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One Dimensional Viscous Conservation Laws with Discontinuous Initial Data

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1 Introduction

I summarize the results concerning the existence and the decay rates obtained for the viscous conservation laws with discontinuous initial data. An interesting case is the hyperbolic-parabolic case where the system contains both hyperbolic equations and parabolic equations.

What we consider is the initial value problem where there is a discontinuity in the initial data at $x=0$. It is well-known that the discontinuities in the initial data persist along $x=0$ for $t>0$. To study the decay rates we use the Laplace and inverse transforms to obtain the solution representations, which involve the Green functions. We first apply the Laplace transform to obtain a system of ordinary differential equations for $x>0$ and $x<0$. We use the Rankine-Hugoniot conditions as the boundary conditions to solve the system of ordinary differential equations. Then, the inverse Laplace transform is applied to obtain the solution representations, from which we obtain the existence and the decay rates of solutions.

There are numerous results for the viscous conservation laws with smooth initial data. On the other hand, there are only a few results with discontinuous initial data. Hoff [1, 2, 3, 4, 5] studied the compressible Navier-Stokes equations. Hoff and Khodja [6] and Pego [17] discussed the nonmonotone case. Using the Green's functions to study the existence, stability, or decay rates of solutions is widely practiced. Kawashima [7, 8] studied the large time behavior of hyperbolic-parabolic equations. Nishihara [15] studied the wave equations with damping. Nishihara, Wang, and Yang [16] studied the L^p-convergence rate for the p-system with damping. The decay rates of smooth solutions to traveling wave solutions or diffusion waves for hyperbolic-parabolic conservation laws have been discussed in various literature. Liu and Zeng [10, 11] and Zeng [18] discussed the decay rates of solutions to the diffusion waves. Liu [9] also studied the decay rates of solutions to the traveling wave solutions. The solutions are represented through the Fourier transform. Zumbrun and Howard [20, 21] and Mascia and Zumbrun [12, 13, 14] studied the stability of shock profiles with decay rates using the Laplace transform.

This paper consists of four sections. In Section 2 we describe the problem and the solution representations are obtained in Section 3. The existence and decay results are stated in Section 4. Remarks about the results and some future problems are given in Section 5.

2 Description of the problem

The system we consider is given by

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - f(v)_x &= u_{xx}, \end{aligned} \tag{2.1}$$

where f, v, and u are the stress, strain, and velocity, respectively. We assume that f is a smooth function of v. f can be monotone or nonmonotone. In the nonmonotone case the graph of f is given in Fig. 1.1.

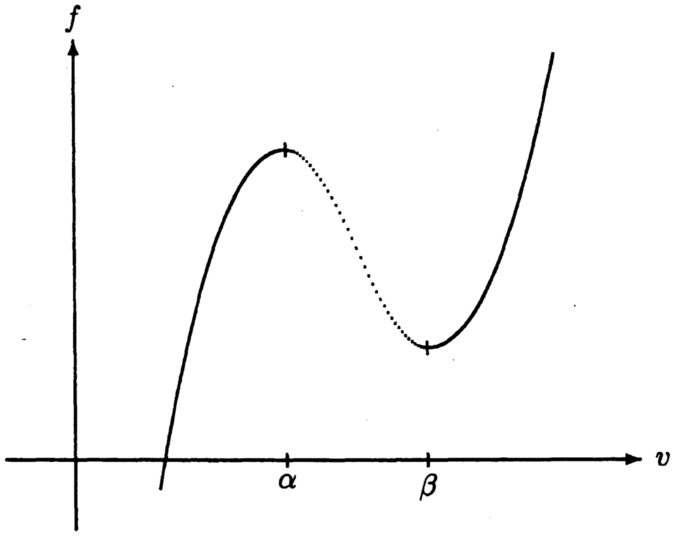

Figure 1.1: Graph of a nonmonotone stress strain relation.

