ON RICCATI INEQUALITY
(RICCATI 不等式について)

Jun Ichi Fujii (藤井淳一, 大阪教育大学)
Masatoshi Fujii (藤井正俊, 大阪教育大学)
Ritsuo Nakamoto (中本律男, 筑城大学)

1. Introduction.
As stated in our preceding discussion, the algebraic Riccati equation

\[ X^*B^{-1}X - T^*X - X^*T = C \quad (B, C; \text{positive definite matrices}) \]

has solutions given by \( X = W + BT \) for some solution \( W \) of

\[ W^*B^{-1}W = C + T^*BT \]

because

\[ X^*B^{-1}X - T^*X - X^*T = W^*B^{-1}W - T^*BT. \]

Namely the operator equation

(1) \[ X^*B^{-1}X = C \]

is essential, so we call it the Riccati equation.

Related to this, we recall Ando's definition of operator geometric mean [2]: For positive operators \( B, C \) on a Hilbert space,

(2) \[ B \# C = \max \{X \geq 0; X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\}. \]

is called the geometric mean of \( B \) and \( C \). If \( B \) is invertible, it is expressed by

(3) \[ B \# C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}}. \]

It is known that \( X_0 = B \# C \) is the unique positive solution of the Riccati equation \( X^*B^{-1}X = C \), see [1,4,6,11].

Very recently, Izumino and Nakamura [5] gave some interesting consideration to weakly positive operators introduced by Wigner [12]. An operator \( T \) is weakly positive if \( T = SCS^{-1} \) for some \( S, C > 0 \), where \( X > 0 \) means it is positive and invertible. It is equivalent to be of form \( T = AB \) for some \( A, B > 0 \). (Take \( A = S^2 \) and \( B = S^{-1}CS^{-1} \).) They pointed out that the square root \( T^{\frac{1}{2}} \) of a weakly positive operator \( T = SCS^{-1} = AB \) can be defined by \( T^{\frac{1}{2}} = SC^{\frac{1}{2}}S^{-1} = A^{\frac{1}{2}}(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}A^{-\frac{1}{2}} \), and

\[ A^{-1} \# B = A^{-1}(AB)^{\frac{1}{2}}. \]

As an easy consequence, \( A^{-1} \# B \) is a (unique) positive solution of Riccati equation \( XAX = B \) for given \( A, B > 0 \).

Inspired by Ando's work (2) and Izumino-Nakamura's consideration, we would like to introduce a Riccati inequality by the positivity of an operator matrix, i.e.,

\[ \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \text{for given } B, A > 0 \]
2. Riccati inequality.

In this section, we investigate solutions of Riccati inequality, in which we characterize them by factorization.

The following lemma is well-known, but important.

**Lemma 1.** Let $A$ be a positive operator. Then

\[
\begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \geq 0 \quad \text{if and only if} \quad A \geq X^*X.
\]

**Proof.** Since

\[
\begin{pmatrix} 1 & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -X^* & 1 \end{pmatrix} \begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix},
\]

it follows that

\[
\begin{pmatrix} 1 & X \\ X^* & A \end{pmatrix} \geq 0 \quad \text{if and only if} \quad A \geq X^*X.
\]

We here note the existence of the maximum of the geometric mean (2), as an application of an idea in Lemma 1: We may assume that $B$ is invertible. Then

\[
\begin{pmatrix} B & X \\ X^* & C \end{pmatrix} \geq 0 \quad \text{if and only if} \quad C \geq X^*B^{-1}X
\]

because

\[
\begin{pmatrix} 1 & 0 \\ -X^*B^{-1} & 1 \end{pmatrix} \begin{pmatrix} B & X \\ X^* & C \end{pmatrix} \begin{pmatrix} 1 & -B^{-1}X \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & C - X^*B^{-1}X \end{pmatrix}.
\]

Hence, if $C \geq XB^{-1}X$, then

\[
B^{-\frac{1}{2}}CB^{-\frac{1}{2}} \geq (B^{-\frac{1}{2}}XB^{-\frac{1}{2}})^2.
\]

By Lowner-Heinz inequality, we have

\[
(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^\frac{1}{2} \geq B^{-\frac{1}{2}}XB^{-\frac{1}{2}},
\]

so that

\[
B \parallel C = B^{\frac{1}{2}}(B^{-\frac{1}{2}}CB^{-\frac{1}{2}})^{\frac{1}{2}}B^{\frac{1}{2}} \geq X.
\]

Consequently, the maximum

\[
\max\{X \geq 0; \begin{pmatrix} B & X \\ X & C \end{pmatrix} \geq 0\} = \max\{X \geq 0; C \geq XB^{-1}X\}
\]

is given by $B \parallel C$.

