数理解析研究所講究録
1536 巻 2007 年 87-96

Some computer assisted proofs on the bifurcation structure of solutions for the Rayleigh-Bénard problem

九州大学情報基盤センター
渡部 善隆 (Yoshitaka Watanabe)
Computing and Communications Center, Kyushu University

1 The Rayleigh-Bénard problem

Consider a plane horizontal layer (see Figure 1) of an incompressible viscous fluid heated from the bottom. At the lower boundary: $z=0$ the layer of fluid is maintained at temperature $T+\delta T$ and the temperature of the upper boundary ($z=h$) is T.

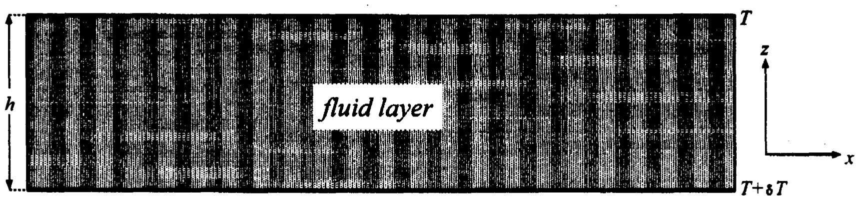

Figure 1. Model of fluid layer

As well known, under the vanishing assumption in y-direction, the two-dimensional (x-z) heat convection model can be described as the following Oberbeck-Boussinesq approximations [1, 3]:

$$
\begin{array}{rcl}
u_t + uu_x + wu_z &=& p_x + \mathcal{P}\Delta u, \\
w_t + uw_x + ww_z &=& p_z - \mathcal{P}\mathcal{R}\theta + \mathcal{P}\Delta w, \\
u_x + w_z &=& 0, \\
\theta_t + w + u\theta_x + w\theta_z &=& \Delta\theta.
\end{array}
\tag{1}
$$

Here, u and w are velocity in x and z, respectively, p, θ are pressure and temperature field reprensating deviation from the linear profile, $*_\xi{:=}\partial/\partial\xi(\xi=x,z,t)$, $\Delta := \partial^2/\partial x^2 + \partial^2/\partial z^2$, $\mathcal{R}$ is Rayleigh number and $\mathcal{P}$ is Prandtl number.

In previous results[8, 9], the authors considered the Rayleigh-Bénard problem (1) and proposed an approach to prove the exsistence of the steady-state solutions based on the infinite dimensional fixed-point theorem using Newton-like operator with the spectral approximation and the constructive error estimates. For the given Prandtl and Rayleigh numbers, several exact non-trivial solutions have been verified.

This paper will present a computer assisted proof of the existence for a symmetry-breaking bifurcation point which is an important information to clarify the global bifurcation structure.

2 Fixed-point formulation of problem

This section describes on a basic concept of our numerical verification method to prove the exsistence of the steady-state solutions. Since we only consider the the *steady-state solutions*, u_t, w_t and θ_t vanish in (1). And also assume that all fluid motion is confined to the rectangular region

$$\Omega := \{0 < x < 2\pi/a,\ 0 < z < \pi\}$$

for a given wave number $a > 0$.

Let us impose periodic boundary condition (period $2\pi/a$) in the horizontal direction, stress-free boundary conditions ($u_z = w = 0$) for the velocity field and Dirichlet boundary conditions ($\theta = 0$) for the temperature field on the surfaces $z = 0, \pi$, respectively. Furthermore, we assume the following evenness and oddness conditions [2]:

$$u(x,z) = -u(-x,z), \quad w(x,z) = w(-x,z), \quad \theta(x,z) = \theta(-x,z).$$

We use the stream function Ψ satisfying

$$u = -\Psi_z, \quad w = \Psi_x$$

so that $u_x + w_z = 0$. By some simple calculations in (1) with setting $\Theta := \sqrt{\mathcal{PR}}\theta$, we obtain

$$\begin{aligned} \mathcal{P}\Delta^2\Psi &= \sqrt{\mathcal{PR}}\,\Theta_x - \Psi_z\Delta\Psi_x + \Psi_x\Delta\Psi_z, \\ -\Delta\Theta &= -\sqrt{\mathcal{PR}}\,\Psi_x + \Psi_z\Theta_x - \Psi_x\Theta_z. \end{aligned} \tag{2}$$

