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# Microfunctions and a transfer operator for complex dynamical systems.

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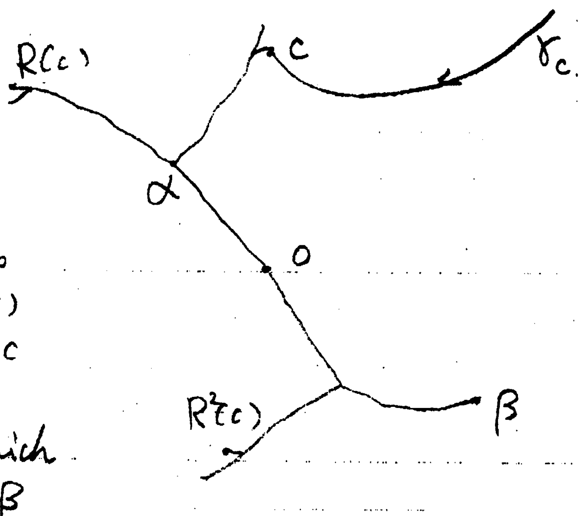
## §1. Functions with regular singularities

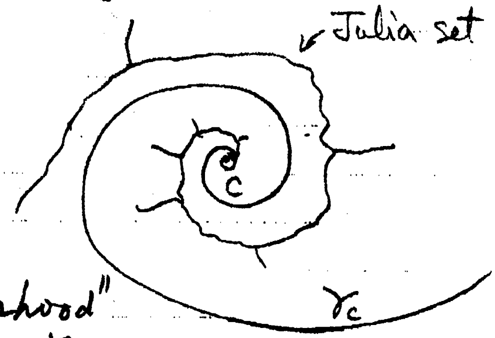
Let us begin with the following situation. Let $R(z) = z^2 + c$ be a quadratic polynomial on the Riemann sphere. We assume that the complex dynamical system defined by this quadratic polynomial is postcritically finite, i.e., the forward orbit $\{f^n(0) \mid n = 1, 2, \cdots\}$ of the critical point $0$ of $R(z)$ is a finite set. For the sake of simplicity, we denote by $F$ the Fatou set of $R(z)$ and by $J$ the Julia set $\hat{\mathbb{C}} \setminus F$. In order to illustrate the situation, we consider especially the case $c = i$. Then the critical point is $0$ and its forward orbit is $\{0, i, i-1, -i\}$ and $R(c)$ and $R^2(c)$ form a periodic cycle of period 2. $R(z)$ has two fixed points, which we denote by $\alpha$ and $\beta$ as in the picture.

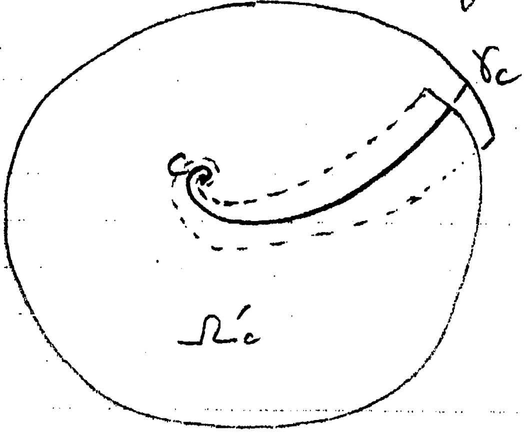



These two fixed points are so-called the $\alpha$-fixed point and the $\beta$-fixed point.

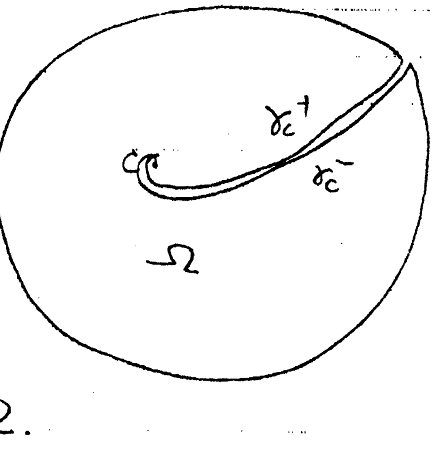
As $R(z)$ is postcritically finite, there exists an external ray, say $\gamma_c$, landing at the critical value $c$. We give an orientation to this curve as $\infty \to c$. Note that this external ray is spiraling near $c$.

Let $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$ be a domain in the complex plane. This domain $\Omega_c$ has smooth boundaries along the external ray $\gamma_c$. We consider an abstract "neighborhood" $\Omega'_c$ of $\Omega_c$ doubly sheeted near $\gamma_c$.




The domain $\Omega_c$ has two smooth curves on the boundary. We add two curves $\gamma_c^+$ and $\gamma_c^-$ to this domain to each sides of the external ray $\gamma_c$, and we denote this set by $\overline{\Omega}_c$. $\Omega'_c$ is an openset containing $\overline{\Omega}_c$. For domain $\Omega$, we denote by $\mathcal{O}(\Omega)$ the set of holomorphic functions on $\Omega$.




Let $f : \Omega_c \to \mathbb{C}$ be a holomorphic function on $\Omega_c$ which can be extended holomorphically to some "neighborhood" $\Omega'_c$ of $\overline{\Omega}_c$. Such a function $f$ is said to be equivalent to $g : \Omega_c \to \mathbb{C}$ which is a holomorphic function on $\Omega_c$ and extendable to some "neighborhood" $\Omega''_c$ of $\overline{\Omega}_c$ holomorphically, if there exists a "neighborhood" $\Omega'''_c$ of $\overline{\Omega}_c$ such that $f$ and $g$ coincides on $\Omega'''_c$. This equivalence relation defines a concept of germ. Note that $f : \Omega_c \to \mathbb{C}$ itself gives a representative of its germ since analytic continuation is unique if it exists. We call such function $f$ a (general) premicrofunction along $\gamma_c$. A (general) microfunction at $c$ along $\gamma_c$ is defined by an equivalence class of germs of (general) pre-microfunctions $f: \Omega_c \to \mathbb{C}$ at $c$ modulo germs of holomorphic functions at $c$. More precisely, (general) pre-microfunctions $f: \Omega_c \to \mathbb{C}$ and $g: \Omega_c \to \mathbb{C}$ defines the same microfunction at $c$ if there exists an open neighborhood $U$ of $c$ in the complex plane $\mathbb{C}$ and a holomorphic function $h: U \to \mathbb{C}$ such that $f(z) - g(z) = h(z)$ holds for $z \in U \cap \Omega_c$. The above definition of (general) microfunction is so general that the singularities of such functions at $c$ are too much complicated. So, we restrict our singularities to "regular singularities" defined as follows.

<u>Definition</u> 1.1. Pre-microfunction $f: \Omega_c \to \mathbb{C}$ is said to have a regular singularity at $c$ if there exist positive numbers $\varepsilon$ and $k$ such that inequality

$$|f(z)| < k\,|z-c|^{-1+\varepsilon}$$

holds near $c$.

<u>Definition</u> 1.2. Pre-microfunction $f: \Omega_c \to \mathbb{C}$ is said to have a regular singularity at $\infty$ if there exist positive numbers $\varepsilon$ and $k$ such that inequality

$$|f(z)| < k|z|^{-\varepsilon}$$

holds near the infinity.

We denote by $\mathcal{M}_c$ the set of pre-microfunctions along $\gamma_c$ with regular singularities both at $c$ and $\infty$. More precisely we denote $\mathcal{M}_{\gamma_c}$ instead of $\mathcal{M}_c$ when there are are more than one external rays landing at $c$. The space of equivalence classes of germs of pre-microfunctions with regular singularities at $c$ modulo the space of germs of holomorphic functions $\mathcal{O}(c)$ at $c$, will be denoted by $\tilde{\mathcal{M}}_c$.

Let $P(R)$ denote the postcritical set. For each point

$p \in \mathcal{P}(R)$, the space of pre-microfunctions along its external rays with regular singularities at both $p$ and $\infty$ is defined in a similar manner and will be denoted by $\mathcal{M}_p$ for simplicity and by $\mathcal{M}_{\gamma_p}$ when it is necessary to indicate the external ray.