**Lemma 2.** Let $A$ and $B$ be positive operators. Then

\[
\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \quad \text{implies} \quad \text{ran} W \subseteq \text{ran} B^{\frac{3}{2}}.
\]

and so $X = B^{-\frac{3}{2}}W$ is well-defined as a mapping.
Proof. Let $S = \begin{pmatrix} a & b \\ b^* & d \end{pmatrix}$ be the square root of $R = \begin{pmatrix} B & W \\ W^* & A \end{pmatrix}$. Then

$$R = S^2 = R = \begin{pmatrix} a^2 + bb^* & ab + bd \\ b^*a + db^* & b^*b + d^2 \end{pmatrix},$$

that is,

$$B = a^2 + bb^* \text{ and } W = ab + bd.$$ 
Therefore ran $B^{\frac{1}{2}}$ contains both ran $a$ and ran $b$, so that it contains ran $a+$ ran $b$. Moreover ran $W$ is contained in ran $a+$ ran $b$ by $W = ab + bd$.

Under the preparation of Lemmas 1 and 2, Riccati inequality can be solved as follows:

**Theorem 3.** Let $A$ and $B$ be positive operators on $K$ and $H$ respectively, and $W$ be an operator from $K$ to $H$. Then $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$ if and only if $W = B^{\frac{1}{2}}X$ for some operator $X$ from $K$ to $H$ and $A \geq X^*X$.

Proof. Suppose that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$. Since ran$W \subseteq$ ran$B^{\frac{1}{2}}$ by Lemma 2, Douglas' majorization theorem [3] says that $W = B^{\frac{1}{2}}X$ for some operator $X$. Moreover we restrict $X$ by $P_B X = X$, where $P_B$ is the range projection of $B$. Noting that $y \in$ ran$B$ if and only if $y = B^{\frac{1}{2}}x$ for some $x \in$ ran$B^{\frac{1}{2}}$, the assumption implies that

$$\left( \begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix} \right) = \left( \begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix} \right) \geq 0$$

for all $y \in$ ran$B$ and $z \in$ $K$. This means that $\begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \geq 0$, and so

$$\begin{pmatrix} P_B & 0 \\ 0 & A - X^*X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \left( \begin{pmatrix} P_B & X \\ X^* & A \end{pmatrix} \begin{pmatrix} 1 & -X^* \\ X^* & 1 \end{pmatrix} \right) \geq 0,$$

that is, $A \geq X^*X$, as required. The converse is easily checked.

The following factorization theorem [2; Theorem 1.1] is led by Theorem 3 and Douglas' factorization theorem [3].

**Theorem 4.** Let $A$ and $B$ be positive operators. Then

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0 \text{ if and only if } W = B^{\frac{1}{2}}VA^{\frac{1}{2}} \text{ for some contraction } V.$$  

Proof. Suppose that $\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \geq 0$. Then it follows from Theorem 3 that $W = B^{\frac{1}{2}}X$ for some bounded $X$ satisfying $A \geq X^*X$. Hence we can find a contraction $V$ with $X = VA^{\frac{1}{2}}$ by [3], so that $W = B^{\frac{1}{2}}VA^{\frac{1}{2}}$ is shown.

The converse is proved by Lemma 1 as follows:

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} = \begin{pmatrix} B & B^{\frac{1}{2}}VA^{\frac{1}{2}} \\ A^{\frac{1}{2}}V^*B^{\frac{1}{2}} & A \end{pmatrix} = \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ V & 1 \end{pmatrix} \begin{pmatrix} B^{\frac{1}{2}} & 0 \\ 0 & A^{\frac{1}{2}} \end{pmatrix} \geq 0.$$
3. An exact expression of the harmonic mean. Recall that the harmonic mean is defined by

$$B \! \! C = \max \{ X \geq 0; \begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \preceq \begin{pmatrix} X & X \\ X & X \end{pmatrix} \}.$$ 

To obtain the exact expression of the harmonic mean, we need the following lemma which is more explicit than Theorem 3.

**Lemma 5.** If $$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \succeq 0,$$ then $$X = B^{-\frac{1}{2}}W$$ is bounded and $$A \geq X^*X.$$

**Proof.** For a fixed vector $$x$$, we put $$x_1 = B^{-\frac{1}{2}}Wx$$. Since $$B^{\frac{1}{2}}x_1 = Wx$$, we may assume $$x_1 \in (\ker B^\frac{1}{2})^\perp$$. So it follows that

$$\|B^{-\frac{1}{2}}Wx\| = \sup\{|(Wx, v); \|v\| = 1\} = \sup\{|(B^{-\frac{1}{2}}Wx, B^{\frac{1}{2}}u); \|B^{\frac{1}{2}}u\| = 1\} = \sup\{|(Wx, u); (Bu, u) = 1\}.$$ 

On the other hand, since

$$\begin{pmatrix} B & W \\ W^* & A \end{pmatrix} \begin{pmatrix} u \\ tx \end{pmatrix}, \begin{pmatrix} u \\ tx \end{pmatrix} = \|t\|^2(Ax, x) + 2\text{Re}(Wx, u) + (Bu, u) \geq 0$$

for all scalars $$t$$, we have

$$|(Wx, u)|^2 \leq (Ax, x)(Bu, u).$$

Hence it follows that

$$\|B^{-\frac{1}{2}}Wx\|^2 = \sup\{|(Wx, u); (Bu, u) = 1\} \leq (Ax, x),$$

which implies that $$X = B^{-\frac{1}{2}}W$$ is bounded and $$A \geq X^*X$$.

