数理解析研究所講究録
1536巻 2007年 122-131

Large-time behavior of solutions for the compressible viscous fluid in a half space

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In this paper, it is considered a large-time behavior of solutions to the isentropic and compressible Navier–Stokes equations in a half space. Precisely, we obtain a convergence rate toward the stationary solution for the outflow problem. In §1, we consider the one dimensional half space problem. For the supersonic flow at spatial infinity, we obtain an algebraic or an exponential decay rate. Namely, if an initial perturbation decays with the algebraic or the exponential rate in the spatial asymptotic point, the solution converges to the corresponding stationary solution with the same rate in time as time tends to infinity. An algebraic convergence rate is also obtained for the transonic flow. In §2, we study the same problem in a two dimensional half space and obtain the algebraic and exponential convergence rate toward the planar stationary wave for the supersonic flow.

1. One dimensional half space problem

1.1. **Main results.** This section is devoted to consider an asymptotic behavior of a solution to the initial boundary value problem for the compressible Navier-Stokes equation in one dimensional half space $\mathbb{R}_+ := (0,\infty)$. We especially study a convergence rate toward a corresponding stationary solution for the problem in which fluid blows out through a boundary. An isentropic or isothermal model of the compressible viscous fluid is formulated in the Eulerian coordinate as

$$\rho_t + (\rho u)_x = 0, \tag{1.1a}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = \mu u_{xx}. \tag{1.1b}$$

In the equations (1.1), $x \in \mathbb{R}_+$ and $t > 0$ mean a space variable and a time variable, respectively. The unknown functions are a mass density $\rho(x,t)$ and a fluid velocity $u(x,t)$. A constant μ is called a viscosity coefficient. A pressure $p(\rho)$ is given by $p(\rho) = K\rho^\gamma$ where $K > 0$ and $\gamma \geq 1$ are constants. The initial condition is prescribed by

$$(\rho,u)(0,x) = (\rho_0,u_0)(x), \tag{1.2a}$$

$$\lim_{x\to\infty}(\rho_0,u_0)(x) = (\rho_+,u_+), \quad \inf_{x\in\mathbb{R}_+}\rho_0(x) > 0, \quad \rho_+ > 0, \tag{1.2b}$$

where ρ_+ and u_+ are constants. The main concern of the present paper is a phenomena in which the gas brows out from the boundary. This is called an outflow problem in [6]. Thus, we adopt a Dirichlet boundary condition

$$u(t,0) = u_b < 0. \tag{1.3}$$

Note that only one boundary condition (1.3) is necessary and sufficient for the wellposedness of this problem since the characteristic $u(t,x)$ of the hyperbolic equation (1.1a) is negative around the boundary $\{x=0\}$ due to the condition (1.3).

It is shown in the paper [4] that the solution to the problem (1.1), (1.2) and (1.3) converges to the corresponding stationary solution as time tends to infinity. Here we summarize the results in [4]. The stationary solution $(\tilde{\rho},\tilde{u})(x)$ is a solution to the system (1.1) independent of a time variable t, satisfying the same conditions (1.2b) and (1.3). Therefore, the stationary solution $(\tilde{\rho},\tilde{u})$ satisfies the system of equations

$$(\tilde{\rho}\tilde{u})_x = 0, \tag{1.4a}$$

$$(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho}))_x = \mu\tilde{u}_{xx} \tag{1.4b}$$

and the boundary and the spatial asymptotic conditions

$$\tilde{u}(0) = u_b, \quad \lim_{x\to\infty}(\tilde{\rho},\tilde{u})(x) = (\rho_+,u_+), \quad \inf_{x\in\mathbb{R}_+}\tilde{\rho}(x) > 0. \tag{1.5}$$

To summerize the solvability result for the problem (1.4) and (1.5), define c_+ and M_+ be a sound speed and a Mach number at the spatial asymptotic states, respectively. Then they are given by

$$c_+ := \sqrt{p'(\rho_+)} = \sqrt{\gamma K\rho_+^{\gamma-1}}, \quad M_+ := \frac{|u_+|}{c_+}.$$

Lemma 1.1 ([4]). *There exists a constant w_c such that the boundary value problem* (1.4) *and* (1.5) *has a unique smooth solution $(\tilde{\rho},\tilde{u})$ if and only if*

$$u_+ < 0,\ M_+ \geq 1 \text{ and } w_c u_+ > u_b. \tag{1.6}$$

If $M_+ > 1$, there exist positive constants λ and C such that the stationary solution satisfies the estimate

$$|\partial_x^k(\tilde{\rho}(x)-\rho_+,\tilde{u}(x)-u_+)| \leq C\delta_S e^{-\lambda x} \text{ for } k = 0,1,2,\cdots, \tag{1.7a}$$

where $\delta_S := |u_b - u_+|$. If $M_+ = 1$, the stationary solution satisfies

$$|\partial_x^k(\tilde{\rho}(x)-\rho_+,\tilde{u}(x)-u_+)| \leq C\frac{\delta_S^{k+1}}{(1+\delta_S x)^{k+1}} \text{ for } k = 0,1,2,\cdots, \tag{1.7b}$$

where C is a positive constant.