For $p \in J$ with multiple external rays, say $\gamma_1, \ldots, \gamma_r$, landing at $p$, we define the space $\mathcal{M}_p$ by the direct sum

$$\mathcal{M}_p = \bigoplus_{k=1}^{r} \mathcal{M}_{\gamma_k}.$$

Where the sum is taken as a formal sum, since each component belongs to different spaces. However each element of $\mathcal{M}_p$ defines a function holomorphic in the intersection of the domains of definitions and the decomposition of a holomorphic function

$$f : \mathbb{C} \setminus \left( \left( \bigcup_{k=1}^{r} \gamma_k \right) \cup \{p\} \right) \longrightarrow \mathbb{C}$$

defined by an element of $\mathcal{M}_p$ into components $f_k$ in $\mathcal{M}_{\gamma_k}$,

$$f_k : \mathbb{C} \setminus (\gamma_k \cup \{p\}) \longrightarrow \mathbb{C}$$

is unique since we are considering the pre-microfunctions with regular singularities at the infinity. We denote

$$\mathcal{M}_+ = \bigoplus_{p \in \mathcal{P}(R)} \mathcal{M}_p$$

$$\mathcal{M}_0 = \mathcal{M}_{\gamma_0^+} \oplus \mathcal{M}_{\gamma_0^-}$$

$$\mathcal{M}_- = \bigoplus_{k=1}^{\infty} \bigoplus_{p \in R^{-k}(0)} \mathcal{M}_p$$

and

$$\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_0 \oplus \mathcal{M}_-.$$

Here, the origin $0$ is the critical point of our quadratic map $R(z)$ and there are two external rays landing at $0$, which are pre-images of the external ray $\gamma_c$. $\gamma_0^+$ and $\gamma_0^-$ denotes the external angles $\frac{1}{12}$ and $\frac{7}{12}$ respectively. Note that $\gamma_c$ is the external angle $\frac{1}{6}$, since it is mapped to period two cycle of external rays with angles $\frac{1}{3}$ and $\frac{2}{3}$. Here, the infinite direct sum is only in a formal sense.

## §2 Difference operator and an exact sequence

Let $\mathcal{O}(\gamma_c)$ denote the space of holomorphic functions in a neighbourhood of the external ray $\gamma_c$. An element of $\mathcal{O}(\gamma_c)$ is represented by a continuous function $f : \gamma_c \to \mathbb{C}$ which can be extended to some neighborhood of $\gamma_c$ holomorphically. The space of holomorphic functions along the external ray $\gamma_c$ with regular singularities at both $c$ and the infinity is defined by the following.

$$\mathcal{O}_0(\gamma_c) = \left\{ f \in \mathcal{O}(\gamma_c) \;\middle|\; \begin{array}{l} \exists \varepsilon > 0,\ \exists k > 0,\ \exists \text{ nbd of } \gamma_c \text{ s.t.} \\ |f(z)| < k|z-c|^{-1+\varepsilon} \quad \text{near } c \\ \text{and } |f(z)| < k|z|^{-\varepsilon} \quad \text{near } \infty \end{array} \right\}$$

Now, we define a difference operator along an external ray.

<u>Definition</u> 2.1 Difference operator $\Delta_c : \mathcal{M}_c \to \mathcal{O}_0(\gamma_c)$ is defined by the difference of boundary values along $\gamma_c$

$$\Delta_c \varphi(z) = \varphi(z) - \varphi((z-c)e^{-2\pi i} + c).$$

Here $z \in \gamma_c$ is consider as a point in the boundary of $\Omega_c$ of the clockwise side and $(z-c)e^{-2\pi i} + c$ represents the same point but considered as a point in the boundary of $\Omega_c$ of the counterclockwise side.

For each $p \in J$ and its external ray $\gamma_p$, difference operator $\Delta_p : \mathcal{M}_p \to \mathcal{O}_0(\gamma_p)$ is defined in a similar way. We denote $\Delta_{\gamma_p}$ instead of $\Delta_p$ if there are more than two external rays landing at $p$ and we need to indicate it.

<u>Remark</u> Difference operator can be defined for functions holomorphic along $\gamma_c$ in a doubly sheeted domains. The domain of definition of such a holomorphic function need not be connected.

Let us fix a double sheeted neighborhood $\Omega_c'$ of our domain $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$, and let $S_c$ denote the neighborhood of $\gamma_c$ where $\Omega_c'$ is double sheeted.

<u>Theorem</u> 2.2 The following sequence is exact.

$$0 \to \mathcal{O}(\mathbb{C}\setminus\{c\}) \hookrightarrow \mathcal{O}(\Omega_c') \xrightarrow{\Delta_c} \mathcal{O}(S_c) \longrightarrow 0.$$

<u>Proof</u> We gave an orientation to the external ray $\gamma_c$ defining an order to the points in $\gamma_c$ so that $\infty < p < c$. Take points $r_j, s_j \in \gamma_c$ for $j \in \mathbb{Z}$ ordered along $\gamma_c$ as

$$\infty < \cdots < r_j < s_{j-1} < r_{j+1} < s_j < \cdots < c$$

and

$$\lim_{k\to-\infty} s_k = \lim_{k\to-\infty} r_k = \infty,$$

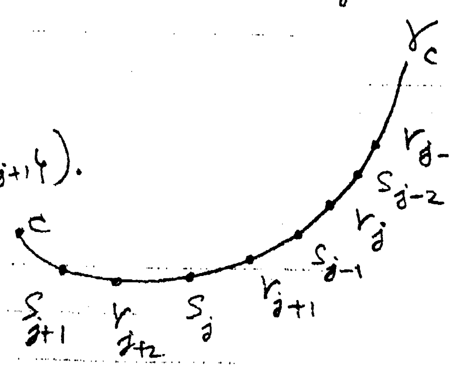
$$\lim_{k\to\infty} s_k = \lim_{k\to\infty} r_k = c.$$

Then open arcs $\widehat{r_j s_j}$ $(j \in \mathbb{Z})$ form an open covering of the external ray $\gamma_c$. The space $\mathcal{O}(\mathbb{C}\setminus\{c\})$ of holomorphic functions on $\mathbb{C}\setminus\{c\}$ can be injectively embedded in the space of holomorphic functions on $\Omega_c'$ and the values of such a function on the two sheets of $\Omega_c'$ coincide on the overlapped sector $S_c$, hence the difference of these values vanishes. So, we need only to prove the ontoness of the difference operator $\Delta_c$. For $\varphi \in \mathcal{O}(S_c)$, we want to construct a holomorphic function in $\mathcal{O}(\Omega_c')$. Note that such a function is not unique since the kernel of $\Delta_c$ contains $\mathcal{O}(\mathbb{C}\setminus\{c\})$.

Let

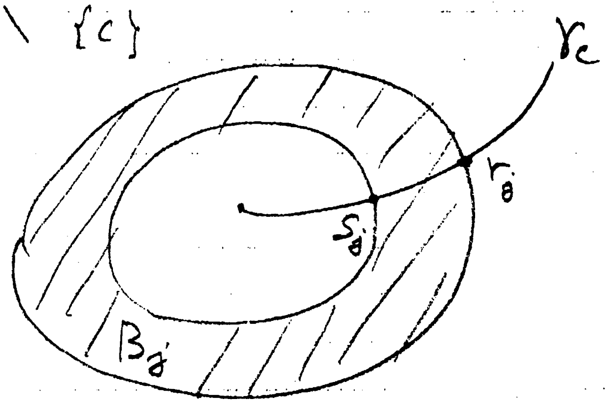
$$F_j(z) = \frac{1}{2\pi i}\int_{r_{j-1}}^{s_{j+1}} \frac{\varphi(\tau)}{\tau - z}\, d\tau$$