The initial data are given by

$$(v,u)(x,0) = (v_0,u_0)(x) = \begin{cases} (v_l,u_l)(x), & x < 0, \\ (v_r,u_r)(x), & x > 0, \end{cases} \tag{2.2}$$

where there is a discontinuity at $x = 0$. We assume that the initial data satisfy that

$$C_0 = \sup_{1 \le p \le \infty} \|v_0 - \bar{v}, u_0 - \bar{u}\|_{L^p} + \sup_{1 \le p \le p_0} \left\| \frac{\partial v_0}{\partial x}, \frac{\partial u_0}{\partial x} \right\|_{L^p} \tag{2.3}$$

is finite, where p_0 is a constant larger than or equal to two, $\underline{L}^p$ is the piecewise L^p norm in space given by

$$\|v\|_{\underline{L}^p} = (\int_{x<0} v(x,t)^p dx)^{1/p} + (\int_{x>0} v(x,t)^p dx)^{1/p} = \|v\|_{L^p_-} + \|v\|_{L^p_+},$$

and $(\bar{v}, \bar{u})$ are piecewise constant states such that $\bar{v}$ is the piecewise constant strain given by

$$\bar{v} = \begin{cases} \bar{v}_-, & x < 0, \\ \bar{v}_+, & x > 0 \end{cases} \tag{2.4}$$

and $\bar{u}$ is a constant velocity. In the nonmonotone case, we choose $\bar{v}_- \in (0, \alpha]$ and $\bar{v}_+ \in [\beta, \infty)$ so that $f(\bar{v}_-) = f(\bar{v}_+)$ is satisfied. In the hyperbolic case, we need to assume $\bar{v}_- = \bar{v}_+$. One important difference is that in the case of monotone f, the strength of discontinuity decays exponentially while in the case of nonmonotone f, it does not and it is not small by any means. It is well known that the discontinuity in v persists for $t > 0$ along $x = 0$ whether we deal with nonmonotone f or monotone f ($f' > 0$) while u becomes continuous for $t > 0$. The Rankine-Hugoniot condition along $x = 0$ is given by

$$\begin{aligned} u(0_+, t) &= u(0_-, t), \\ -f(v(0_+, t)) + f(v(0_-, t)) &= u_x(0_+, t) - u_x(0_-, t). \end{aligned} \tag{2.5}$$

3 Solution representations

Substituting $(v_\pm, u_\pm) = (w_\pm + \bar{v}_\pm, z_\pm + \bar{u})$ in (2.1) and applying the Laplace transform, we obtain the systems of ordinary differential equations for $x > 0$ and $x < 0$. Then, solving the ordinary differential equations with the Rankine-Hugoniot condition (2.5) as the boundary conditions at $x = 0$ and applying the inverse transforms, we obtain the following solution representations.

$$\begin{aligned} & w_+(x,t) \\ = \; & f'(\bar{v}_-) \int_{-\infty}^0 T_{30}^{2-}(x,\eta,t) w(\eta,0) d\eta - \int_{-\infty}^0 T_{20}^{2-}(x,\eta,t) z(\eta,0) d\eta \\ & + \int_0^t \int_{-\infty}^0 T_{31}^{2-}(x,\eta,t-s) \tilde{g}_2(\eta,s) d\eta ds \\ & + f'(\bar{v}_+) \int_0^\infty R_{30}^+(x,\eta,t) w(\eta,0) d\eta + \int_0^\infty R_{20}^+(x,\eta,t) z(\eta,0) d\eta \\ & + \int_0^t \int_0^\infty R_{31}^+(x,\eta,t-s) \tilde{g}_2(\eta,s) d\eta ds \\ & + e^{-f'(\bar{v}_+)t} w_0(x) - \int_0^t e^{-f'(\bar{v}_+)(t-s)} \tilde{g}_2(x,s) ds \\ & + f'(\bar{v}_+) \int_0^x P_{30}^+(x-\eta,t) w(\eta,0) d\eta - \int_0^x P_{20}^+(x-\eta,t) z(\eta,0) d\eta \\ & + \int_0^t \int_0^x P_{31}^+(x-\eta,t-s) \tilde{g}_2(\eta,s) d\eta ds \end{aligned}$$

$$+f'(\bar{v}_+)\int_x^\infty P^+_{30}(\eta-x,t)w(\eta,0)d\eta+\int_x^\infty P^+_{20}(\eta-x,t)z(\eta,0)d\eta$$
$$+\int_0^t\int_x^\infty P^+_{31}(\eta-x,t-s)\tilde{g}_2(\eta,s)d\eta ds,$$