In Lemma 1.1, the constant w_c is determined as follows. For the case $M_+ > 1$, w_c is one root of the equation $K\rho_+^\gamma(w_c^{-\gamma}-1) + \rho_+u_+^2(w_c-1) = 0$ satisfying $w_c > 1$. For the case $M_+ = 1$, w_c is equal to 1.

The asymptotic stability of the stationary solution is proved by Kawashima, Nishibata and Zhu in [4]. Thus, the main purpose of the present section is to investigate the convergence rate of the solution (ρ,u) toward the the stationary solution $(\tilde{\rho},\tilde{u})$ under the assumption that the initial perturbation decays exponentially or algebraically fast in the spatial direction.

Theorem 1.2. *Suppose that the condition* (1.6) *hold. In addition, the initial data* (ρ_0, u_0) *is supposed to satisfy the condition*

$$(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \in H^1(\mathbb{R}_+), \quad (\rho_0, u_0) \in \mathscr{B}^{1+\sigma}(\mathbb{R}_+) \times \mathscr{B}^{2+\sigma}(\mathbb{R}_+)$$

for a certain constant $\sigma \in (0,1)$ *and the condition* $\|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_1 + \delta_S < \varepsilon_0$ *for a certain positive constant* ε_0.

(i) *Suppose that* $M_+ > 1$ *holds. If the initial data satisfies* $(1+x)^{\alpha/2}(\rho_0 - \tilde{\rho})$, $(1+x)^{\alpha/2}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$ *for a certain positive constant* α, *then the solution* (ρ, u) *to* (1.1), (1.2) *and* (1.3) *satisfies the decay estimate*

$$\|(\rho,u)(t) - (\tilde{\rho}, \tilde{u})\|_\infty \leq C(1+t)^{-\alpha/2}. \tag{1.8}$$

On the other hand, if the initial data satisfies $e^{(\zeta/2)x}(\rho_0 - \tilde{\rho})$, $e^{(\zeta/2)x}(u_0 - \tilde{u}) \in L^2(\mathbb{R}_+)$ *for a certain positive constant* ζ, *then there exists a positive constant* α *such that the solution* (ρ, u) *to* (1.1), (1.2) *and* (1.3) *satisfies*

$$\|(\rho,u)(t) - (\tilde{\rho}, \tilde{u})\|_\infty \leq Ce^{-\alpha t}. \tag{1.9}$$

(ii) *Suppose that* $M_+ = 1$ *holds. There exists a positive constant* ε_0 *such that if the initial data satisfies* $\|(1+x)^{\alpha/2}(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_1 < \varepsilon_0$ *for a certain constant* α *satisfying* $\alpha \in [2, \alpha^*)$, *where* α^* *is a constant defined by*

$$\alpha^*(\alpha^* - 2) = \frac{4}{\gamma + 1} \quad \textit{and} \quad \alpha^* > 0, \tag{1.10}$$

then the solution (ρ, u) *to* (1.1), (1.2) *and* (1.3) *satisfies*

$$\|(\rho,u)(t) - (\tilde{\rho}, \tilde{u})\|_\infty \leq C(1+t)^{-\alpha/4}. \tag{1.11}$$

Remark 1.3. We see that the convergence rate (1.11) for the transonic flow is not as fast as the supersonic flow. Moreover, we assume the condition $\alpha < \alpha^*$, which is necessary for the derivation of the weighted estimate (1.31). Also, this type of assumption is used in [7] for the analysis of the convergence rate toward the traveling wave for a scalar viscous conservation law. It is still open problem whether the assumption $\alpha < \alpha^*$ can be removed or not.

Related results. For the one dimensional half space problem to the compressible Navier-Stokes equation, Matsumura in [6] expects that the asymptotic states of the solutions are classified into more than twenty cases subject to the boundary condition and the spatial asymptotic data. Several problems in this classification have been already studied. For example, Matsumura and Nishihara in [8] consider the case when the asymptotic state becomes one of stationary solutions, rarefaction waves and superposition of them for the inflow problem. The research [4] by Kawashima, Nishibata and Zhu shows the asymptotic stability of the stationary solution for the outflow problem. Following [4], the present paper investigates the convergence rate toward the stationary solution for the outflow problem.

For the multi-dimensional half space problem, Kagei and Kawashima in [1] study the outflow problem and prove the asymptotic stability of a planar stationary wave. Recently, the authors have obtained the convergence rate for this problem. This result also will be published soon.

Notations in the present section. For a non-negative integer $l \geq 0$, $H^l(\mathbb{R}_+)$ denotes the l-th order Sobolev space over $\mathbb{R}_+$ in the L^2 sense with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. The norm $\|\cdot\|_\infty$ means the L^∞-norm over $\mathbb{R}_+$. For $\alpha \in (0,1)$, $\mathscr{B}^{k+\alpha}(\mathbb{R}_+)$ denotes the Hölder space of bounded functions over $\mathbb{R}_+$ which have the k-th order derivatives of Hölder continuity with exponent α. Its norm is $|\cdot|_{k+\alpha}$. For a domain $Q_T \subseteq [0,T] \times \mathbb{R}_+$, $\mathscr{B}^{\alpha,\beta}(Q_T)$ denotes the space of the Hölder continuous functions with the Hölder exponents α and β with respect to t and x, respectively. For integers k and l, $\mathscr{B}^{k+\alpha,l+\beta}(Q_T)$ denotes the space of the functions satisfying $\partial_t^i u, \partial_x^j u \in \mathscr{B}^{\alpha,\beta}(Q_T)$ for arbitrary integers $i \in [0,k]$ and $j \in [0,l]$. We abbreviate $\mathscr{B}^{k+\alpha,l+\beta}([0,T] \times \mathbb{R}_+)$ by $\mathscr{B}_T^{k+\alpha,l+\beta}$.