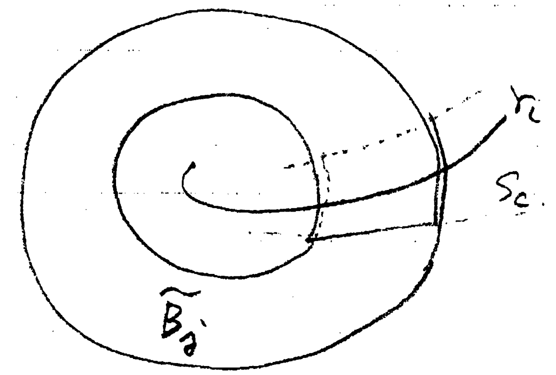
for $j \in \mathbb{Z}$. Such integration is called a Cousin's integral along the arc $\widehat{r_{j-1} s_{j+1}}$. Note that this arc includes the arc $\widehat{r_j s_j}$ in its interior. The function $F_j(z)$ is holomorphic in $\mathbb{C}\setminus(\widehat{r_{j-1} s_{j+1}} \cup \{r_{j-1}\} \cup \{s_{j+1}\})$. By deforming the path of integration of the Cousin's integral we see that $F_j(z)$ can be holomorphically extended beyond the arc from both sides into the other sides, except



at $r_{j-1}$ and $s_{j+1}$. Next let us take a family of annuli in $\mathbb{C} \setminus \{c\}$ separating $c$ and $\infty$ with smooth boundaries as follows. We take anulus $B_j$ for each $j \in \mathbb{Z}$ so that the intersection of $B_j$ with the external ray $\gamma_c$ is the arc $\overline{r_j s_j}$, and $r_j, s_j$ belong to the outer and inner boundary of $B_j$ respectively. Furthermore, we for $j, k \in \mathbb{Z}$, $B_j \cap B_k$ is empty if $|k - j| > 1$ hold, and for each $j \in \mathbb{Z}$, $B_j \cap B_{j+1}$ is an an annulus. We impose that

$$\bigcup_{j \in \mathbb{Z}} B_j = \mathbb{C} \setminus \{c\}$$




For each $j$, we denote by $\widetilde{B}_j$ a covering of $B_j$ such that $\widetilde{B}_j$ covers twice on the sector $S_c \cap B_j$. Our function $F_j(z)$ defined by Cousin's integration can be extended holomorphically to $\widetilde{B}_j$. It is further extendable to a wider domain $\widetilde{B}_{j-1} \cup \widetilde{B}_j \cup \widetilde{B}_{j+1}$. Hence $F_j(z)$ is Bonded in $\widetilde{B}_j$. As is easily verified by considering the integration, we have

$$F_j(z) - F_j\left((z-c)e^{-2\pi i} + c\right) = \varphi(z)$$

for $z \in S_c \cap B_j$.

For $j, k \in \mathbb{Z}$ with $B_j \cap B_j \neq \phi$, define a holomorphic function

$$H_{jk} : B_j \cap B_k \to \mathbb{C}$$

by

$$H_{jk}(z) = F_j(z) - F_k(z).$$

$F_j(z)$ and $F_k(z)$ are holomorphic on $\widetilde{B}_j \cap \widetilde{B}_j$. But, as we have

$$F_j((z-c)e^{-2\pi i} + c) - F_k((z-c)e^{-2\pi i} + c)$$
$$= F_j(z) - F_k(z)$$

along $\gamma_c$, $H_{jk}((z-c)e^{-2\pi i}+c) = H_{jk}(z)$ holds on $S_c \cap B_j \cap B_k$, so that $H_{jk}(z)$ is well defined and holomorphic on the annulus $B_j \cap B_k$. This family of holomorphic functions $\{H_{jk}\}$ forms a "Cousin data", i.e. for $i, j, k \in \mathbb{Z}$,

$$H_{ij} + H_{jk} + H_{ki} = 0 \quad \text{on } B_i \cap B_j \cap B_k.$$

As we assumed $B_j \cap B_k = \phi$ if $|j-k|>1$, this above fact is easily verified.

For each $j \in \mathbb{Z}$, take a loop $l_j$ in $B_j \cap B_{j+1}$, making a clockwise turn once and define $h_j(z)$ and $k_j(z)$ by

$$h_j(z) = \frac{1}{2\pi i}\int_{l_j} \frac{H_{i,i+1}(\tau)}{\tau - z}\,d\tau$$

defined and holomorphic in $\bigcup_{k=-\infty}^{j} B_k \cup \{\infty\}$ (outside of the annulus), and

$$k_{j+1}(z) = \frac{1}{2\pi i}\int_{\gamma_j} \frac{H_{i,i+1}(\tau)}{\tau - z}\,d\tau$$

define an holomorphic in $\bigcup_{k=j+1}^{\infty} B_k \cup \{c\}$ (inside of the annulus).

By deforming the integration path we see that they are well defined and we have

$$H_{j,j+1}(z) = h_j(z) - k_{j+1}(z) \quad (z \in B_j \cap B_{j+1})$$

By Runge's theorem, $k_{j+1} : \bigcup_{k=j+1}^{\infty} B_k \cup \{c\} \to \mathbb{C}$ can be approximated by polynomials in the sense of uniform convergence on compact sets, and $h_j : \bigcup_{k=-\infty}^{j} B_k \cup \{\infty\}$ can be approximated by rational functions with poles only at $c$.

For each $j \geq 0$, find a rational function $g_j : \bar{\mathbb{C}} \setminus \{c\} \to \mathbb{C}$ such that

$$|g_j(z) - h_j(z)| < \frac{1}{2^j} \quad \text{for } z \in \bigcup_{i=-\infty}^{j} B_i \cup \{\infty\}.$$

And for each $j < 0$, find a polynomial $g_j : \mathbb{C} \to \mathbb{C}$ such that

$$|g_j(z) - k_{j+1}(z)| < \frac{1}{2^{|j|}} \quad \text{for } z \in \bigcup_{i=j+1}^{\infty} B_i \cup \{c\}.$$

Note that these functions $g_j$ are all holomorphic in $\mathbb{C} \setminus \{c\}$.

Let $\quad \tilde{h}_j = h_j - g_j : \bigcup_{i=-\infty}^{j} B_i \cup \{\infty\} \to \mathbb{C}$
for $j \geq 0$, and
$$\tilde{k}_{j+1} = k_{j+1} - g_j : \bigcup_{i=j+1}^{\infty} B_i \cup \{c\} \to \mathbb{C},$$
for $j < 0$. Then we have
$$|\tilde{h}_j| < \frac{1}{2^j} \qquad (j \geq 0)$$
and
$$|\tilde{k}_{j+1}| < \frac{1}{2^{|j|}} \qquad (j < 0).$$
We still have
$$H_{j,j+1}(z) = \tilde{h}_j(z) - \tilde{k}_{j+1}(z) \quad \text{for } z \in B_j \cap B_{j+1}.$$
Now, we set
$$H_j(z) = -\sum_{i=-\infty}^{j} \tilde{k}_i(z) - \sum_{i=j}^{\infty} \tilde{h}_i(z).$$
For $i \leq j$, $\tilde{k}_i(z)$ is holomorphic in $\bigcup_{l=i}^{\infty} B_l \cup \{c\}$, hence they are all holomorphic in the smallest disk $\bigcup_{l=j}^{\infty} B_l \cup \{c\}$ and that we have the estimate of the supremum of the functions, the sum of $\tilde{k}_i$'s is uniformly convergent on $B_j$. Similarly, the sum of $\tilde{h}_i$'s converge uniformly convergent on $B_j$, too. Hence $H_j(z)$ is holomorphic in $B_j$.

In the ovelapping annulus $B_j \cap B_{j+1}$, we have
$$\begin{aligned} H_{j+1} - H_j &= -\sum_{i=-\infty}^{j+1} \tilde{k}_i - \sum_{i=j+1}^{\infty} \tilde{h}_i + \sum_{i=-\infty}^{j} \tilde{k}_i + \sum_{i=j}^{\infty} \tilde{h}_i \\ &= \tilde{h}_j - \tilde{k}_{j+1} = H_{j,j+1}. \end{aligned}$$
Finally, in $\widehat{B}_j$, let $G_j(z) = H_j(z) + F_j(z)$. These functions $\{G_j\}$ on $\widehat{B}_j$ defines a holomorphic function
$$G : \Omega_c' \to \mathbb{C}$$
define on the overlapped neighborhood $\Omega_c'$ of $\Omega_c$. We can verify that these functions coincide and $G$ is well defined by an immediate calculations as follows.
In $\widehat{B}_j \cap \widehat{B}_{j+1}$,
$$\begin{aligned} G_{j+1}(z) &= H_{j+1}(z) + F_{j+1}(z) = H_j(z) + H_{j,j+1}(z) + F_{j+1}(z) \\ &= H_j(z) + F_j(z) - F_{j+1}(z) + F_{j+1}(z) = H_j(z) + F_j(z) = G_j(z) \end{aligned}$$

Thus, we conclude that $G \in \mathcal{O}(\Omega_c')$ and

$$\Delta_c G = \varphi$$

holds. This completes the proof the our Theorem 2.2.