$$\begin{aligned}
& z_+(x,t) \\
= \; & -f'(\bar{v}_-)\int_{-\infty}^0 T^{1-}_{20}(x,\eta,t)w(\eta,0)d\eta+\int_{-\infty}^0 T^{1-}_{10}(x,\eta,t)z(\eta,0)d\eta \\
& -\int_0^t\int_{-\infty}^0 T^{1-}_{21}(x,\eta,t-s)\tilde{g}_2(\eta,s)d\eta ds \\
& -f'(\bar{v}_+)\int_0^\infty R^+_{20}(x+\eta,t)w(\eta,0)d\eta-\int_0^\infty R^+_{10}(x+\eta,t)z(\eta,0)d\eta \\
& -\int_0^t\int_0^\infty R^+_{21}(x+\eta,t-s)\tilde{g}_2(\eta,s)d\eta ds \\
& -f'(\bar{v}_+)\int_0^x P^+_{20}(x-\eta,t)w(\eta,0)d\eta+\int_0^x P^+_{10}(x-\eta,t)z(\eta,0)d\eta \\
& -\int_0^t\int_0^x P^+_{21}(x-\eta,t-s)\tilde{g}_2(\eta,s)d\eta ds \\
& +f'(\bar{v}_+)\int_x^\infty P^+_{20}(\eta-x,t)w(\eta,0)d\eta+\int_x^\infty P^+_{10}(\eta-x,t)z(\eta,0)d\eta \\
& +\int_0^t\int_x^\infty P^+_{21}(\eta-x,t-s)\tilde{g}_2(\eta,s)d\eta ds, \qquad (3.1)
\end{aligned}$$

where $\tilde{g}_2(w_\pm)=[f(\bar{v}_\pm+w_\pm)-f(\bar{v}_\pm)-f'(\bar{v}_\pm)w_\pm]$. The similar expressions can be obtained for w_- and z_-. Here, the subscripts $+$ and $-$ stand for $x>0$ and $x<0$, respectively. Differentiating w_+ and z_+ in x and performing the integration by parts in η, we obtain as the representation for the derivatives

$$\begin{aligned}
& w_{+x}(x,t) \\
= \; & f'(\bar{v}_+)(w_0(0_+)-w_0(0_-))T^{3-}_{30}(x,0,t)-(z(0_-,0)-z(0_+,0))T^{3-}_{20}(x,0,t) \\
& +\int_0^t T^{3-}_{31}(x,0,t-s)\{\tilde{g}_2(w(0_+,s))-\tilde{g}_2(w(0_-,s)\}ds \\
& + \text{similar terms as in } w_+(x,t),
\end{aligned}$$

$$\begin{aligned}
& z_{+x}(x,t) \\
= \; & (z(0_+,0)-z(0_-,0))T^{2-}_{10}(x,0,t)+f'(\bar{v}_+)(w(0_-,0)-w(0_+,0))T^{2-}_{20}(x,0,t) \\
& +\int_0^t T^{2-}_{21}(x,0,t-s)\{\tilde{g}_2(w(0_-,s))-\tilde{g}_2(w(0_+,s))\}ds \\
& + \text{similar terms as in } z_+(x,t). \qquad (3.2)
\end{aligned}$$

In the above expressions

$$T^{l\pm}_{mn}(x,\eta,t)=\frac{1}{2\pi i}\int_\Gamma \frac{\tilde{\mu}^l_\mp}{\tilde{\mu}^{l-1}_\pm(\tilde{\mu}_++\tilde{\mu}_-)}\tilde{\mu}^m_\pm\lambda^n e^{\lambda(t\pm\tilde{\mu}_\mp x\mp\tilde{\mu}_\pm\eta)}d\lambda,$$

$$R^{\pm}_{mn}(x,\eta,t) = \frac{1}{2\pi i}\int_\Gamma \frac{(\tilde{\mu}_\pm - \tilde{\mu}_\mp)}{2(\tilde{\mu}_+ + \tilde{\mu}_-)}\tilde{\mu}^m_\pm \lambda^n e^{\lambda\{t\mp\tilde{\mu}_\pm(x+\eta)\}}d\lambda,$$
$$P^{\pm}_{mn}(x-\eta,t) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{2}\tilde{\mu}^m_\pm \lambda^n e^{\lambda\{t-\tilde{\mu}_\pm(x-\eta)\}}d\lambda,$$

where $\tilde{\mu}_\pm = \frac{1}{\sqrt{\epsilon\lambda + f'(\bar{v}_\pm)}}$ and Γ is a path in the complex λ-plane. $T^{l\pm}_{mn}$ and $R^{\pm}_{mn}$ account for the transmission and reflection, respectively. $P^{\pm}_{mn}$ is basically the same as the usual Green's function.