1.2. **A priori estimate.** In this subsection, we derive the a priori estimate of the solution in the H^1 Sobolev space. To this end, we define the perturbation (φ, ψ) from the stationary solution as

$$(\varphi, \psi)(t,x) = (\rho, u)(t,x) - (\tilde{\rho}, \tilde{u})(x). \tag{1.12}$$

Due to (1.1) and (1.4), we have the system of equations for (φ, ψ) as

$$\varphi_t + u\varphi_x + \rho\psi_x = -(\tilde{u}_x\varphi + \tilde{\rho}_x\psi), \tag{1.13a}$$

$$\rho\psi_t + \rho u\psi_x + p'(\rho)\varphi_x - \mu\psi_{xx} = -(\varphi\psi + \tilde{u}\varphi + \tilde{\rho}\psi)\tilde{u}_x - (p'(\rho) - p'(\tilde{\rho}))\tilde{\rho}_x. \tag{1.13b}$$

The initial and the boundary conditions to (1.13) are derived from (1.2a), (1.3) and (1.5) as

$$(\varphi, \psi)(0,x) = (\varphi_0, \psi_0)(x) := (\rho_0, u_0)(x) - (\tilde{\rho}, \tilde{u})(x), \tag{1.14}$$

$$\psi(t,0) = 0. \tag{1.15}$$

To obtain the weighted energy estimates, we use the norms $|\cdot|_{2,\omega}$, $\|\cdot\|_{\mathrm{a},\alpha}$ and $\|\cdot\|_{\mathrm{e},\alpha}$ defined by

$$|f|_{2,\omega} := \left\{ \int_0^\infty \omega(x) f(x)^2 \, dx \right\}^{1/2}, \quad \|f\|_{\mathrm{a},\alpha} := |f|_{2,(1+x)^\alpha}, \quad \|f\|_{\mathrm{e},\alpha} := |f|_{2,e^{\alpha x}}.$$

1.2.1. *Supersonic flow.* we first derive the weighted energy estimate of the solution for the case when $M_+ > 1$ holds. To summarize the a priori estimate, we use the following notations for a weight function $W(t,x) = \chi(t)\omega(x)$ until the end of this subsection:

$$N(t) := \sup_{0 \leq \tau \leq t} \|(\varphi, \psi)(\tau)\|_1, \tag{1.16}$$

$$M(t)^2 := \int_0^t \chi(\tau) \left(\|\varphi_x(\tau)\|^2 + \|\psi_x(\tau)\|_1^2 + \varphi(\tau,0)^2 \right) d\tau, \tag{1.17}$$

$$\begin{aligned} L(t)^2 := \int_0^t & \chi_t(\tau) \left(|(\varphi, \psi)(\tau)|_{2,\omega}^2 + \|(\varphi_x, \psi_x)(\tau)\|^2 \right) \\ & + \chi(\tau) \left(|\psi(\tau)|_{2,\omega_{xx}}^2 + |(\varphi, \psi)(\tau)|_{2,|\tilde{u}_x|\omega}^2 \right) d\tau. \end{aligned} \tag{1.18}$$

Proposition 1.4. *Suppose that $M_+ > 1$ holds. Let (φ, ψ) be a solution to* (1.13), (1.14) *and* (1.15) *satisfying $(\varphi, \psi) \in C([0,T]; H^1(\mathbb{R}_+))$ and $(\varphi, \psi) \in \mathscr{B}_T^{1+\sigma/2,1+\sigma} \times \mathscr{B}_T^{1+\sigma/2,2+\sigma}$ for a certain positive constant and T.*

(i) (*Algebraic decay*) *Suppose that* $(1+x)^{\alpha/2}(\varphi,\psi)\in C([0,T];L^2(\mathbb{R}_+))$ *holds for a certain positive constant* α. *Then there exist positive constants* ε_0 *and* C *such that if* $N(T)+\delta_S<\varepsilon_0$, *then the solution* (φ,ψ) *satisfies the estimate*

$$(1+t)^{\alpha+\varepsilon}\|(\varphi,\psi)(t)\|_1^2+\int_0^t(1+\tau)^{\alpha+\varepsilon}\left(\|\varphi_x(\tau)\|^2+\|\psi_x(\tau)\|_1^2+|(\varphi,\varphi_x)(\tau,0)|^2\right)d\tau \\ \leq C\left(\|(\varphi_0,\psi_0)\|_1^2+\|(\varphi_0,\psi_0)\|_{a,\alpha}^2\right)(1+t)^{\varepsilon} \quad (1.19)$$

for arbitrary $t\in[0,T]$ *and* $\varepsilon>0$.

