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ON RICCATI INEQUALITY
(RICCATI 不等式について)

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1. Introduction.

As stated in our preceding discussion, the algebraic Riccati equation

$$X^*B^{-1}X - T^*X - X^*T = C \quad (B, C; \text{positive definite matrices})$$

has solutions given by $X = W + BT$ for some solution W of

$$W^*B^{-1}W = C + T^*BT$$

because

$$X^*B^{-1}X - T^*X - X^*T = W^*B^{-1}W - T^*BT.$$

Namely the operator equation

(1) $$X^*B^{-1}X = C$$

is essential, so we call it the Riccati equation.

Related to this, we recall Ando's definition of operator geometric mean [2]: For positive operators B, C on a Hilbert space,

(2) $$B \sharp C = \max\{X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\}.$$

is called the geometric mean of B and C. If B is invertible, it is expressed by

(3) $$B \sharp C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}.$$

It is known that $X_0 = B \sharp C$ is the unique positive solution of the Riccati equation $X^*B^{-1}X = C$, see [1,4,6,11].

Very recently, Izumino and Nakamura [5] gave some interesting consideration to weakly positive operators introduced by Wigner [12]. An operator T is weakly positive if $T = SCS^{-1}$ for some S, $C > 0$, where $X > 0$ means it is positive and invertible. It is equivalent to be of form $T = AB$ for some A, $B > 0$. (Take $A = S^2$ and $B = S^{-1}CS^{-1}$.) They pointed out that the square root $T^{\frac{1}{2}}$ of a weakly positive operator $T = SCS^{-1} = AB$ can be defined by $T^{\frac{1}{2}} = SC^{\frac{1}{2}}S^{-1} = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}A^{-\frac{1}{2}}$, and

$$A^{-1} \sharp B = A^{-1}(AB)^{\frac{1}{2}}.$$

As an easy consequence, $A^{-1} \sharp B$ is a (unique) positive solution of Riccati equation $XAX = B$ for given A, $B > 0$.

Inspired by Ando's work (2) and Izumino-Nakamura's consideration, we would like to introduce a Riccati inequality by the positivity of an operator matrix, i.e.,

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \text{for given} \quad B, A \geq 0$$

2. Riccati inequality.

In this section, we investigate solutions of Riccati inequality, in which we characterize them by factorization.

The following lemma is well-known, but important.

Lemma 1. *Let A be a positive operator. Then*

$$\begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \geq 0 \quad \textit{if and only if} \quad A \geq X^*X.$$

Proof. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -X^* & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix},$$

it follows that

$$\begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \geq 0 \quad \text{if and only if} \quad A \geq X^*X.$$

We here note the existence of the maximum of the geometric mean (2), as an application of an idea in Lemma 1: We may assume that B is invertible. Then

$$\begin{pmatrix} B & X \\ X^* & C \end{pmatrix} \geq 0 \quad \text{if and only if} \quad C \geq X^*B^{-1}X$$

because

$$\begin{pmatrix} 1 & 0 \\ -X^*B^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & X \\ X^* & C \end{pmatrix} \begin{pmatrix} 1 & -B^{-1}X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & C - X^*B^{-1}X \end{pmatrix}.$$

Hence, if $C \geq XB^{-1}X$, then

$$B^{-\frac{1}{2}}CB^{-\frac{1}{2}} \geq (B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^2.$$

By Lowner-Heinz inequality, we have

$$(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}} \geq B^{-\frac{1}{2}}XB^{-\frac{1}{2}},$$

so that

$$B \,\sharp\, C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}} \geq X.$$

Consequently, the maximum

$$\max\{X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\} = \max\{X \geq 0; C \geq XB^{-1}X\}$$

is given by $B \,\sharp\, C$.

Lemma 2. *Let A and B be positive operators. Then*

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \textit{implies} \quad \mathrm{ranW} \subseteq \mathrm{ranB}^{\frac{1}{2}}.$$

and so $X = B^{-\frac{1}{2}}W$ is well-defined as a mapping.