4 Existence and decay estimates

After performing the integrations in λ for $T^{l\pm}_{mn}$, $R^{\pm}_{mn}$, and $P^{\pm}_{mn}$, we obtain the following estimates.

Lemma 4.1 *For $n = 0$ and $m = 1$ or $n = 1$, $T^{l\pm}_{mn}$, $R^{\pm}_{mn}$, and $P^{\pm}_{mn}$ satisfy the estimates*

$$\begin{aligned}
|T^{l\pm}_{mn}(x,\eta,t)| &= O(1)t^{-\frac{n+1}{2}}e^{-K\frac{\left(t-\frac{|x|}{\sqrt{f'(\bar{v}_\mp)}}-\frac{|\eta|}{\sqrt{f'(\bar{v}_\pm)}}\right)^2}{t}} + O(1)t^{-\frac{n+1}{2}}e^{-K\frac{(x-\eta)^2}{t}} + \mathcal{R},\\
|R^{\pm}_{mn}(x+\eta,t)| &= O(1)t^{-\frac{n+1}{2}}e^{-K\frac{\left(t-\frac{|x+\eta|}{\sqrt{f'(\bar{v}_\pm)}}\right)^2}{t}} + O(1)t^{-\frac{n+1}{2}}e^{-K\frac{(x+\eta)^2}{t}} + \mathcal{R},\\
|P^{\pm}_{mn}(x-\eta,t)| &= O(1)t^{-\frac{n+1}{2}}e^{-K\frac{\left(t-\frac{|x-\eta|}{\sqrt{f'(\bar{v}_\pm)}}\right)^2}{t}} + O(1)t^{-\frac{n+1}{2}}e^{-K\frac{(x-\eta)^2}{t}} + \mathcal{R},
\end{aligned}$$

where K is a positive constant and $\mathcal{R}$ represents residual terms decaying faster than the first two terms.

For $T^{l\pm}_{mn}$, if $n = 0$ and $m = 2$, using the convolution theorem, we obtain

$$\begin{aligned}
|T^{l\pm}_{mn}(x,\eta,t)| &= \left|\frac{1}{2\pi i}\int_\Gamma \tilde{\mu}_\pm \frac{\tilde{\mu}^l_\mp}{\tilde{\mu}^{l-1}_\pm(\tilde{\mu}_+ + \tilde{\mu}_-)}\tilde{\mu}_\pm e^{\lambda t\pm\mu_\mp x\mp\mu_\pm\eta}d\lambda\right|\\
&\le O(1)\int_0^t (t-s)^{-\frac{1}{2}}e^{-f'(\bar{v}_+)(t-s)}|T^{l\pm}_{10}(x,\eta,s)|ds. \qquad (4.1)
\end{aligned}$$

Similarly, if $n = 0$ and $m = 3$,

$$\begin{aligned}
|T^{l\pm}_{mn}(x,\eta,t)| &= \left|\frac{1}{2\pi i}\int_\Gamma \tilde{\mu}^2_\pm \frac{\tilde{\mu}^l_\mp}{\tilde{\mu}^{l-1}_\pm(\tilde{\mu}_+ + \tilde{\mu}_-)}\tilde{\mu}_\pm e^{\lambda t\pm\mu_\mp x\mp\mu_\pm\eta}d\lambda\right|\\
&\le O(1)\int_0^t e^{-f'(\bar{v}_+)(t-s)}|T^{l\pm}_{10}(x,\eta,s)|ds. \qquad (4.2)
\end{aligned}$$

Similar estimates can be obtained for $R^{\pm}_{mn}$, and $P^{\pm}_{mn}$.