(ii) (*Exponential decay*) *Suppose that* $e^{(\zeta/2)x}(\varphi,\psi)\in C([0,T];L^2(\mathbb{R}_+))$ *for a certain positive constant* ζ. *Then there exist positive constants* ε_0, C, $\beta\,(<\zeta)$ *and* α *satisfying* $\alpha\ll\beta$ *such that if* $N(T)+\delta_S<\varepsilon_0$, *then the solution* (φ,ψ) *satisfies*

$$e^{\alpha t}\left(\|(\varphi,\psi)(t)\|_1^2+\|(\varphi,\psi)(t)\|_{e,\beta}^2\right)+\int_0^t e^{\alpha\tau}\left(\|\varphi_x(\tau)\|^2+\|\psi_x(\tau)\|_1^2+|(\varphi,\varphi_x)(\tau,0)|^2\right)d\tau \\ +\int_0^t e^{\alpha\tau}\left(\|(\varphi,\psi)(\tau)\|_{e,\beta}^2+\|\psi_x(\tau)\|_{e,\beta}^2\right)d\tau\leq C\left(\|(\varphi_0,\psi_0)\|_1^2+\|(\varphi_0,\psi_0)\|_{e,\beta}^2\right). \quad (1.20)$$

To prove Proposition 1.4, we first derive the basic energy estimate. To this end, we define an energy form $\mathscr{E}$, as in [4], by

$$\mathscr{E}:=\frac{1}{2}\psi^2+K\tilde{\rho}^{\gamma-1}\omega\left(\frac{\tilde{\rho}}{\rho}\right),\quad \omega(s):=s-1-\int_1^s\eta^{-\gamma}d\eta. \quad (1.21)$$

Owing to Proposition 1.1, we see that the energy form $\mathscr{E}$ is equivalent to $|(\varphi,\psi)|^2$. Namely, there exist positive constants c and C such that

$$c(\varphi^2+\psi^2)\leq\mathscr{E}\leq C(\varphi^2+\psi^2). \quad (1.22)$$

We also have positive bounds of ρ as

$$0<c\leq\rho(t,x)\leq C. \quad (1.23)$$

Lemma 1.5. *Suppose that the same assumptions as in Proposition 1.4 hold. Then there exists a positive constant* ε_0 *such that if* $N(T)+\delta_S<\varepsilon_0$, *it holds that*

$$\chi(t)|(\varphi,\psi)(t)|_{2,\omega}^2+\int_0^t\chi(\tau)\left(|(\varphi,\psi)(\tau)|_{2,\omega_x}^2+|\psi_x(\tau)|_{2,\omega}^2+\varphi(\tau,0)^2\right)d\tau \\ \leq C|(\varphi_0,\psi_0)|_{2,\omega}^2+CL(t)^2. \quad (1.24)$$

Next, we obtain the estimate for the first order derivatives of the solution (φ,ψ). As the existence of the higher order derivatives of the solution is not supposed, we need to use the difference quotient for the rigorous derivation of the higher order estimates. Since the argument using the difference quotient is similar to that in the paper [4], we omit the details and proceed with the proof as if it verifies

$$(\varphi,\psi)\in C([0,T];H^2(\mathbb{R}_+)),\ \varphi_x\in L^2(0,T;H^1(\mathbb{R}_+)),\ \psi_x\in L^2(0,T;H^2(\mathbb{R}_+)).$$

Lemma 1.6. *There exists a positive constant* ε_0 *such that if* $N(T)+\delta_S<\varepsilon_0$, *then*

$$\chi(t)\|(\varphi_x,\psi_x)(t)\|^2+\int_0^t\chi(\tau)\left(\|\varphi_x(\tau)\|^2+\|\psi_{xx}(\tau)\|^2+\varphi_x(\tau,0)^2\right)d\tau \\ \leq C\left(\|(\varphi_{0x},\psi_{0x})\|^2+|(\varphi_0,\psi_0)|_{2,\omega}^2\right)+CL(t)^2+C(N(t)+\delta_S)M(t)^2. \quad (1.25)$$

Summing up the estimates (1.24) and (1.25), and taking $N(T)+\delta_S$ suitably small with the aid of the Poincaré type inequality

$$|\varphi(t,x)| \leq |\varphi(t,0)| + \sqrt{x}\|\varphi_x(t)\| \tag{1.26}$$

which is proved by the similar computation as in [3, 9], we get the estimates (1.19) and (1.20).