We remark that such a function $G$ satisfying $\Delta_c G = \varphi$ is not unique since $\operatorname{Ker} \Delta_c = \mathcal{O}(\mathbb{C} \setminus \{c\})$.

## §3. Cousin's integral operator and decomposition of pro-microfunctions.

In the previous section, we discussed the surjectivity of the difference operator $\Delta_c$. In this section, we restrict the space of (general) pre-microfunctions to the space of pre-microfunctions with regular singularities, and consider an inverse operator of $\Delta_c$, which we call a Cousin's integral operator.

Definition 3.1 $\quad \mathcal{I}_c : \mathcal{O}_0(\gamma_c) \longrightarrow \mathcal{M}_c$ is defined by

$$\mathcal{I}_c[\varphi](z) = \frac{1}{2\pi i} \int_{\gamma_c} \frac{\varphi(\tau)}{\tau - z}\, d\tau$$

for $\varphi \in \mathcal{O}_0(\gamma_c)$.

Here, we use notation $\mathcal{I}_c[\varphi]$ as $\mathcal{I}_c : \mathcal{O}_0(\gamma_c) \to \mathcal{M}_c$ is an operator and we want to emphasise it, i.e. the arguement of the operator is a function and not its value.

Definition 3.2 Let $f$ be bi-valued function defined in a neighborhood of $\gamma_c$, both of the two branches are holomorphic and the differece $\Delta_c f$ of $f$ has regular singularities at $c$ and at $\infty$, i.e. $\Delta_c f \in \mathcal{O}_0(\gamma_c)$. The $\mathcal{M}_c$ component of $f$ is defined as

$$[f]_c = [f]_{\gamma_c} = \mathcal{I}_c[\Delta_c f].$$

This mapping $[\ ]_c$ is a projection map onto $\mathcal{M}_c$.

We have the following identities.

Theorem 3.3
$$\mathcal{I}_c \circ \Delta_c = id \quad \text{on } \mathcal{M}_c,$$
$$\Delta_c \circ \mathcal{I}_c = id \quad \text{on } \mathcal{O}_0(\gamma_c).$$

Proof These identities are easily verified.

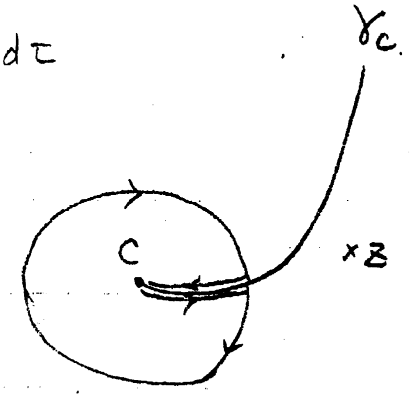
For each point $p \in J$ (and an external ray $\gamma_p$ landing at $p$), projection $[\ ]_p$ is similarly defined.

Let $\mathcal{O}_0(\Omega_c)$ denote the space of holomorphic functions $f : \Omega_c \to \mathbb{C}$ such that $f$ can be extendable holomorphically to some double sheeted neighborhood $\Omega_c'$ and satisfies $\Delta_c f \in \mathcal{O}_0(\gamma_c)$. Function $f \in \mathcal{O}_0(\Omega_c)$ is holomorphic in $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$ and has singularities at $c$ and at the infinity together with it's difference along $\gamma_c$.

Let $\mathcal{H}_c$ denote the space of hyperfunctions supported at $c$, i.e. $\varphi \in \mathcal{H}_c$ if and only if $\varphi$ is holomorphic in $(\mathbb{C} \cup \{\infty\}) \setminus \{c\}$. The space of entire functions is denoted by $\mathcal{O}(\mathbb{C})$. Let us define the operators that extract singularities of $f$.

Definition 3.4. Operator $\Gamma_c : \mathcal{O}_0(\Omega_c) \to \mathcal{H}_c$ is defined by

$$\Gamma_c[f](z) = \frac{1}{2\pi i} \int_{|\tau - c| = \varepsilon} \frac{f(\tau)}{\tau - z}\, d\tau - \frac{1}{2\pi i} \int_{r_\varepsilon}^{c} \frac{(\Delta_c f)(\tau)}{\tau - z}\, d\tau$$




where $z \in \mathbb{C} \setminus \{c\}$, $\varepsilon > 0$ is chosen sufficiently small so that the $\varepsilon$-ball around $c$ does not contain $z$, and $r_\varepsilon \in \gamma_c$ is the intersection point of $\gamma_c$ and the circle $|z - c| = \varepsilon$. The orientation of the path of integration along the circle is the counterclockwise with respect to $z$.

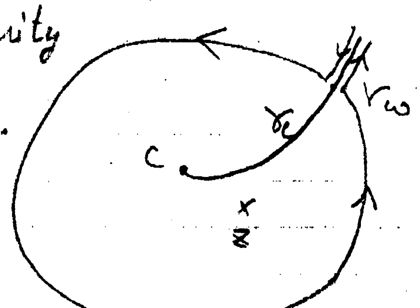
As $\Delta_c f$ has a regular singularity at $c$, this defines a holomorphic function on $(\mathbb{C}\cup\{\infty\})\setminus\{c\}$. That is, $\Gamma_c[f] \in \mathcal{H}_c$.

<u>Definition</u> 3.5. Operator $\Gamma_\infty : \mathcal{O}_0(\Omega_c) \longrightarrow \mathcal{O}(\mathbb{C})$ is defined by

$$\Gamma_\infty[f](z) = \frac{1}{2\pi i}\int_{|\tau - c| = w} \frac{f(\tau)}{\tau - z}\,d\tau + \frac{1}{2\pi i}\int_{r_w}^{\infty} \frac{(\Delta_c f)(\tau)}{\tau - z}\,d\tau$$

wher $w>0$ is taken sufficiently large for each $z$, so that the circle of integration path surrounds $z$, and $r_w \in \gamma_c$ is the intersection point of $\gamma_c$ and the big circle. The orientation is taken as the counterclockwise with respect to $z$.

As $\Delta_c f$ has a regular singularity at $\infty$, this defines an entire function. Hence $\Gamma_\infty[f] \in \mathcal{O}(\mathbb{C})$.




Just for the sake of consistence of notation we define

$$\Gamma_M : \mathcal{O}_0(\Omega_c) \longrightarrow \mathcal{M}_c$$

by $\Gamma_M[f] = [f]_c$. We have the following decomposition.

<u>Theorem</u> 3.6. $\mathcal{O}_0(\Omega_c) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C})$

and $\Gamma_c : \mathcal{O}_0(\Omega_c) \rightarrow \mathcal{H}_c$, $\Gamma_M : \mathcal{O}_0(\Omega_c) \rightarrow \mathcal{M}_c$

$\Gamma_\infty : \mathcal{O}_0(\Omega_c) \rightarrow \mathcal{O}(\mathbb{C})$

gives the projections to components.

<u>Proof</u> Clearly, the kernel of the defference operator $\Delta_c$ is $\mathcal{O}(\mathbb{C}\setminus\{c\})$ and $\mathcal{O}(\mathbb{C}\setminus\{c\}) = \mathcal{H}_c \oplus \mathcal{O}(\mathbb{C})$.

Note that these operators can be difined if $f$ is defined and holomorphic in a double covered neighborhood of $\gamma_c$. In this case, $\Gamma_c + \Gamma_M + \Gamma_\infty$ defines a projection to $\mathcal{O}_0(\Omega_c) = \mathcal{H}_c \oplus \mathcal{M}_c \oplus \mathcal{O}(\mathbb{C})$ if $\Delta_c f \in \mathcal{O}_0(\gamma_c)$.