Using the solution representations obtained in the previous section, we construct an iteration scheme in the Banach space X with the norm given by

$$\begin{aligned}
\|(v,u)(\cdot,t)\|_X = \sup_{1\le p\le\infty, 1\le q\le p_0, 0\le t<\infty} &\{(t+1)^{\frac{1}{2p}-\frac{1}{2}}\|(v-\bar{v}, u-\bar{u})(\cdot,t)\|_{L^p}\\
&+(t+1)^{\frac{1}{2q}-\frac{1}{2}}\|(v,u)_x(\cdot,t)\|_{L^q}\}.
\end{aligned}$$

The iteration can be constructed by putting upper-subscripts $(k+1)$ to the variables on the left hand sides and (k) to those on the right hand sides in (3.1), (3.2), and etc. After estimating the terms in the solution representations with the help of Lemma 4.1, we obtain

Lemma 4.2 *Let*

$$\begin{aligned} N^{(k)}(t) &= \sup_{1\le p\le\infty, 1\le q\le p_0, 0\le\tau<t} \{ \left\| w^{(k)}, z^{(k)} \right\|_{L^p} (1+\tau)^{\frac{1}{2}-\frac{1}{2p}} \\ &+ \left\| w_x^{(k)}, z_{\pm x}^{(k)} - (z(0_+,0) - z(0_-,0)) T_{10}^{2\mp}(x,0,t) \right\|_{\underline{L}^q} (1+\tau)^{\frac{1}{2}-\frac{1}{2q}} \}. \end{aligned}$$

Then, the following inequality

$$N^{(k+1)}(t) \le C_1 C_0 + C_2 C_3 N^{(k)}(t)^2$$

holds, where C_0 is the constant defined in (2.3), C_1 and C_2 are positive constants from the Green's functions, and $C_3 = \max_{|w|\le 1} |f''(\bar{v}_+ + w)|$.

From the above lemma and the contraction mapping, we can easily obtain

Theorem 4.3 *There exists a constant $\bar{C}_0$ such that if the initial data satisfy $C_0 \le \bar{C}_0$, where C_0 is given in (2.3), the global solutions exist in X and satisfy the decay rates given by*

$$\begin{aligned} \| v - \bar{v}, u - \bar{u} \|_{L^p} &= O(1)(t+1)^{-\frac{1}{2}+\frac{1}{2p}}, \ 1 \le p \le \infty, \\ \left\| v_x, u_{\pm x} - (u_0(0_+) - u_0(0_-)) T_{10}^{2\mp}(x,0,t) \right\|_{\underline{L}^p} &= O(1)(t+1)^{-\frac{1}{2}+\frac{1}{2p}}, \ 1 \le p \le p_0, \end{aligned}$$

where $T_{10}^{2\mp}(x,0,t)$ are diffusion waves singular at $t=0$. Also, the Rankine-Hugoniot condition is satisfied across $x=0$.

5 Concluding remarks

In this concluding section we discuss some observations and the possible future work. The discontinuity in the initial data brings various interesting differences between the hyperbolic variable v and the parabolic variable u. They are summarized as follows.

(1) The first two terms on the right hand side of each expression in $w_{+x}(x,t)$ and $z_{+x}(x,t)$ are from the discontinuity in the initial data. They are diffusion waves. T_{10}^{2-} behaves like a heat kernel approaching a delta function as $t \downarrow 0$. On the other hand, T_{20}^{2-}, T_{20}^{3-}, and T_{30}^{3-} are not singular as $t \downarrow 0$. The parabolic variable has a singularity in the derivative which behaves like a heat kernel. This captures the way the parabolic variable becomes continuous for $t>0$. On the other hand the hyperbolic variable remains discontinuous. Therefore, there in no singular wave in the derivative.

(2) The main effect near $t=0$ is given by the exponentially decaying term $e^{-f'(\bar{v}_+)t} w_0(x)$ for the hyperbolic variable and the diffusive terms $\int_0^x P_{10}^+(x-\eta,t) z(\eta,0) d\eta + \int_x^\infty P_{10}^+(\eta - x,t) z(\eta,0) d\eta$ for the parabolic variable.

(3) m and n for the hyperbolic variable and the parabolic variable are different. This affects the behavior of these variables especially near $t = 0$.

The result obtained should be extended for the following prototype systems. First, in the Lagrangian coordinates, the viscosity terms are nonlinear as shown below.

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - f(v)_x &= (\varepsilon \frac{u_x}{v})_x, \end{aligned}$$

It is important to discuss the changes necessary to treat the nonlinear nature of the viscous term. Also, it is interesting to extend the result to the nonisothermal case with various viscosity terms. Another interesting problem is to extend to the initial data with different values at $x = \pm\infty$. This will include the Riemann problems for viscous conservation laws. The difficulty is the fact that the solutions will not approach piecewise constant states. We need to find the approximate solutions to which the solutions will approach.

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