1.2.2. *Transonic flow.* This subsection is devoted to prove the algebraic decay estimate for the transonic case $M_+ = 1$ in Theorem 1.2. To state the a priori estimate of the solution precisely, we use the notations:

$$N_1(t) := \sup_{0\leq\tau\leq t} \|((1+x)^{\alpha/2}\varphi, (1+x)^{\alpha/2}\psi)(\tau)\|_1,$$

$$M_1(t)^2 := \int_0^t (1+\tau)^{\xi}\|(\varphi_x, \psi_x, \psi_{xx})(\tau)\|^2_{a,\beta}\,d\tau.$$

Proposition 1.7. *Suppose that $M_+ = 1$ holds. Let (φ,ψ) be a solution to* (1.13), (1.14) *and* (1.15) *satisfying $(1+x)^{\alpha/2}(\varphi,\psi) \in C([0,T]; H^1(\mathbb{R}_+))$ and $(\varphi,\psi) \in \mathscr{B}_T^{1+\sigma/2,1+\sigma} \times \mathscr{B}_T^{1+\sigma/2,2+\sigma}$ for certain positive constants T and $\alpha \in [2,\alpha^*)$, where α^* is defined in* (1.10). *Then there exist positive constants ε_0 and C such that if $N_1(T)+\delta_S < \varepsilon_0$, then the solution (φ,ψ) satisfies the estimate*

$$\begin{aligned}(1+t)^{\alpha/2+\varepsilon}\|(\varphi,\psi)\|_1^2 + \int_0^t (1+\tau)^{\alpha/2+\varepsilon}\left(\|\varphi_x\|^2 + \|\psi_x\|_1^2 + |(\varphi,\varphi_x)(\tau,0)|^2\right)d\tau \\ \leq C\|(\varphi_0,\psi_0,\varphi_{0x},\psi_{0x})\|^2_{a,\alpha}(1+t)^{\varepsilon}.\end{aligned} \tag{1.27}$$

In order to prove Proposition 1.7, we need estimates for $\tilde{u}$ and the Mach number $\tilde{M}$ on the stationary solution $(\tilde{\rho},\tilde{u})$ defined by

$$\tilde{M}(x) := \frac{|\tilde{u}(x)|}{\sqrt{p'(\tilde{\rho}(x))}}. \tag{1.28}$$

Lemma 1.8. *The stationary solution $\tilde{u}(x)$ and the Mach number $\tilde{M}(x)$ satisfy*

$$\tilde{u}_x(x) \geq A\left(\frac{u_+}{u_b}\right)^{\gamma+2}\frac{\delta_S^2}{(1+Bx)^2}, \quad A := \frac{(\gamma+1)\rho_+}{2\mu}, \quad B := \delta_S A, \tag{1.29}$$

$$\frac{\gamma+1}{2|u_+|}\frac{\delta_S}{1+Bx} - C\frac{\delta_S^2}{(1+Bx)^2} \leq \tilde{M}(x) - 1 \leq C\frac{\delta_S}{1+Bx}. \tag{1.30}$$

for $x \in (0,\infty)$.

By using Lemma 1.8, we obtain the weighted L^2 estimate of (φ,ψ).

Lemma 1.9. *There exists a positive constant ε_0 such that if $N_1(T)+\delta_S < \varepsilon_0$, then*

$$\begin{aligned}(1+t)^{\xi}\|(\varphi,\psi)\|^2_{a,\beta} + \int_0^t (1+\tau)^{\xi}\left(\varphi(\tau,0)^2 + \beta\delta_S^2\|(\varphi,\psi)\|^2_{a,\beta-2} + \|\psi_x\|^2_{a,\beta}\right)d\tau \\ \leq C\|(\varphi_0,\psi_0)\|^2_{a,\beta} + C\xi\int_0^t (1+\tau)^{\xi-1}\|(\varphi,\psi)\|^2_{a,\beta}\,d\tau + C\delta_S\int_0^t (1+\tau)^{\xi}\|\varphi_x\|^2\,d\tau\end{aligned} \tag{1.31}$$

for $\beta \in [0,\alpha]$ and $\xi \geq 0$.

Next, we obtain the weighted estimate of (φ_x, ψ_x).

Lemma 1.10. *There exists a positive constant ε_0 such that if $N_1(T) + \delta_S < \varepsilon_0$, then*

$$\begin{aligned}(1+t)^{\xi}\|(\varphi_x, \psi_x)\|_{a,\beta}^2 + \int_0^t (1+\tau)^{\xi}\left(\varphi_x(\tau,0)^2 + \|(\varphi_x, \psi_{xx})(\tau)\|_{a,\beta}^2\right) d\tau \\ \leq C\|(\varphi_0, \psi_0, \varphi_{0x}, \psi_{0x})\|_{a,\beta}^2 + C\xi \int_0^t (1+\tau)^{\xi-1}\|(\varphi, \psi, \varphi_x, \psi_x)(\tau)\|_{a,\beta}^2 d\tau \quad (1.32)\end{aligned}$$

for $\beta \in [0, \alpha]$ and $\xi \geq 0$.

By the same inductive argument as in deriving (1.19), we can prove Proposition 1.7 which immediately yields the decay estimate (1.11).