From the boundary conditions, the functions Ψ and Θ can be assumed to have the following double Fourier series:

$$\Psi = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(amx)\sin(nz), \qquad \Theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cos(amx)\sin(nz). \tag{3}$$

We now define the following function spaces for integers $k \geq 0$:

$$X^k := \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(amx) \sin(nz) \mid A_{mn} \in \mathbf{R}, \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} ((am)^{2k} + n^{2k}) A_{mn}^2 < \infty \right\},$$

$$Y^k := \left\{ \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} B_{mn} \cos(amx) \sin(nz) \mid B_{mn} \in \mathbf{R}, \quad \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} ((am)^{2k} + n^{2k}) B_{mn}^2 < \infty \right\}.$$

In order to get the enclosure of the exact solutions for the problem (2), we need some appropriate finite dimensional subspaces. For $M_1, N_1, M_2 \geq 1$ and $N_2 \geq 0$, we set $N := (M_1, N_1, M_2, N_2)$ and define the finite dimensional approximate subspaces by

$$\begin{aligned} S_N^{(1)} &:= \left\{ \Psi_N = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} \hat{A}_{mn} \sin(amx) \sin(nz) \mid \hat{A}_{mn} \in \mathbf{R} \right\}, \\ S_N^{(2)} &:= \left\{ \Theta_N = \sum_{m=0}^{M_2} \sum_{n=1}^{N_2} \hat{B}_{mn} \cos(amx) \sin(nz) \mid \hat{B}_{mn} \in \mathbf{R} \right\}, \\ S_N &:= S_N^{(1)} \times S_N^{(2)}. \end{aligned}$$

Let denote an approximate solution of (2) by $\hat{u}_N := (\hat{\Psi}_N, \hat{\Theta}_N) \in S_N$. We now set

$$\begin{aligned} f_1(\Psi, \Theta) &:= \sqrt{\mathcal{PR}}\, \Theta_x - \Psi_z \Delta \Psi_x + \Psi_x \Delta \Psi_z, \\ f_2(\Psi, \Theta) &:= -\sqrt{\mathcal{PR}}\, \Psi_x + \Psi_z \Theta_x - \Psi_x \Theta_z, \end{aligned}$$

where

$$\Psi = \hat{\Psi}_N + w^{(1)}, \qquad \Theta = \hat{\Theta}_N + w^{(2)}.$$

Then the problem (2) is rewritten as the following system of equations with respect to $(w^{(1)}, w^{(2)}) \in X^4 \times Y^2$ satisfying

$$\begin{aligned} \mathcal{P}\Delta^2 w^{(1)} &= f_1(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) - \mathcal{P}\Delta^2 \hat{\Psi}_N, \\ -\Delta w^{(2)} &= f_2(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) + \Delta \hat{\Theta}_N, \end{aligned} \tag{4}$$

which is so-called a residual form. Setting

$$\begin{aligned} w &= (w^{(1)}, w^{(2)}), \\ h_1(w) &= f_1(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) - \mathcal{P}\Delta^2 \hat{\Psi}_N, \\ h_2(w) &= f_2(\hat{\Psi}_N + w^{(1)}, \hat{\Theta}_N + w^{(2)}) + \Delta \hat{\Theta}_N, \\ h(w) &= (h_1(w), h_2(w)), \end{aligned}$$

by virtue of the Sobolev embbeding theorem and the definition of f_1 and f_2, h is a bounded continuous map from $X^3 \times Y^1$ to $X^0 \times Y^0$. Moreover, it is easily shown that for all $(g_1, g_2) \in X^0 \times Y^0$, the linear problem:

$$\begin{aligned} \Delta^2 \bar{\Psi} &= g_1, \\ -\Delta \bar{\Theta} &= g_2 \end{aligned} \tag{5}$$

has a unique solution $(\bar{\Psi}, \bar{\Theta}) \in X^4 \times Y^2$. We denote this mapping by $\bar{\Psi} = (\Delta^2)^{-1} g_1$ and $\bar{\Theta} = (-\Delta)^{-1} g_2$, then the operator:

$$\mathcal{K} := (\mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}) : \ X^0 \times Y^0 \to X^3 \times Y^1$$