## §4 Space of pre-microfunctions and a transfer operator

Let us go back to our complex dynamical system $R(z)$. Let $J$ denote the Julia set of $R(z)$ and let $F$ denote the Fatou set of $R(z)$. We suppose $R(z)$ is postcritically finite, especially the case of $c=i$. We denote by $\mathcal{O}(J)$ the space of germs of continuous functions $f: J \to \mathbb{C}$ which are holomorphic in some neighborhood of $J$. The space of holomorphic functions $f: F \to \mathbb{C}$ of the Fatou set satisfying $f(\infty)=0$ will be denoted by $\mathcal{O}_0(F)$. The postcritical set of $R(z)$ is denoted by $P(R)$. The space of pre-microfunctions at $P(R)$ is defined by

$$\mathcal{M}_{P(R)} = \bigoplus_{p \in P(R)} \mathcal{M}_p .$$

and

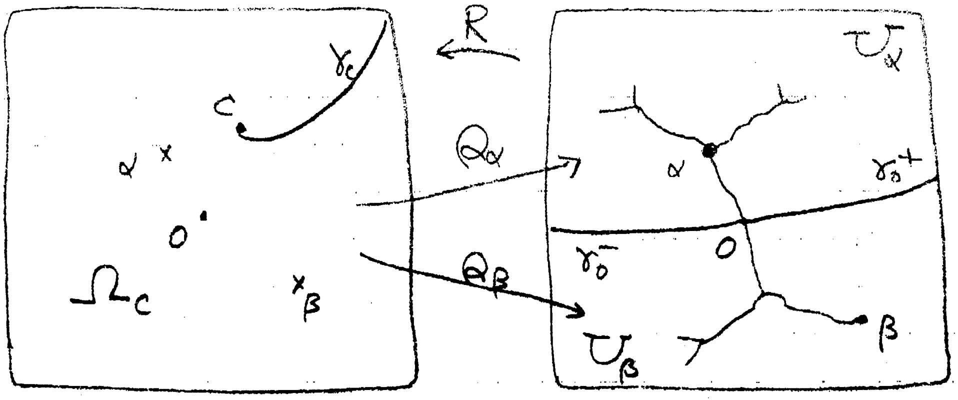
$$\mathcal{M} = \mathcal{M}_{P(R)} \oplus \bigoplus_{k=0}^{\infty} \bigoplus_{p \in R^{k}(0)} \mathcal{M}_p$$

denotes the space of formal sup of premicrofunctions at the grand obit of the critical point $0$.

Let $f \in \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F)$ and $p \in P(R)$ with $\gamma_p$ its external ray. Then the $\mathcal{M}_p$-component $[f]_p$ of $f$ is given by a projection

$$[f]_p(z) = \frac{1}{2\pi i} \int_{\gamma_p} \frac{(\Delta_p f)(z)}{\tau - z} \, d\tau .$$

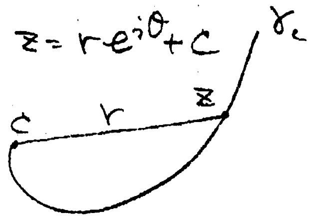
Let us consider the most simple postcritically finite case (except $c=-2$ case) of $R(z) = z^2 + i$. Fixed points of $R$ are denoted by $\alpha$ and $\beta$. The preimage of the external ray $\gamma_c$ consists of two external rays, say $\gamma_0^+$ and $\gamma_0^-$, of the critical point $0$, with external angles $\frac{1}{12}$ and $\frac{7}{12}$ respectively. These external rays are oriented as $\infty \to 0$. Let $U_\alpha$ denote the upper connected component of $\mathbb{C} \setminus (\gamma_0^+ \cup \gamma_0^-)$ which contains the critical value $c=i$. The $\alpha$-fixed point belongs to this domain. We denote the other connected point by $U_\beta$. It contains the $\beta$-fixed point. The quadratic map $R$ is of degree two. The critical value $c$ is a branch point. We denote the two branches of $R^{-1}$ by $Q_\alpha$ and $Q_\beta$ defined in $\Omega_c = \mathbb{C} \setminus (\gamma_c \cup \{c\})$.




$$Q_\alpha : \mathbb{C} \setminus (\gamma_c \cup \{c\}) \longrightarrow U_\alpha$$
$$Q_\beta : \mathbb{C} \setminus (\gamma_c \cup \{c\}) \longrightarrow U_\beta$$

with $Q_\alpha(z) = -\sqrt{z-c}$, $Q_\beta(z) = \sqrt{z-c}$, where the branch of the square root is chosen by assigning $Q_\beta(c+1)=1$. If we regard $Q_\alpha$ and $Q_\beta$ as holomorphic functions on $\Omega_c$, we can naturally consider holomorphic functions $(Q_\alpha(z))^s$ and $(Q_\beta(z))^s$ for $0 < s < 2$. They can be extended to a double sheeted neighborhood $\Omega_c'$ holomorphically. We define a holomorphic function

$$\Psi_s(z) = \frac{1}{(2\,Q_\beta(z))^s}$$




defined in $\Omega_c'$.

For $z = c + re^{i\theta} \in \gamma_c$, we have

$$(\Delta_c \Psi_s)(z) = \Psi_s(z) - \Psi_s((z-c)e^{-2\pi i} + c)$$
$$= \frac{1 - e^{s\pi i}}{\left(2\sqrt{r}\, e^{\frac{\theta}{2} i}\right)^s}$$

Hence
$$|\Delta_c \Psi_s| = \text{const.}\ r^{-\frac{s}{2}}$$
which implies $\Delta_c \Psi_s$ has regular singularities at $c$ and $\infty$ if $0 < s < 2$. Therefore $\Psi_s \in \mathcal{M}_c$.

Now, take a function $f \in \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F)$. Here, we abuse the formal sum of function in different spaces and the sum as a functions defined in the common domain of definition. So, $f$ is defined and holomorphic in $F \setminus \left( \bigcup_{p \in P(R)} \gamma_p \right)$. For an external ray $\gamma_p$, we denote by

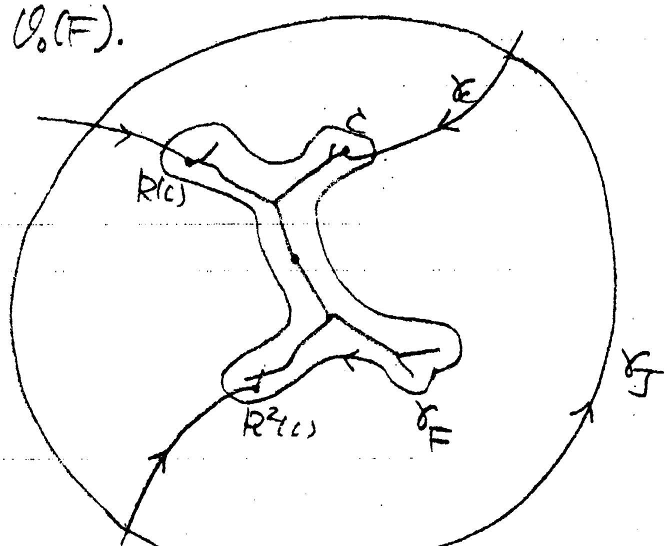
$\tilde{\gamma}_p$ the path of integration coming from $\infty$ to $p$ along $\gamma_p$ taking the clockwise side value of the integrand function and going back from $p$ to $\infty$ along $\gamma_p$ taking the counter-clockwise side value of the integrand function. That is,

$$\int_{\tilde{\gamma}_p} f(z)\,dz = \int_{\gamma_p} f(z)\,dz - \int_{\gamma_p} f((z-p)e^{2\pi i}+p)\,dz$$

$$= \int_{\gamma_p} (\Delta_p f)(z)\,dz.$$

By $\gamma_F$ we represent an integration path along the boundary of the Fatou set, passing near the Julia set and taking values of the function on the Fatou set. And finally by $\gamma_J$ we represent an integration path turning around the Julia set in the counterclockwise direction.

Let $\mathcal{H}_+ = \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_{P(R)} \oplus \mathcal{O}_0(F)$.




**Definition 4.1** Transfer operator $\mathcal{L}_s : \mathcal{H}_+ \to \mathcal{H}_+$ is defined for $0 < s < 2$ and for $f \in \mathcal{H}_+$ by

$$(\mathcal{L}_s f)(z) = \sum_{y \in R^{-1}(z)} \Psi_s(R(y)) f(y).$$

We can rewrite the transfer operator in an integral operator form as

$$(\mathcal{L}_s f)(z) = \frac{1}{2\pi i} \int_{\gamma_F + \tilde{\gamma}_{P(R)} + \gamma_J} \frac{\Psi_s(R(\tau))\, R'(\tau)\, f(\tau)}{R(\tau) - z}\, d\tau$$

and as

$$(\mathcal{L}_s f)(z) = \Psi_s(z)\left(f \circ Q_\alpha(z) + f \circ Q_\beta(z)\right).$$

**Definition 4.2** Push forward operator $R_*$ is defined by

$$R_* f = f \circ Q_\alpha + f \circ Q_\beta.$$

For point $\eta \in \mathbb{C}$, we denote by $\chi_\eta(z) = \frac{1}{z-\eta}$ the unit pole at $\eta$. If $\eta \in J$ then $\chi_\eta \in \mathcal{O}_0(F)$. If $\eta \in \gamma_p$ then $\chi_\eta \in \mathcal{O}(\Omega_p)$. Note that $\chi_z(\eta) = -\chi_\eta(z)$.