2. Two dimensional half space problem

2.1. **Main results.** In this section, we consider the compressible Navier–Stokes equation in the two dimensional half space $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}$,

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{2.1a}$$

$$\rho\{u_t + (u\cdot\nabla)u\} = \mu_1 \Delta u + (\mu_1 + \mu_2)\nabla(\mathrm{div} u) - \nabla p(\rho). \tag{2.1b}$$

In this equations, $(x,y) \in \mathbb{R}_+^2$ is a space variable. The unknown functions are ρ and $u = (u_1, u_2)$. The constants μ_1 and μ_2 are viscosity coefficients satisfying $\mu_1 > 0$ and $\mu_1 + \mu_2 > 0$. We put the initial condition

$$(\rho, u)(0,x,y) = (\rho_0, u_0)(x,y) \tag{2.2}$$

and the outflow boundary condition

$$u(t,0,y) = (u_b, 0), \tag{2.3}$$

where $u_b < 0$ is a constant. We also assume that the spatial asymptotic state in a normal direction of the initial data is a constant:

$$\lim_{x\to\infty} \rho_0(x,y) = \rho_+ > 0, \quad \lim_{x\to\infty} u_0(x,y) = (u_+, 0). \tag{2.4}$$

In the present section, we investigate a convergence rate toward the planar stationary wave under the assumption that the initial perturbation decays in the normal direction.

The planer stationary wave $(\tilde{\rho}(x), \tilde{u}(x))$ is a solution to (2.1) independent of y and t. Moreover, we also assume that $\tilde{u}$ is given by the form $\tilde{u} = (\tilde{u}_1, 0)$ and that $(\tilde{\rho}(x), \tilde{u}(x))$ satisfies the boundary condition (2.3) and the spatial asymptotic condition (2.4). Therefore, $(\tilde{\rho}, \tilde{u}_1)$ is given by the solution to the following boundary value problem:

$$(\tilde{\rho}\tilde{u}_1)_x = 0, \tag{2.5a}$$

$$(\tilde{\rho}\tilde{u}_1^2 + p(\tilde{\rho}))_x = \mu \tilde{u}_{1xx}, \tag{2.5b}$$

$$\tilde{u}_1(0) = u_b, \quad \lim_{x\to\infty}(\tilde{\rho}(x), \tilde{u}_1(x)) = (\rho_+, u_+), \quad \inf_{x\in\mathbb{R}_+} \tilde{\rho}(x) > 0, \tag{2.6}$$

where $\mu > 0$ is a constant defined by $\mu := 2\mu_1 + \mu_2$. Since the problem (2.5) and (2.6) has a same form to the problem (1.4) and (1.5), we can apply the solvability lemma 1.1 to the problem (2.5) and (2.6). Thus, under the condition (1.6), there exists a unique solution $(\tilde{\rho}, \tilde{u}_1)$. Moreover, $(\tilde{\rho}, \tilde{u})$ satisfies the estimate (1.7a) for the case $M_+ > 1$.

For the multi-dimensional half space problem, Kagei and Kawashima prove an asymptotic stability of the planar stationary wave under the smallness assumptions on the initial perturbation and the shock strength δ_S. The main purpose of the present section is to obtain a convergence rate of solutions toward the planar stationary wave by assuming that the initial perturbation decays in the normal direction.

Theorem 2.1. *Suppose that the conditions $M_+ > 1$, (1.6) and $\|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_{H^2} + \delta_S < \varepsilon_0$ hold for a certain positive constant ε_0.*
(i) *If the initial data satisfies $(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \in L^2_\alpha(\mathbb{R}^2_+)$ for a certain constant $\alpha \geq 0$, then the solution (ρ, u) to the initial boundary value problem* (2.1), (2.2) *and* (2.3) *satisfies the estimate*

$$\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\|_{L^\infty} \leq C(1+t)^{-\alpha/2-1/4}. \tag{2.7}$$

(ii) *If the initial data satisfies $(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \in L^{2,\zeta}(\mathbb{R}^2_+)$ for a certain positive constant ζ, then there exists a certain positive constant α such that the solution (ρ, u) to the initial boundary value problem* (2.1), (2.2) *and* (2.3) *satisfies the estimate*

$$\|(\rho, u)(t) - (\tilde{\rho}, \tilde{u})\|_{L^\infty} \leq Ce^{-\alpha t}. \tag{2.8}$$

Notations in the present section. For a constant $\alpha \in \mathbb{R}$, the space $L^2_\alpha(\mathbb{R}^2_+)$ denotes the algebraic weighted L^2 space in the normal direction defined by $L^2_\alpha(\mathbb{R}^2_+) := \{u \in L^2_{\text{loc}}(\mathbb{R}^2_+) \ ; \ |u|_\alpha < \infty\}$ equiped with the norm

$$|u|_\alpha := \|u\|_{L^2_\alpha} := \Big(\iint_{\mathbb{R}^2_+} (1+x)^\alpha |u(x,y)|^2 \, dx\,dy\Big)^{1/2}.$$

The space $L^{2,\alpha}(\mathbb{R}^2_+)$ denotes the exponential weighted L^2 space in the normal direction defined by $L^{2,\alpha}(\mathbb{R}^2_+) := \{u \in L^2_{\text{loc}}(\mathbb{R}^2_+) \ ; \ \|u\|_{L^{2,\alpha}} < \infty\}$ equiped with the norm

$$\|u\|_{L^{2,\alpha}} := \Big(\iint_{\mathbb{R}^2_+} e^{\alpha x} |u(x,y)|^2 \, dx\,dy\Big)^{1/2}.$$