is a compact map because of the compactness of the imbedding $X^4 \hookrightarrow X^3$ and $Y^2 \hookrightarrow Y^1$ and the boundedness of $(\Delta^2)^{-1} : X^0 \to X^4$, $(-\Delta)^{-1} : Y^0 \to Y^2$. Thus, (4) is rewritten by a fixed-point equation:

$$w = Fw \tag{6}$$

for the compact operator $F := \mathcal{K} \circ h$ on $X^3 \times Y^1$. Therefore, by the Schauder fixed-point theorem, if we find a nonempty, closed, bounded and convex set $W \subset X^3 \times Y^1$, satisfying

$$FW \subset W \tag{7}$$

then there exists a solution of (6) in W. The set W in (7) is referred as *a candidate set* of solutions.

The candidate set W is usually constructed by computer as a direct sum of the finite dimensional subset

$$W_N \subset S_N^{(1)} \times S_N^{(1)} \subset X^3 \times Y^1$$

and its orthogonal complement $W_N^{\perp}$ in the space $X^3 \times Y^1$. By using an appropriate projection $P_N : X^3 \times Y^1 \longrightarrow X_N^3 \times Y_N^1$, the decomposed form $P_N FW \subset W_N$ and $(I - P_N)FW \subset W_N^{\perp}$ are numerically verified instead of (7), which implies the verification of a sufficient condition for (7). Here, the former condition is verified by direct computation in the computer, and the latter criterion can be proved by the effective use of the constructive error estimates for the projection. In the present case, the projection P_N can be taken as the finite trunction operator of solutions to (5)[8]. Furthermore, in general, a kind of Newton-type formulation is utilized so that the concerning operator has the retraction property in a neighborhood of the solution(see, e.g., [6, 8] for details).

By using the Newton-like procedure [6], we succeeded to verify various kinds of bifurcating solutions as shown in Figure 2. Here, $\mathcal{R}_C$ implies the critical Rayleigh number which equals 6.75. The vertical axis stands for the absolute value of the coefficient of the approximate solution for Θ. And each dot in Figure 2 means that the existence of an exact solution corresponding to the point was numerically verified.

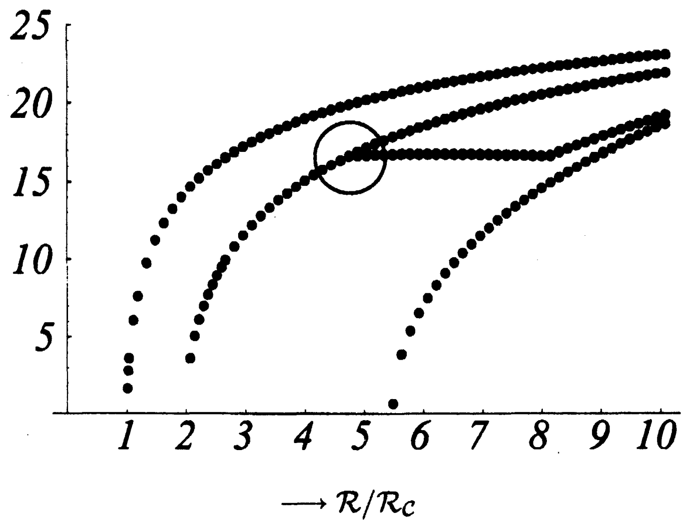

Figure 2. Verified bifurcating solutions

3 Existence of bifurcation point

From the observation of Figure 2, particularly the behaviour around the part enclosed by the circle, we expected that there should exists a secondary bifurcation point. Namely, near "the bifurcation-like point" we found the following two different kinds of approximate solutions. For approximate solutions of the form

$$\Psi_N = \sum_{m=1}^{M_1} \sum_{n=1}^{N_1} A_{mn} \sin(amx) \sin(nz), \qquad \Theta_N = \sum_{m=0}^{M_2} \sum_{n=1}^{N_2} B_{mn} \cos(amx) \sin(nz),$$

we have following two solutions satisfying

$$A_{mn} = B_{mn} = 0, \quad m = 1, 3, 5, 7, \ldots \text{with } \mathcal{R} = 32$$

and

$$A_{mn} \neq 0, \quad B_{mn} \neq 0, \quad m = 1, 3, 5, 7, \ldots \text{with } \mathcal{R} = 33.$$

These approximate results strongly suggest that there should exist a symmetry-breaking bifurcation point between $32 \leq \mathcal{R} \leq 33$.