<u>Proposition</u> 4.3 For $\eta \in J \cup \bigcup_{p \in P(R)} \gamma_p$,

$$\mathcal{L}_s \chi_\eta = R'(\eta)\, \psi_s(R(\eta))\chi_{R(\eta)} + R'(\eta)\left[\psi_s \cdot \chi_{R(\eta)}\right]_c .$$

<u>Proof</u> By a direct computation, we have

$$(\mathcal{L}_s \chi_\eta)(z) = \psi_s(z)\left(\chi_\eta \circ Q_\alpha(z) + \chi_\eta \circ Q_\beta(z)\right) = \psi_s(z) \sum_{y \in R^{-1}(z)} \left(-\chi_y(\eta)\right)$$

$$= \psi_s(z) \sum_{y \in R^{-1}(z)} \frac{1}{y - \eta} .$$

By the residue formula, we have

$$0 = \frac{1}{2\pi i} \int_{\gamma_J} \frac{R'(y)}{(z - R(y))(y-\eta)}\, dy = \sum_{y \in R^{-1}(z)} \frac{1}{y-\eta} + \frac{R'(\eta)}{z - R(\eta)} .$$

Hence $(\mathcal{L}_s \chi_\eta)(z) = \psi_s(z) \dfrac{R'(\eta)}{z - R(\eta)} = \psi_s(z) R'(\eta) \chi_{R(\eta)}(z)$

$$= \psi_s(R(\eta))\, R'(\eta)\chi_{R(\eta)}(z) + \left[R'(\eta) \cdot \psi_s \cdot \chi_{R(\eta)}\right]_c .$$

Here $[R'(\eta)\,\psi_s \cdot \chi_{R(\eta)}]_c(z) = (\psi_s(z) - \psi_s(R(\eta)))\, R'(\eta)\, \chi_{R(\eta)}(z)$ is holomorphic near $R(\eta)$ so that it belongs to $\mathcal{M}_c$. The first term $\psi_s(R(\eta))\, R'(\eta)\, \chi_{R(\eta)}(z)$ is a multiple of unit pole at $R(\eta)$.

Unit poles $\{\chi_\eta\}_{\eta \in J}$ form a basis of function space $\mathcal{O}(F)$ and the family of unit poles $\{\chi_\eta\}_{\eta \in \gamma_p}$ form a basis of space of pre-microfunctions $\mathcal{M}_p$. This splitting of the transfer operator $\mathcal{L}_s$ gives a decomposition of the operator into components of the direct sum decomposition of function spaces.

<u>Theorem</u> 4.4 For $p \in J$ and $g \in \mathcal{M}_p$ we have the following decomposition

$$\mathcal{L}_s g = \left[g \circ Q_p \cdot \psi_s\right]_{R(p)} + \left[\psi_s \cdot [g \circ Q_p]_{R(p)}\right]_c$$

where $Q_p = Q_\alpha$ or $Q_\beta$ according to $p \in U_\alpha$ or $U_\beta$.

<u>Proof</u>. As $g \in \mathcal{M}_p$, $g(z)$ can be expressed in an integration of Cauchy type for $z \in \Omega_p = \mathbb{C} \setminus (\gamma_p \cup \{p\})$ as

$$g(z) = \frac{1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(t)\frac{dt}{t-z} = \frac{1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(t)(-\chi_t(z))dt.$$

Hence, we have

$$\begin{aligned}
(\mathcal{L}_s g)(z) &= \frac{-1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(\eta)(\mathcal{L}_s \chi_\eta)(z)d\eta \\
&= \frac{-1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(\eta)\left(\psi_s(R(\eta))\cdot R'(\eta)\chi_{R(\eta)} + [\psi_s \cdot R'(\eta)\cdot \chi_{R(\eta)}]_c\right)d\eta \\
&= \frac{-1}{2\pi i}\int_{\gamma_p} (\Delta_p g)(\eta)\cdot \psi_s(R(\eta))\cdot R'(\eta)\cdot \chi_{R(\eta)}\,d\eta - \frac{1}{2\pi i}\int_{\gamma_p}(\Delta_p g)(\eta)[\psi_s\cdot R'(\eta)\cdot\chi_{R(\eta)}]_c\,d\eta \\
&= \frac{-1}{2\pi i}\int_{\gamma_{R(p)}} (\Delta_{R(p)}(g\circ Q_p))(\xi)\cdot\psi_s(\xi)\chi_\xi\,d\xi + \left[\psi_s\cdot\frac{-1}{2\pi i}\int_{\gamma_p}(\Delta_p g)(\eta)\cdot R'(\eta)\chi_{R(\eta)}d\eta\right]_c \\
&= [(g\circ Q_p)\cdot\psi_s]_{R(p)} + [\psi_s\cdot[g\circ Q_p]_{R(p)}]_c .
\end{aligned}$$

Note that in the above calculations, $\eta \in \gamma_p$ is a variable along $\gamma_p$ and we changed variables by $\xi = R(\eta)$ and $d\xi = R'(\eta)d\eta$.

This theorem shows that the transfer operator $\mathcal{L}_s$ sends $\mathcal{M}_p$ into $\mathcal{M}_{R(p)} \oplus \mathcal{M}_C$ for $p \in J$. In the case of postcritically finite complex dynamical system case, the space of pre-microfunctions for postcritical set is invariant for $\mathcal{L}_s$, i.e.

<u>Proposition</u> $\mathcal{L}_s(\mathcal{M}_+) \subset \mathcal{M}_+$.

## §5. Decomposition of the transfer operator

Our space of pre-microfunctions $\mathcal{H}_+$ has a direct sum decomposition

$$\mathcal{H}_+ = \mathcal{O}(\mathbb{C}) \oplus \mathcal{M}_+ \oplus \mathcal{O}_0(F)$$

and the transfer operator $\mathcal{L}_s$ maps this space into itself. We denote the components of $\mathcal{L}_s$ in a matrix form as

$$\mathcal{L}_s = \begin{pmatrix} \mathcal{L}_{JJ} & \mathcal{L}_{JM} & \mathcal{L}_{JF} \\ \mathcal{L}_{MJ} & \mathcal{L}_{MM} & \mathcal{L}_{MF} \\ \mathcal{L}_{FJ} & \mathcal{L}_{FM} & \mathcal{L}_{FF} \end{pmatrix}.$$

For $f_J \in \mathcal{O}(C)$ we have the following proposition.

Proposition 5.1 $\quad \mathcal{L}_s f_J = \Psi_s \cdot R_* f_J = \mathcal{L}_{JJ} f_J + \mathcal{L}_{MJ} f_J + \mathcal{L}_{FJ} f_J$
with

$$\mathcal{L}_{JJ} f_J = \Psi_s \cdot R_* f_J - [\Psi_s \cdot R_* f_J]_c,$$

$$\mathcal{L}_{MJ} f_J = [\Psi_s \cdot R_* f_J]_c,$$

$$\mathcal{L}_{FJ} f_J = 0.$$

Proof As $0 < s < 2$, we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{C_\rho} \frac{\Psi_s(R(\tau)) \cdot R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau = 0,$$

where $z \neq C$ and the integration path $C_\rho$ is the circle of radius $\varepsilon$ around the critical point $0$, since the singularity at the origin is of order $1-s$. In the next calculation, integration paths are as explained in the previous section. We have

$$(\mathcal{L}_{JJ} f_J)(z) = \frac{1}{2\pi i} \int_{\gamma_J} \frac{\Psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau$$

$$= \frac{1}{2\pi i} \left\{ \int_{\gamma_J} + \int_{\tilde{\gamma}_0^+ + \tilde{\gamma}_0^-} - \int_{\hat{\gamma}_0^+ + \hat{\gamma}_0^-} + \int_{C_\rho} - \int_{C_\rho} \right\} \frac{\Psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau$$

$$= \Psi_s(z) \sum_{y \in R^{-1}(z)} f_J(y) - \frac{1}{2\pi i} \int_{\hat{\gamma}_0^+ + \hat{\gamma}_0^-} \frac{\Psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z} d\tau$$

$$= \Psi_s(z) \sum_{y \in R^{-1}(z)} f_J(y) - \frac{1}{2\pi i} \int_{\gamma_0^+ + \gamma_0^-} \frac{(\Delta_c \Psi_s)(R(\tau)) f_J(\tau) \cdot R'(\tau) d\tau}{R(\tau) - z}$$

$$= \Psi_s(z) (R_* f_J)(z) - \frac{1}{2\pi i} \int_{\gamma_c} \frac{(\Delta_c \Psi_s)(\sigma) (f_J \circ Q^+(\sigma) + f_J \circ Q^-(\sigma))}{\sigma - c} d\sigma$$