Proof. Let $S = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ be the square root of $R = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix}$. Then

$$R = S^2 = R = \begin{pmatrix} a^2 + bb^* & ab + bd \\ b^*a + db^* & b^*b + d^2 \end{pmatrix},$$

that is,

$$B = a^2 + bb^* \text{ and } W = ab + bd.$$

Therefore ran $B^{\frac{1}{2}}$ contains both ran a and ran b, so that it contains ran $a+$ ran b. Moreover ran W is contained in ran $a+$ ran b by $W = ab + bd$.

Under the preparation of Lemmas 1 and 2, Riccati inequality can be solved as follows:

Theorem 3. *Let A and B be positive operators on K and H respectively, and W be an operator from K to H. Then $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ if and only if $W = B^{\frac{1}{2}}X$ for some operator X from K to H and $A \geq X^*X$.*

Proof. Suppose that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$. Since ranW $\subseteq$ ranB$^{\frac{1}{2}}$ by Lemma 2, Douglas' majorization theorem [3] says that $W = B^{\frac{1}{2}}X$ for some operator X. Moreover we restrict X by $P_BX = X$, where P_B is the range projection of B. Noting that $y \in$ ranB if and only if $y = B^{\frac{1}{2}}x$ for some $x \in$ ranB$^{\frac{1}{2}}$, the assumption implies that

$$\left(\begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left(\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix} \right) \geq 0$$

for all $y \in$ ranB and $z \in K$. This means that $\begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \geq 0$, and so

$$\begin{pmatrix} P_B & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -X^* & A \end{pmatrix} \begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} \geq 0,$$

that is, $A \geq X^*X$, as required. The converse is easily checked.

The following factorization theorem [2; Theorem I.1] is led by Theorem 3 and Douglas' factorization theorem [3].

Theorem 4. *Let A and B be positive operators. Then $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ if and only if $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ for some contraction V.*

Proof. Suppose that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$. Then it follows from Theorem 3 that $W = B^{\frac{1}{2}}X$ for some bounded X satisfying $A \geq X^*X$. Hence we can find a contraction V with $X = VA^{\frac{1}{2}}$ by [3], so that $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ is shown.
The converse is proved by Lemma 1 as follows:

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} = \begin{pmatrix} B & B^{\frac{1}{2}}VA^{\frac{1}{2}} \\ A^{\frac{1}{2}}V^*B^{\frac{1}{2}} & A \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & V \\ V^* & 1 \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \geq 0.$$

3. An exact expression of the harmonic mean. Recall that the harmonic mean is defined by

$$B\ !\ C = \max\{X \geq 0; \begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \geq \begin{pmatrix} X & X \\ X & X \end{pmatrix}\}.$$

To obtain the exact expression of the harmonic mean, we need the following lemma which is more explicit than Theorem 3.

Lemma 5. *If* $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$, *then* $X = B^{-\frac{1}{2}}W$ *is bounded and* $A \geq X^*X$.

Proof. For a fixed vector x, we put $x_1 = B^{-\frac{1}{2}}Wx$. Since $B^{\frac{1}{2}}x_1 = Wx$, we may assume $x_1 \in (\ker B^{\frac{1}{2}})^{\perp}$. So it follows that

$$\begin{aligned} \|B^{-\frac{1}{2}}Wx\| &= \sup\{|(Wx, v)|; \|v\| = 1\} \\ &= \sup\{|(B^{-\frac{1}{2}}Wx, B^{\frac{1}{2}}u)|; \|B^{\frac{1}{2}}u\| = 1\} \\ &= \sup\{|(Wx, u)|; (Bu, u) = 1\}. \end{aligned}$$

On the other hand, since

$$\left(\begin{pmatrix} B & W \\ W^* & A \end{pmatrix}\begin{pmatrix} u \\ tx \end{pmatrix}, \begin{pmatrix} u \\ tx \end{pmatrix}\right) = |t|^2(Ax, x) + 2\mathrm{Re}\ t(Wx, u) + (Bu, u) \geq 0$$

for all scalars t, we have

$$|(Wx, u)|^2 \leq (Ax, x)(Bu, u).$$

Hence it follows that

$$\|B^{-\frac{1}{2}}Wx\|^2 = \sup\{|(Wx, u)|; (Bu, u) = 1\} \leq (Ax, x),$$

which implies that $X = B^{-\frac{1}{2}}W$ is bounded and $A \geq X^*X$.