In order to obtain the enclosure of the bifurcation point, we set and an operator $S : X^0 \times Y^0 \longrightarrow X^0 \times Y^0$ by

$$\begin{aligned} Su = S(\Psi, \Theta) &= (S_1\Psi, S_2\Theta) \\ &= (\Psi(x + \pi/a, z), \Theta(x + \pi/a, z)), \end{aligned}$$

then using this "symmetric" operator S, X^k and Y^k can be decomposed as

$$X^k = X_s^k \oplus X_a^k, \qquad Y^k = Y_s^k \oplus Y_a^k,$$

where

$$X_s^k = \{\Psi \in X^k \mid S_1\Psi = \Psi\}, \qquad X_a^k = \{\Psi \in X^k \mid S_1\Psi = -\Psi\},$$
$$Y_s^k = \{\Theta \in Y^k \mid S_2\Theta = \Theta\}, \qquad Y_a^k = \{\Theta \in Y^k \mid S_2\Theta = -\Theta\}.$$

Also, setting

$$Z := X^3 \times Y^1, \qquad G := I - F,$$

$SGw = GSw$ holds and Z is decomposed as

$$Z = Z_s \oplus Z_a,$$

where $Z_s = \{w \in Z; Sw = w\}$ and $Z_a = \{w \in Z; Sw = -w\}$.

Next, considering $\mathcal{R}$ as a variable, let $\mathcal{G}$ be a map on $Z_s \times Z_a \times \mathbf{R}$ defined by

$$\mathcal{G}(w, v, \mathcal{R}) := \begin{pmatrix} G(w, \mathcal{R}) \\ D_wG[w, \mathcal{R}]v \\ \mathcal{L}(v) - 1. \end{pmatrix}. \tag{8}$$

Here $\mathcal{L}$ is an appropriate functional on Z_a. Then the following Lemma presented by Kawanago [4] can be applied.

Lemma 1 $(w_0, \mathcal{R}_0) \in Z_s \times \mathbf{R}$ is a symmetry-breaking bifurcation point of $G(w, \mathcal{R}) = 0$ if

1. Extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ has an isolated solution $(w_0, v_0, \mathcal{R}_0) \in Z_s \times Z_a \times \mathbf{R}$.
2. $D_uG[w_0, \mathcal{R}_0]|_{X_s^4 \times Y_s^2} : X_s^4 \times Y_s^2 \longrightarrow X_s^0 \times Y_s^0$ is bijective.

First, we tried to prove that the extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ has an isolated solution $(w_0, v_0, \mathcal{R}_0) \in Z_s \times Z_a \times \mathbf{R}$ by a computer-assisted approach using our verification principle in the section 2. The equation $\mathcal{G}(w, v, \mathcal{R}) = 0$ means the problem to find out

$$[\Psi, \Theta, \Xi, \Upsilon, \mathcal{R}] \in Z_s \times Z_a \times \mathbf{R}$$

satisfying

$$\begin{array}{rcl} \mathcal{P}\Delta^2\Psi - \sqrt{\mathcal{PR}}\,\Theta_x - J(\Psi,\Delta\Psi) &=& 0,\\ -\Delta\Theta + \sqrt{\mathcal{PR}}\,\Psi_x + J(\Psi,\Theta) &=& 0,\\ \mathcal{P}\Delta^2\Xi - \sqrt{\mathcal{PR}}\,\Upsilon_x - J(\Psi,\Delta\Xi) - J(\Xi,\Delta\Psi) &=& 0,\\ -\Delta\Upsilon + \sqrt{\mathcal{PR}}\,\Xi_x + J(\Psi,\Upsilon) + J(\Xi,\Theta) &=& 0,\\ \mathcal{L}(v) - 1 &=& 0. \end{array} \tag{9}$$