Here we made a change of variables by $\sigma = R(\tau)$ and $d\sigma = R'(\tau) d\tau$. $Q^+ : \gamma_c \to \gamma_0^+$ and $Q^- : \gamma_c \to \gamma_0^-$ denotes the inverse branches of $R$ along $\gamma_c$. $R_* f_J = f_J \circ Q^+ + f_J \circ Q^-$ holds along $\gamma_c$ and $R_* f_J = f_J \circ Q_\alpha + f_J \circ Q_\beta$ elsewhere. We continue the calculation.

$$(\mathcal{L}_{JJ} f_J)(z) = \Psi_s(z) \cdot (R_* f_J)(z) - \mathcal{I}_c [(\Delta_c \Psi_s) \cdot R_* f_J]$$

$$= \Psi_s(z) \cdot R_* f_J(z) - [\Psi_s \cdot R_* f_J]_c.$$

This completes the first line. Note that $\mathcal{L}_s f_J = \Psi_s \cdot R_* f_J$ has singularities along $\gamma_c$ only. Next, compute the component $\mathcal{L}_{MJ} f_J$ as follows.

$$(\mathcal{L}_{MJ} f_J)(z) = \frac{1}{2\pi i}\int_{\gamma_M} \frac{\psi_s(R(\tau))\, R'(\tau) f_J(\tau)}{R(\tau) - z}\, d\tau$$

$$= [\psi_s \cdot R_* f_J]_c$$

and

$$(\mathcal{L}_{FJ} f_J)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\psi_s(R(\tau)) R'(\tau) f_J(\tau)}{R(\tau) - z}\, d\tau$$

$$= \frac{1}{2\pi i}\int_{|\sigma - c| = \rho} \frac{\psi_s(\sigma)(R_* f_J)(\sigma)}{\sigma - z}\, d\sigma$$

$$= 0$$

since $\quad |\mathcal{L}_{FJ} f_J| \leq \dfrac{2\pi\rho}{2\pi}\ \dfrac{|2\rho|^{1-s}\,|f_J(0) + f_J'(0)\rho|}{|z-c|-\rho} \longrightarrow 0$ $\quad (\text{as } \rho \to 0)$

For the second column of the operator matrix, we have the following.

<u>Proposition</u> 5.2. For $f_M \in \mathcal{M}_p \subset \mathcal{M}_+ \quad (p \neq 0)$,

$$\mathcal{L}_{JM} f_M = 0$$
$$\mathcal{L}_{MM} f_M = [\psi_s \cdot R_* f_M]_c + [\psi_s \cdot f_M \circ Q_p]_{R(p)}$$
$$\mathcal{L}_{FM} f_M = 0$$

<u>Remark</u> $\quad \mathcal{L}_s f_M \in \mathcal{M}_c \oplus \mathcal{M}_{R(p)}$ if $f_M \in \mathcal{M}_p$

<u>Proof</u> As $f_M \in \mathcal{M}_p$, there exists a positive number $\varepsilon$ and a positive constant $K$ such that

$$|f_M(z)| < K|z|^{-\varepsilon}$$

holds near the infinity. Hence we have

$$|(\mathcal{L}_{JM} f_M)(z)| \leq \frac{1}{2\pi}\int_{\gamma_J} \frac{|\psi_s(R(\tau)) R'(\tau) f_M(\tau)|}{|R(\tau) - z|}\, d\tau$$

$$\leq \frac{2\pi|\tau|}{2\pi}\, |2\tau|^{1-s} \frac{1}{|\tau|^2 |1 + \frac{c}{\tau^2} - \frac{z}{\tau^2}|} K |\tau|^{-\varepsilon}$$

$$\leq \text{Const.}\, |\tau|^{-s-\varepsilon} \longrightarrow 0 \quad (|\tau| \to \infty)$$

and $\quad \mathcal{L}_{JM} f_M = 0.$

Next, we show that $\mathcal{L}_{FM} = 0$ in the following.

$$(\mathcal{L}_{FM} f_M)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\Psi_s(R(\tau)) R'(\tau) f_M(\tau)}{R(\tau) - z} d\tau$$

$$= \frac{1}{2\pi i}\left\{\int_{|\tau-p|=\rho} + \int_{|\tau|=\rho}\right\} \frac{\Psi_s(R(\tau)) R'(\tau) f_M(\tau)}{R(\tau) - z} d\tau.$$

This goes to zero as $\rho \longrightarrow 0$.
For, $\Psi_s$ belongs to $M_c$ so that the second term vanishes and the first term vanishes since $f_M$ belongs to $M_p$. Note that in this computation, $z$ is taken from the Fatou set and the integration path $\gamma_F$ runs near the Julia set. Note that this argument cannot be applied if $p = 0$ since the integrand might have a singularity at $p$ which not regular. We need regularity of the singular points to have this kind of integral vanish. Finally,

$$(\mathcal{L}_{MM} f_M)(z) = \frac{1}{2\pi i}\int_{\gamma_M} \frac{\Psi_s(R(\tau)) R'(\tau) f_M(\tau)}{R(\tau) - z} d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_0^+ + \gamma_0^-} \frac{\Psi_s(R(\tau)) R'(\tau) f_M(\tau)}{R(\tau) - z} d\tau + \frac{1}{2\pi i}\int_{\gamma_p} \frac{\Psi_s(R(\tau)) R'(\tau) f_M(\tau)}{R(\tau) - z} d\tau$$

$$= I_c[(\Delta_c \Psi_s)\cdot R_* f_M] + I_{R(p)}[\Psi_s \cdot (\Delta_p f_M)\circ Q_a]$$

$$= [\Psi_s \cdot R_* f_M]_c + [\Psi_s \cdot f_M \circ Q_p]_{R(p)}.$$ This completes the proof of Proposition 5.2.

<u>Remark</u> If $f_M \in M_0$, then the integrand may have a non-regular singularity, since we have a product of two regular singularities at $p=0$. Hence.

$$\mathcal{L}_{FM} f_M = [\Psi_s \cdot R_* f_M]_F$$
$$\mathcal{L}_{MM} f_M = [\Psi_s \cdot R_* f_M]_c$$
$$\mathcal{L}_{JM} f_M = [\Psi_s \cdot R_* f_M]_J.$$

For $f_F \in \mathcal{O}_0(F)$, we have the following

<u>Proposition</u> 5.3

$$\mathcal{L}_{JF} f_F = 0$$
$$\mathcal{L}_{MF} f_F = [\Psi_s \cdot R_* f_F]_c$$
$$\mathcal{L}_{FF} f_F = \Psi_s \cdot R_* f_F - [\Psi_s \cdot R_* f_F]_c$$

Proof As $f_F \in \mathcal{O}_0(F)$, we have an estimate $|f_F(\tau)| < k|\tau|^{-1}$ near $\infty$. Hence

$$\left|(\mathcal{L}_{JF} f_F)(z)\right| \leq \frac{2\pi|\tau|}{2\pi} \frac{|2\tau|^{-s}\,|2\tau|\cdot k|\tau|^{-1}}{|\tau^2|\,|1+\frac{c}{\tau^2}+\frac{z}{\tau^2}|} \longrightarrow 0 \quad (\text{as } \tau \to \infty)$$

Therefore we have $\mathcal{L}_{JF} f_F = 0$. For the second component,

$$\mathcal{L}_{MF} f_F = I_c[(\Delta_c \psi_s)\cdot R_* f_F] = [\psi_s \cdot R_* f_F]_c.$$

And the third component is computed similarly.