Theorem 6. *Let* B, C *be positive operators. Then*

$$B\ !\ C = 2(B - [(B+C)^{-\frac{1}{2}}B]^*[(B+C)^{-\frac{1}{2}}B]).$$

In particular, if $B + C$ *is invertible, then*

$$B\ !\ C = 2(B - B(B+C)^{-1}B) = 2B(B+C)^{-1}C.$$

Proof. First of all, the inequality $\begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \geq \begin{pmatrix} X & X \\ X & X \end{pmatrix}$ is equivalent to

$$\begin{pmatrix} 2(B+C) & -2B \\ -2B & 2B - X \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2B - X & -X \\ -X & 2C - X \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \geq 0.$$

Then it follows from Lemma 5 that $D = [2(B+C)]^{-\frac{1}{2}}(-2B)$ is bounded and $D^*D \leq 2B - X$. Therefore we have the explicit expression of $B\ !\ C$ even if both B and C are non-inbertible:

$$B\ !\ C = \max\{X \geq 0; D^*D \leq 2B - X\} = 2B - D^*D.$$

In particular, if $B + C$ is invertible, then

$$B\ !\ C = 2B - D^*D = 2(B - B(B+C)^{-1}B) = 2B(B+C)^{-1}C.$$

4. Pedersen-Takesaki theorem. Finally we review a work of Pedersen and Takesaki [8] from the viewpoint of Riccati inequality; we add another equivalent condition to their theorem:

Theorem 7. *Let B and C be positive operators and B be nonsingular. Then the following statements are mutually equivalent:*

(1) *Riccati equation $XBX = C$ has a positive solution.*

(2) *$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \le kB$ for some $k > 0$.*

(3) *There exists the minimum of $\{X \ge 0; C \le XBX\}$.*

(3') *There exists the minimum of $\{X \ge 0; \begin{pmatrix} 1 & C^{\frac{1}{2}} \\ C^{\frac{1}{2}} & XBX \end{pmatrix} \ge 0\}$.*

Proof. We first note that (3) and (3') are equivalent by Lemma 1.

Now we suppose (1), i.e., $X_0BX_0 = C$ for some $X_0 \ge 0$. If $X \ge 0$ satisfies $C \le XBX$, then

$$(B^{\frac{1}{2}}X_0B^{\frac{1}{2}})^2 = B^{\frac{1}{2}}CB^{\frac{1}{2}} \le (B^{\frac{1}{2}}XB^{\frac{1}{2}})^2$$

and so

$$B^{\frac{1}{2}}X_0B^{\frac{1}{2}} \le B^{\frac{1}{2}}XB^{\frac{1}{2}}.$$

Since B is nonsingular, we have $X_0 \le X$, namely (3) is proved.

Next we suppose (3). Since $C \le XBX$ for some X, we have

$$B^{\frac{1}{2}}CB^{\frac{1}{2}} \le (B^{\frac{1}{2}}XB^{\frac{1}{2}})^2$$

and so

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} \le B^{\frac{1}{2}}XB^{\frac{1}{2}} \le \|X\|B,$$

which shows (2).

The implication (2) $\to$ (1) has been shown by Pedersen-Takesaki [8] and Nakamoto [7], but we sketch it for convenience. By Douglas' majorization theorem [3], we have

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{4}} = ZB^{\frac{1}{2}}$$

for some Z, so that

$$(B^{\frac{1}{2}}CB^{\frac{1}{2}})^{\frac{1}{2}} = B^{\frac{1}{2}}Z^*ZB^{\frac{1}{2}} \quad \text{and} \quad B^{\frac{1}{2}}CB^{\frac{1}{2}} = B^{\frac{1}{2}}(Z^*ZBZ^*Z)B^{\frac{1}{2}}.$$

Since B is nonsingular, Z^*Z is a solution of $XBX = C$.

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