Setting the functional $\mathcal{L}$ by

$$\mathcal{L}(v) = (\,\Xi,\Xi_0\,)_{L^2} + (\,\Upsilon,\Upsilon_0\,)_{L^2}, \quad \Xi_0 := \frac{2a}{\pi^2}\sin(ax)\sin(z), \quad \Upsilon_0 := \frac{2a}{\pi^2}\cos(ax)\sin(z),$$

denoting a fixed approximate solution of (9) by $[\Psi_N,\Theta_N,\Xi_N,\Upsilon_N,R_N]$ and using the residual variables defined by

$$\Psi = \Psi_N + u^{(1)}, \quad \Theta = \Theta_N + u^{(2)}, \quad \Xi = \Xi_N + u^{(3)}, \quad \Upsilon = \Upsilon_N + u^{(4)}, \quad \mathcal{R} = \mathcal{R}_N + u^{(5)},$$

equation (9) can be rewritten as

$$\begin{array}{rcl} \mathcal{P}\Delta^2 u^{(1)} &=& \sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Theta_N + u^{(2)})_x + J(\Psi_N + u^{(1)}, \Delta(\Psi_N + u^{(1)})) - \mathcal{P}\Delta^2\Psi_N,\\ -\Delta u^{(2)} &=& -\sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Psi_N + u^{(1)})_x - J(\Psi_N + u^{(1)}, \Theta_N + u^{(2)}) + \Delta\Theta_N,\\ \mathcal{P}\Delta^2 u^{(3)} &=& \sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Upsilon_N + u^{(4)})_x + J(\Psi_N + u^{(1)}, \Delta(\Xi_N + u^{(3)}))\\ && +J(\Xi_N + u^{(3)}, \Delta(\Psi_N + u^{(1)})) - \mathcal{P}\Delta^2\Xi_N,\\ -\Delta u^{(4)} &=& -\sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Xi_N + u^{(3)})_x - J(\Psi_N + u^{(1)}, \Upsilon_N + u^{(4)})\\ && -J(\Xi_N + u^{(3)}, \Theta_N + u^{(2)}) + \Delta\Upsilon_N,\\ u^{(5)} &=& -(\,\Xi_N + u^{(3)}, \Xi_0\,)_{L^2} - (\,\Upsilon_N + u^{(4)}, \Upsilon_0\,)_{L^2} + 1 + u^{(5)}. \end{array} \tag{10}$$

We now define the nonlinear function of $\mathbf{u} := (u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}, u^{(5)})$ by

$$\begin{aligned} h_1(\mathbf{u}) &:= \sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Theta_N + u^{(2)})_x + J(\Psi_N + u^{(1)}, \Delta(\Psi_N + u^{(1)})) - \mathcal{P}\Delta^2\Psi_N,\\ h_2(\mathbf{u}) &:= -\sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Psi_N + u^{(1)})_x - J(\Psi_N + u^{(1)}, \Theta_N + u^{(2)}) + \Delta\Theta_N,\\ h_3(\mathbf{u}) &:= \sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Upsilon_N + u^{(4)})_x + J(\Psi_N + u^{(1)}, \Delta(\Xi_N + u^{(3)}))\\ &\qquad + J(\Xi_N + u^{(3)}, \Delta(\Psi_N + u^{(1)})) - \mathcal{P}\Delta^2\Xi_N,\\ h_4(\mathbf{u}) &:= -\sqrt{\mathcal{P}(R_N + u^{(5)})}\,(\Xi_N + u^{(3)})_x - J(\Psi_N + u^{(1)}, \Upsilon_N + u^{(4)})\\ &\qquad - J(\Xi_N + u^{(3)}, \Theta_N + u^{(2)}) + \Delta\Upsilon_N,\\ h_5(\mathbf{u}) &:= -(\,\Xi_N + u^{(3)}, \Xi_0\,)_{L^2} - (\,\Upsilon_N + u^{(4)}, \Upsilon_0\,)_{L^2} + \beta + u^{(5)}, \end{aligned}$$

and set $h(\mathbf{u}) := (h_1(\mathbf{u}), h_2(\mathbf{u}), h_3(\mathbf{u}), h_4(\mathbf{u}), h_5(\mathbf{u}))$.