2.2. **A priori estimates.** To prove Theorem 2.1, we obtain the a priori estimates of the perturbation in H^2 and weighted L^2 spaces. To this end, we employ the perturbation (φ, ψ) by

$$(\varphi, \psi)(t,x,y) := (\rho, u)(t,x,y) - (\tilde{\rho}, \tilde{u})(x).$$

Owing to equations (2.1) and (2.5), the perturbation (φ, ψ) satisfies the system of equations

$$\varphi_t + u \cdot \nabla\varphi + \rho \operatorname{div}\psi = f, \tag{2.9a}$$

$$\rho\{\psi_t + (u\cdot\nabla)\psi\} - L\psi + p'(\rho)\nabla\varphi = g, \tag{2.9b}$$

where

$$\begin{aligned} L\psi &:= \mu_1 \Delta\psi + (\mu_1 + \mu_2)\nabla(\operatorname{div}\psi), \\ f &:= -\operatorname{div}\tilde{u}\varphi - \nabla\tilde{\rho}\cdot\psi, \\ g &:= -\rho(\psi\cdot\nabla)\tilde{u} - \varphi(\tilde{u}\cdot\nabla)\tilde{u} - (p'(\rho) - p'(\tilde{\rho}))\nabla\tilde{\rho}. \end{aligned}$$

The initial and the boundary condition for (φ, ψ) are prescribed by

$$(\varphi, \psi)(0,x,y) = (\varphi_0, \psi_0)(x,y) := (\rho_0, u_0)(x,y) - (\tilde{\rho}, \tilde{u})(x), \tag{2.10}$$

$$\psi(t,0,y) = 0. \tag{2.11}$$

To summerize the a priori estimate for (φ, ψ), we introduce the following notations for $\ell = 0, 1$:

$$N_\ell(t) := \sup_{0\le\tau\le t} E_\ell(\tau), \quad E_\ell(t) := \Big(\sum_{j=0}^{\ell} (1+t)^j |||\partial_y^j \Phi(t)|||_{2-j}^2\Big)^{1/2},$$

$$D_\ell(t) := \Big\{\sum_{j=0}^{\ell} (1+t)^j \Big(\sum_{i=1}^{2-j} |[\partial_y^j \varphi(t)]|_i^2 + \sum_{i=1}^{3-j} |[\partial_y^j \psi(t)]|_i^2 + \sum_{i=0}^{1-j} \|\nabla^i \partial_y^j \varphi(t)|_{x=0}\|_{L^2(\mathbb{R})}^2\Big)\Big\}^{1/2},$$

where $\Phi := (\varphi, \psi)$ and $\Phi_0 := (\varphi_0, \psi_0)$. We also define $|||\cdot|||_m$ and $|[\cdot]|_m$ by

$$|||u|||_m := \Big(\sum_{i=0}^{m} |[u]|_i^2\Big)^{1/2}, \quad |[u]|_m := \Big(\sum_{k=0}^{[m/2]} \|\nabla^{m-2k} \partial_t^k u\|^2\Big)^{1/2},$$

where $[x]$ denotes the greatest integer which does not exceed x.

Proposition 2.2. *Suppose that $M_+ > 1$ holds. Let (φ, ψ) be a solution to* (2.9), (2.10) *and* (2.11) *satisfying $(\varphi, \psi) \in C([0,T]\,;\, H^2(\mathbb{R}_+))$ for a certain positive constant T.*
(i) (*Algebraic decay*) *Suppose that $(\varphi, \psi) \in C([0,T]\,;\, L^2_\alpha(\mathbb{R}^2_+))$ holds for a certain constant $\alpha \ge 0$. Then there exist positive constants ε_0 and C such that if $N_1(t) + \delta_S < \varepsilon_0$, then the solution $\Phi = (\varphi, \psi)$ satisfies, for arbitrary $t \in [0,T]$, $\lambda \in [0, \alpha]$ and $\varepsilon > 0$,*

$$(1+t)^{\lambda+\varepsilon} |\Phi(t)|_{\alpha-\lambda}^2 + \int_0^t (1+\tau)^{\lambda+\varepsilon} \{(\alpha-\lambda)|\Phi(\tau)|_{\alpha-\lambda-1}^2 + |\nabla \psi(\tau)|_{\alpha-\lambda}^2\}\, d\tau$$
$$+ (1+t)^{\lambda+\varepsilon} E_1(t)^2 + \int_0^t (1+\tau)^{\lambda+\varepsilon} D_1(\tau)^2\, d\tau \le C(|\Phi_0|_\alpha^2 + \|\Phi_0\|_{H^2}^2)(1+t)^\varepsilon. \tag{2.12}$$

(ii) (*Exponential decay*) *Suppose that $(\varphi, \psi) \in C([0,T]\,;\, L^{2,\zeta}(\mathbb{R}^2_+))$ holds for a certain positive constant ζ. Then there exist positive constants ε_0, C, $\beta\,(\ll \zeta)$ and α satisfying $\alpha \ll \beta$ such that if $N_0(t) + \delta_S < \varepsilon_0$, then the solution $\Phi = (\varphi, \psi)$ satisfies, for arbitrary $t \in [0,T]$,*

$$e^{\alpha t}(\|\Phi(t)\|_{L^{2,\beta}}^2 + E_0(t)^2) + \int_0^t e^{\alpha\tau}(\|\Phi(\tau)\|_{L^{2,\beta}}^2 + \|\nabla\psi(\tau)\|_{L^{2,\beta}}^2 + D_0(\tau)^2)\, d\tau$$
$$\le C(\|\Phi_0\|_{L^{2,\beta}}^2 + \|\Phi_0\|_{H^2}^2). \tag{2.13}$$

Since the derivation of (2.13) is almost same to that of (2.12), we only show the key lemmas to obtain the a priori estimate (2.12).

