. Given $\alpha : \alpha_0, \alpha_1, \alpha_2$ $(0 < \alpha_0 < \alpha_1 < \alpha_2)$, let

$$v_0 := \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}, \quad v_1 := \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}, \quad v_2 := \begin{bmatrix} \gamma_2 \\ \gamma_3 \end{bmatrix}.$$

The vectors v_0 and v_1 are linearly independent in $\mathbb{R}^2$, so there exists a unique $\Psi = (\Psi_0, \Psi_1) \in \mathbb{R}^2$ such that $v_2 = \Psi_0 v_0 + \Psi_1 v_1$. In fact,

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

Let $\widehat{\gamma} := \gamma_n$ $(0 \le n \le 1)$ and let $\widehat{\gamma}_n := \Psi_1\widehat{\gamma}_{n-1} + \Psi_0\widehat{\gamma}_{n-2}$ $(n \ge 2)$. Then $\widehat{\alpha}_n := \sqrt{\widehat{\gamma}_{n+1}/\widehat{\gamma}_n}$ $(n \ge 0)$ (so that $\widehat{\alpha}_n = \alpha_n$ for $0 \le n \le 2$) and the coefficients of a recursively generated weighted shift is $\widehat{\alpha}_n^2 = \Psi_1 + \Psi_0/\widehat{\alpha}_{n-1}^2$ $(n \ge 1)$. Such a recusively weight sequence is written by $(\alpha_0, \alpha_1, \alpha_2)^\wedge$.

This note will be appeared in some other journal as a full version.

2. Striving extremal values. We consider recursively generated weighted shifts of the general form W_α with a weight sequence $\alpha : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$ and $0 < a \le x \le y$. In special case, we focus on the weighted shift W_α having the norm one which, however, involves without loss of generality.

We begin with the following elementary lemma.

Lemma 2.1. *Let $0 < a \le b \le c$. Then $\sqrt{s} \cdot W_{(\sqrt{a},\sqrt{b},\sqrt{c})^\wedge} = W_{(\sqrt{sa},\sqrt{sb},\sqrt{sc})^\wedge}$ for any $s \in (0, \infty)$.*

The following corollary follows immediately from Lemma 2.1.

Corollary 2.2. *Let $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \cdots, \sqrt{\alpha_{n-1}}, (\sqrt{\alpha_n}, \sqrt{\alpha_{n+1}}, \sqrt{\alpha_{n+2}})^\wedge$ with $0 < \alpha_{i-1} \le \alpha_i$ for all $i \ge 1$. Then the unilateral weighted shift W_α has norm $\sqrt{\delta}$ if and only if the shift W_α' with $\alpha' : \sqrt{\frac{\alpha_0}{\delta}}, \ldots, \sqrt{\frac{\alpha_{n-1}}{\delta}}, (\sqrt{\frac{\alpha_n}{\delta}}, \sqrt{\frac{\alpha_{n+1}}{\delta}}, \sqrt{\frac{\alpha_{n+2}}{\delta}})^\wedge$ has norm 1.*

Theorem 2.3 *Let W_α be a recursively generated weighted shift with* $\alpha : \sqrt{a}, (\sqrt{a}, \sqrt{x}, \sqrt{y})^\wedge$, $0 < a < x < y \le 1$, *and* $\|W_\alpha\| = 1$. *Then W_α is quadratically hyponormal if and only if* $x \in (a, r_a]$ *where r_a is the root of $f(x) = 0$, where* $f(x) = \sum_{i=0}^4 c_i x^i$ with

$$\begin{aligned} c_0 &:= a > 0, \\ c_1 &:= -(a^5 - a^4 - a^3 + 3a^2 + 1) < 0, \\ c_2 &:= a(2a^4 - 3a^3 + a^2 + 3) > 0, \\ c_3 &:= -a^2(a^3 - 2a^2 - a + 3) < 0, \\ c_4 &:= a^3(1 - a) > 0. \end{aligned}$$

(Note that $0 < r_a < 1$.)

Remark 2.4. By a simple computation we have that

$$r_a = -\frac{c_3}{4c_4} - \frac{1}{2}G - \frac{1}{2}\sqrt{\frac{c_3^2}{2c_4^2} - \frac{4c_2}{3c_4} - A - B - \frac{t}{4G}},$$

where

$$\begin{aligned} A &= \frac{2^{\frac{1}{3}} q}{3c_4(p + \sqrt{-4q^3 + p^2})^{\frac{1}{3}}}, \\ B &= \frac{(p + \sqrt{-4q^3 + p^2})^{\frac{1}{3}}}{32^{\frac{1}{3}} c_4}, \\ G &= \sqrt{\frac{c_3^2}{4c_4^2} - \frac{2c_2}{3c_4} + A + B}, \\ t &= -\frac{c_3^3}{c_4^3} + \frac{4c_2c_3}{c_4^2} - \frac{8c_1}{c_4}, \\ p &= 2c_2^3 - 9c_1c_2c_3 + 27c_1^2c_4 + 27c_0c_3^2 - 72c_0c_2c_4, \\ q &= c_2^2 - 3c_1c_3 + 12c_0c_4. \end{aligned}$$

Example 2.5. If we consider $a = \frac{1}{2}$, then $f(x) = \frac{1}{16}x^4 - \frac{17}{32}x^3 + \frac{3}{2}x^2 - \frac{51}{32}x + \frac{1}{2}$ and so

$$r_a = \frac{1}{8}(17 - \sqrt{17} - \sqrt{2(41 - \sqrt{17})}).$$

Hence W_α is quadratically hyponormal if and only if $1/2 < x \le \frac{1}{8}(17 - \sqrt{17} - \sqrt{2(41 - \sqrt{17})})$.

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