Furthermore, defining

$$\begin{aligned}\mathcal{K} &:= (\mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}, \mathcal{P}^{-1}(\Delta^2)^{-1}, (-\Delta)^{-1}, I),\\ \mathcal{H} &:= \mathcal{K}h,\end{aligned}$$

the equation (10) can be represented as the fixed equation:

$$\mathbf{u} = \mathcal{H}\mathbf{u}$$

on $Z_s \times Z_a \times \mathbf{R}$. We applied a numerical verification method based on Banach's fixed-point theorem[7, 10] incorporated with the interval arithmetic on Sun ONE Studio 7 Compiler Collection Fortran 95 on FUJITSU PRIMEPOWER850 (CPU: SPARC64V 1.35GHz, OS: Solaris8), and proved that there exists an isolated solution of $\mathcal{G}(w_0, v_0, \mathcal{R}_0) = 0$. Here

$$\mathcal{R}_0 \in 32.04265510708193 + [-2.910, 2.910] \times 10^{-10}.$$

Figure 5-8 shows the shape of approximate solutions for extended system $\mathcal{G}(w, v, \mathcal{R}) = 0$ such that

$$w \approx (\Psi_N, \Theta_N), \qquad v \approx (\Xi_N, \Upsilon_N).$$

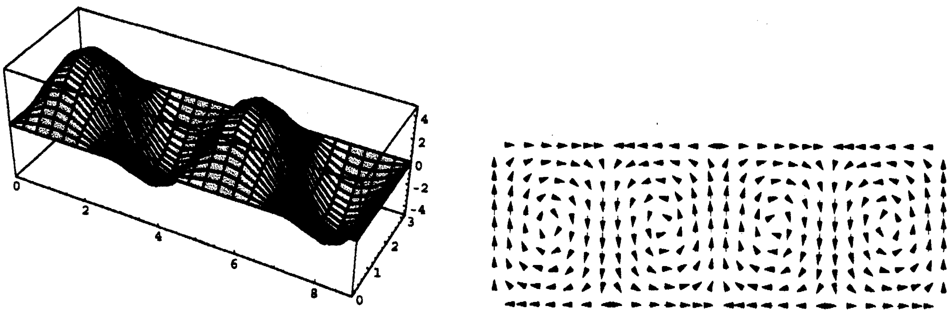

Figure 5. Shape of Ψ_N and velocity field $[-(\Psi_N)_z, (\Psi_N)_x)]^T$

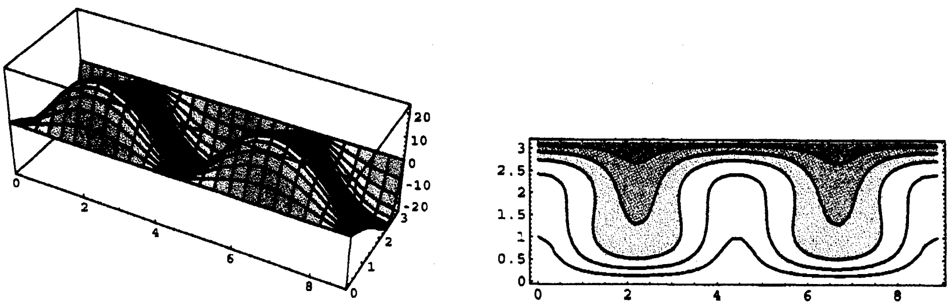

Figure 6. Shape of Θ_N and contour field of
$\delta T(1 - z/\pi - \Theta_N/\sqrt{\mathcal{R}\mathcal{P}}\pi) + T(T = 0, \delta T = 5)$