First, we derive the time weighted L^2_β estimate for $\beta \in [0, \alpha]$. To do this, we introduce an energy form $\mathscr{E}$, in the same way to (1.21), by

$$\mathscr{E} := \frac{1}{2}|\psi|^2 + K\tilde{\rho}^{\gamma-1}\omega\Big(\frac{\tilde{\rho}}{\rho}\Big), \quad \omega(s) := s - 1 - \int_1^s \eta^{-\gamma}\, d\eta.$$

Using $\mathscr{E}$ and a weighted energy method, we obtain the estimate in L^2_β space.

Lemma 2.3. *There exists a positive constant ε_0 such that if $N_1(t)+\delta_S<\varepsilon_0$, then it holds that*

$$(1+t)^{\xi}|\Phi(t)|_{\beta}^{2}+\int_0^t(1+\tau)^{\xi}\left(\beta|\Phi(\tau)|_{\beta-1}^{2}+|\nabla\psi(\tau)|_{\beta}^{2}+\|\varphi(\tau)|_{x=0}\|_{L^2(\mathbb{R})}^{2}\right)d\tau$$
$$\leq C|\Phi_0|_{\beta}^{2}+C\xi\int_0^t(1+\tau)^{\xi-1}|\Phi(\tau)|_{\beta}^{2}\,d\tau+C\delta_S\int_0^t(1+\tau)^{\xi}\|\nabla\varphi(\tau)\|^2\,d\tau \quad (2.14)$$

for arbitrary constants $\beta\in[0,\alpha]$ and $\xi\geq 0$.

To complete the proof of derivation of (2.12), we need to obtain estimates for the higher order derivatives. Namely, we get a time weighted H^2 estimate.

Lemma 2.4. *There exists a positive constant ε_0 such that if $N_1(t)+\delta_S<\varepsilon_0$, then it holds that*

$$(1+t)^{\xi}E_{\ell}(t)^2+\int_0^t(1+\tau)^{\xi}D_{\ell}(\tau)^2\,d\tau$$
$$\leq C\|\Phi_0\|_{H^2}^{2}+C\sum_{j=0}^{\ell}(\xi+j)\int_0^t(1+\tau)^{\xi+j-1}|||\partial_y^j\Phi(\tau)|||_{2-j}^{2}\,d\tau \quad (2.15)$$

for $\ell=0,1$ and $\xi\geq 0$.

Summing up the estimates (2.14) and (2.15), and taking $N_1(t)+\delta_S$ suitably small with the aid of the induction for β and ξ, we obtain Proposition 2.2. Moreover, using the Sobolev inequality

$$\|\Phi\|_{L^\infty}\leq C\left(\|\Phi\|\|\Phi_x\|\|\Phi_y\|\|\Phi_{xy}\|\right)^{1/4},$$

we get the decay estimate in Theorem 2.1.

References

[1] Y. Kagei and S. Kawashima, *Stability of planar stationary solutions to the compressible Navier–Stokes equation on the half space*, to appear.

[2] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Commun. Math. Phys. **101** (1985), 97–127.

[3] S. Kawashima and S. Nishibata, *Stationary waves for the discrete Boltzmann equation in the half space with the reflective boundaries*, Commun. Math. Phys. **211** (2000), 183–206.

[4] S. Kawashima, S. Nishibata, and P. Zhu, *Asymptotic stability of the stationary solution to the compressible Navier-Stokes equations in the half space*, Commun. Math. Phys. **240** (2003), 483–500.

[5] T.-P. Liu, A. Matsumura, and K. Nishihara, *Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves*, SIAM J. Math. Anal. **29** (1998), 293–308.

[6] A. Matsumura, *Inflow and outflow problems in the half space for a one-dimensional isentropic model system for compressible viscous gas*, Proceedings of IMS Conference on Differential Equations from Mechanics, Hong Kong (1999).

[7] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Commun. Math. Phys. **165** (1994), 83–96.

[8] A. Matsumura and K. Nishihara, *Large-time behaviors of solutions to an inflow problem in the half space for a one-dimensional system of compressible gas*, Commun. Math. Phys. **222** (2001), 449–474.

[9] Y. Nikkuni and S. Kawashima, *Stability of stationary solutions to the half-space problem for the discrete Boltzmann equation with multiple collisions*, Kyushu J. Math. **54** (2000), 233–255.

[10] M. Nishikawa, *Convergence rate to the traveling wave for viscous conservation laws*, Funkcial. Ekvac. **41** (1998), 107–132.