**Theorem 6.** Let $$B$$, $$C$$ be positive operators. Then

$$B \! \! C = 2(B - [(B + C)^{-\frac{1}{2}}B]^*[B + C]^{-\frac{1}{2}}B)).$$

In particular, if $$B + C$$ is invertible, then

$$B \! \! C = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C.$$ 

**Proof.** First of all, the inequality $$\begin{pmatrix} 2B & 0 \\ 0 & 2C \end{pmatrix} \succeq \begin{pmatrix} X & X \\ X & X \end{pmatrix}$$ is equivalent to

$$\begin{pmatrix} 2(B + C) & -2B \\ -2B & 2B - X \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2B - X & -X \\ -X & 2C - X \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \succeq 0.$$ 

Then it follows from Lemma 5 that $$D = [2(B + C)]^{-\frac{1}{2}}(-2B)$$ is bounded and $$D^*D \leq 2B - X$$. Therefore we have the explicit expression of $$B \! \! C$$ even if both $$B$$ and $$C$$ are non-invertible:

$$B \! \! C = \max\{X \geq 0; D^*D \leq 2B - X\} = 2B - D^*D.$$ 

In particular, if $$B + C$$ is invertible, then

$$B \! \! C = 2B - D^*D = 2(B - B(B + C)^{-1}B) = 2B(B + C)^{-1}C.$$
4. Pedersen-Takesaki theorem. Finally we review a work of Pedersen and Takesaki [8] from the viewpoint of Riccati inequality; we add another equivalent condition to their theorem:

**Theorem 7.** Let $B$ and $C$ be positive operators and $B$ be nonsingular. Then the following statements are mutually equivalent:

1. Riccati equation $XBX = C$ has a positive solution.
2. $(B^\frac{1}{2}CB^\frac{1}{2})^\frac{1}{2} \leq kB$ for some $k > 0$.
3. There exists the minimum of $\{X \geq 0; C \leq XBX\}$.

(3') There exists the minimum of $\{X \geq 0; \left(\begin{array}{c} \frac{1}{2} \\ C^\frac{1}{2} \end{array}, \frac{1}{2} XBX \right) \geq 0\}$.

**Proof.** We first note that (3) and (3') are equivalent by Lemma 1.

Now we suppose (1), i.e., $X_0BX_0 = C$ for some $X_0 \geq 0$. If $X \geq 0$ satisfies $C \leq XBX$, then

$$(B^\frac{1}{2}X_0B^\frac{1}{2})^2 = B^\frac{1}{2}CB^\frac{1}{2} \leq (B^\frac{1}{2}XB^\frac{1}{2})^2$$

and so

$$B^\frac{1}{2}X_0B^\frac{1}{2} \leq B^\frac{1}{2}XB^\frac{1}{2}.$$  

Since $B$ is nonsingular, we have $X_0 \leq X$, namely (3) is proved.

Next we suppose (3). Since $C \leq XBX$ for some $X$, we have

$$B^\frac{1}{2}CB^\frac{1}{2} \leq (B^\frac{1}{2}XB^\frac{1}{2})^2$$

and so

$$(B^\frac{1}{2}CB^\frac{1}{2})^\frac{1}{2} \leq B^\frac{1}{2}XB^\frac{1}{2} \leq \|X\|B,$$

which shows (2).

The implication (2) $\rightarrow$ (1) has been shown by Pedersen-Takesaki [8] and Nakamoto [7], but we sketch it for convenience. By Douglas' majorization theorem [3], we have

$$(B^\frac{1}{2}CB^\frac{1}{2})^\frac{1}{2} = ZB^\frac{1}{2}$$

for some $Z$, so that $B^\frac{1}{2}Z^*ZB^\frac{1}{2} = B^\frac{1}{2}(Z^*ZB^*Z)B^\frac{1}{2}$. Since $B$ is nonsingular, $Z^*Z$ is a solution of $XBX = C$.

**Acknowledgement.** We remark that similar discussion to ours in Section 2 has been done by several authors, e.g. [9] and [10]. The authors would like to express their thanks to Professor Tomiyama and Professor Kosaki for thier kind suggestions on the above.
REFERENCES


Department of Information Sciences, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.

fuji@cc.osaka-kyoiku.ac.jp

Department of Mathematics, Osaka Kyoiku University, Asahigaoka, Kashiwara, Osaka 582-8582, Japan.

mfuji@cc.osaka-kyoiku.ac.jp

Faculty of Engineering, Ibaraki University, Hitachi, Ibaraki 316-8511, Japan.

nakamoto@base.ibaraki.ac.jp