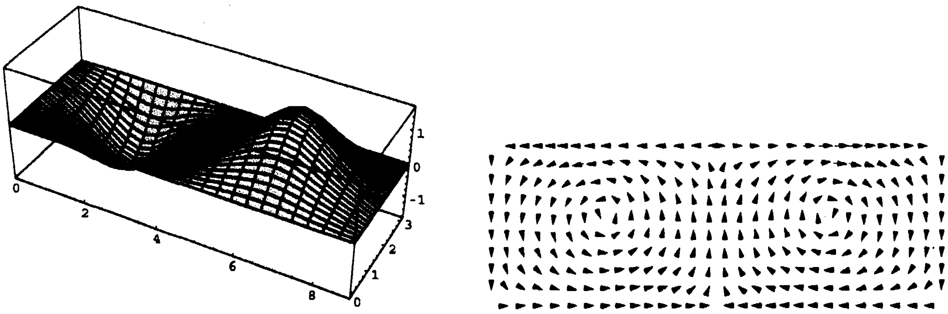

Figure 7. Shape of Ξ_N and velocity field $[-(\Xi_N)z, (\Xi_x)]^T$

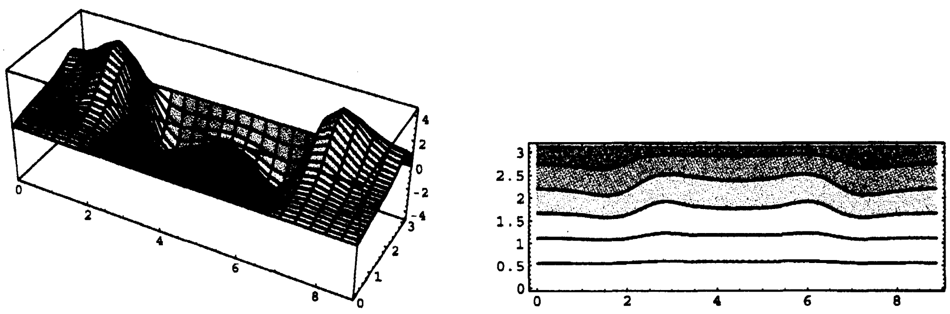

Figure 8. Shape of Υ_N and contour field of
$\delta T(1 - z/\pi - \Upsilon_N/\sqrt{\mathcal{R}\mathcal{P}}\pi) + T(T = 0, \delta T = 5)$

Therefore, from the bifurcation theorem, it implies that there exists an actual bifurcation point in this interval if $D_w G[w_0, \mathcal{R}_0]$ is invertible on Z_s.
On the other hand, from the Fredholm alternative, the invertibility of $D_w G[w_0, \mathcal{R}_0]$ is assured when

$$\begin{aligned} \mathcal{P}\Delta^2\Xi - \sqrt{\mathcal{P}\mathcal{R}_0}\,\Upsilon_x - J(\Psi_0, \Delta\Xi) - J(\Xi, \Delta\Psi_0) &= 0, \\ -\Delta\Upsilon + \sqrt{\mathcal{P}\mathcal{R}_0}\,\Xi_x + J(\Psi_0, \Upsilon) + J(\Xi, \Theta_0) &= 0 \end{aligned}$$

has a unique trivial solution $[\Xi, \Upsilon] = [0, 0]$ in Z_s, where $w_0 = [\Psi_0, \Theta_0]$. We actually succeeded in the verification of the invertibility by using a method similar to that an eigenvalue excluding technique [5]. Thus, it was numerically proved that there exists a symmetry-breaking bifurcation point in the above interval.

References

[1] Chandrasekhar, S. (1961), *Hydrodynamic and Hydromagnetic Stability,* Oxford University Press.

[2] Curry, J.H. (1979), "Bounded solutions of finite dimensional approximations to the Boussinesq equations", SIAM J. Math. Anal., Vol.10, pp.71–79.

[3] Getling, A.V. (1998), *Rayleigh-Bénard Convection: structures and dynamics*, Advanced series in nonlinear dynamics Vol.11, World Scientific.

[4] Kawanago, T. (2004), "A symmetry-breaking bifurcation theorem and some related theorems applicable to maps having unbounded derivatives", Japan Journal of Industrial and Applied Mathematics, Vol.21, pp.57–74.

[5] Nagatou, K., Yamamoto, N. and Nakao, M.T. (1999), "An approach to the numerical verification of solutions for nonlinear elliptic problems with local uniqueness", Numerical Functional Analysis and Optimization, Vol.20, pp.543–565.

[6] Nakao, M.T. (2001), "Numerical verification methods for solutions of ordinary and partial differential equations", Numerical Functional Analysis and Optimization, Vol.22, No.3&4, pp.321–356.

[7] Nakao, M.T., Hashimoto, K. and Watanabe, Y. (2005), "A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems", Computing, Vol.75, pp.1–14.

[8] Watanabe, Y., Yamamoto, N., Nakao, M.T. and Nishida, T. (2004), "A numerical verification of nontrivial solutions for the heat convection problem", Journal of Mathematical Fluid Mechanics, Vol.6, No.1, pp.1–20.

[9] Nakao, M.T., Watanabe, Y., Yamamoto, N., Nishida, T. (2003), "Some computer assisted proofs for solutions of the heat convection problems", Reliable Computing, Vol.9, No.5, pp.359–372.

[10] Yamamoto, N. (1998), "A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed point theorem", SIAM J. Numer. Anal., Vol.35, pp.2004–2013.