$$(\mathcal{L}_{FF} f_F)(z) = \frac{1}{2\pi i}\int_{\gamma_F} \frac{\psi_s(R(\tau))\,R'(\tau)\,f_F(\tau)}{R(\tau)-z}\,d\tau$$
$$= \frac{1}{2\pi i}\int_{\gamma_F} \frac{\psi_s(\sigma)\,(R_* f_F)(\sigma)}{\sigma - z}\,d\sigma$$
$$= \psi_s(z)\cdot(R_* f_F)(z) - [\psi_s \cdot R_* f_F]_c.$$

Putting the above propositions together, we have the dcomposition of our transfer operator into components.

$$\begin{pmatrix} \mathcal{L}_{JJ} & \mathcal{L}_{JM} & \mathcal{L}_{JF} \\ \mathcal{L}_{MJ} & \mathcal{L}_{MM} & \mathcal{L}_{MF} \\ \mathcal{L}_{FJ} & \mathcal{L}_{FM} & \mathcal{L}_{FF} \end{pmatrix} \begin{pmatrix} f_J \\ f_M \\ f_F \end{pmatrix} = \begin{pmatrix} \psi_s \cdot R_* f_J - [\psi_s \cdot R_* f_J]_c & [\psi_s \cdot R_* f_M]_J & 0 \\ [\psi_s \cdot R_* f_J]_c & [\psi_s \cdot R_* f_M]_c + [\psi_s \cdot f_M \circ \Theta_p]_{R(p)} & [\psi_s \cdot R_* f_F]_c \\ 0 & [\psi_s \cdot R_* f_M]_F & \psi_s \cdot R_* f - [\psi_s \cdot R_* f_F]_c \end{pmatrix}$$

Note that if $f_J$ or $f_F$ are not identically zero, then $[\psi_s \cdot R_* f_J]_c \neq 0$ or $[\psi_s \cdot R_* f_F]_c \neq 0$. Hence, there is no eigenfunction in subspace $\mathcal{O}(\mathbb{C}) \oplus \mathcal{O}_0(F)$.

## §6. Invariant subspace of the transfer operator

Our transfer operator $\mathcal{L}_s : \mathcal{M}_+ \longrightarrow \mathcal{M}_+$ maps the space of pre-microfunctions supported on the forward orbit of the critical point. For $h_0 \in \mathcal{M}_c$, $h_0(z)$ can be written in a form of Cauchy integral.

$$h_0(z) = \frac{-1}{2\pi i}\int_{\gamma_c} (\Delta_c h_0)(t)\,\chi_t(z)\,dt = \frac{1}{2\pi i}\int_{\gamma_c} (\Delta_c h_0)(t)\,\chi_z(t)\,dt.$$

In the following, we denote as $\psi_{s,k}(z) = \psi_s(R_k(z))\cdot\psi_s(R_{k-1}(z))\cdots\psi_s(R(z))$

Where $R_k(z) = R \circ R_{k-1}(z)$ denote the $k$-th iteration of $R(z)$. Note that $\Psi_s \circ R_k \in \mathcal{M}_{R^{-k+1}(c)}$ and that $\Psi_{s,k}$ are regular on $\gamma_{P(R)}$.

For each $k = 1, 2, \cdots$, we consider a pre-microfunction $h_k$ in $\mathcal{M}_{R_k(c)}$ expressed in terms of a pre-microfunction $g_k$ in $\mathcal{M}_c$. For $g_k \in \mathcal{M}_c$, let

$$G_k(\zeta) = g_k(\zeta) \cdot \Psi_{s,k}(\zeta)$$

and define $h_k \in \mathcal{M}_{R_k(c)}$ by

$$h_k(z) = \frac{-1}{2\pi i} \int_{\gamma_c} (\Delta_c G_k)(t)\, R_k'(t)\, \chi_{R_k(t)}(z)\, dt.$$

Let $Q_k = (R_k|_{\gamma_c})^{-1} : \gamma_{R_k(c)} \longrightarrow \gamma_c$ be the inverse branch of $R_k$. By a change of variables $\sigma = R_k(t)$, $d\sigma = R_k'(t)\,dt$ and $t = Q_k(\sigma)$, we have

$$h_k(z) = \frac{-1}{2\pi i} \int_{\gamma_{R_k(c)}} \left(\Delta_{R_k(c)}[G_k \circ Q_k]\right)(\sigma)\, \chi_\sigma(z)\, d\sigma$$

This implies $h_k = [G_k \circ Q_k]_{R_k(c)}$ and $h_k \in \mathcal{M}_{R_k(c)}$.
We have

$$\Delta_c G_k = (\Delta_{R_k(c)} h_k) \circ R_k \quad \text{along } \gamma_c.$$

So, this correspondence induces an isomorphism $\mathcal{M}_{R_k(c)} \simeq \mathcal{M}_c$. As $h_k \in \mathcal{M}_{R_k(c)}$, we have $\mathcal{L}_s h_k \in \mathcal{M}_{R_{k+1}(c)} \oplus \mathcal{M}_c$. More precisely, we have the following explicit formula.

<u>Proposition</u> 6.1 If $h_k = [G_k \circ Q_k]_{R_k} \in \mathcal{M}_{R_k(c)}$ with $G$ as above, we have the following decomposition.

$$\mathcal{L}_s h_k = [G_k \circ Q_{k+1} \cdot \Psi_s]_{R_{k+1}(c)} + [\Psi_s \cdot [G_k \circ Q_{k+1}]_{R_{k+1}(c)}]_c$$

<u>Proof</u> This is immediately verified by applying Theorem 4.4. By an immediate calculation, we can obtain the proof as follows. First component of $\mathcal{L}_s h_k$ is given by,

$$[\mathcal{L}_s h_k]_{R_{k+1}(c)} = \left[ \mathcal{L}_s \left[ \frac{-1}{2\pi i} \int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\, \chi_\sigma(z)\, d\sigma \right] \right]_{R_{k+1}(c)}$$

$$= \left[ \frac{-1}{2\pi i} \int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\, \mathcal{L}_s[\chi_\sigma]\, d\sigma \right]_{R_{k+1}(c)}$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\left[\mathcal{L}_s \chi_\sigma\right]_{R_{k+1}(c)} d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\cdot \psi_s(R(\sigma))\, R'(\sigma)\,\chi_{R(\sigma)}\, d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)}[G_k \circ Q_{k+1}])(\rho)\,\psi_s(\rho)\,\chi_\rho\, d\rho$$

$$= [\psi_s \cdot G_k \circ Q_{k+1}]_{R_{k+1}(c)},$$

where we made change of variables $\rho = R(\sigma)$, $d\rho = R'(\sigma)d\sigma$. Similarly, the second component is computed as follows.

$$[\mathcal{L}_s h_k]_c = \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\,[\mathcal{L}_s \chi_\sigma]_c\, d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_k(c)}} (\Delta_{R_k(c)}[G_k \circ Q_k])(\sigma)\cdot R'(\sigma)\,[\psi_s \cdot \chi_{R(\sigma)}]_c\, d\sigma$$

$$= \frac{-1}{2\pi i}\int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)}[G_k \circ Q_{k+1}])(\rho)\cdot[\psi_s \cdot \chi_\rho]_c\, d\rho$$

$$= \left[\psi_s \cdot \frac{-1}{2\pi i}\int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)}[G_k \circ Q_{k+1}])(\rho)\,\chi_\rho\, d\sigma\right]_c$$

$$= \left[\psi_s \cdot [G_k \circ Q_{k+1}]_{R_{k+1}(c)}\right]_c.$$

## §7 Eigenvalue problem

In this section, we consider the eigenvalue problem for our transfer operator $\mathcal{L}_s$ restricted to an invariant subspace of pre-microfunctions $\mathcal{M}_{P(R)}$ defined in section 4 as

$$\mathcal{M}_{P(R)} = \sum_{k=0}^{\infty} \mathcal{M}_{R_k(c)}.$$

Here, the sum is considered as formal sum. In the case of postcritically finite maps, the post critical set $P(R)$ is a finite set and the sum is finite. In this case

$$\mathcal{M}_{P(R)} = \bigoplus_{p\in P(R)} \mathcal{M}_p.$$

In order to analize the eigenvalue problem of $\mathcal{L}_s$, we consider a formal sum of pre-microfunctions.

$$h = \sum_{k=0}^{\infty} h_k \quad \text{with} \quad h_k \in \mathcal{M}_{R_k(c)}$$

<u>Proposition</u> 7.1. If $h$ is an eigenfunction of $\mathcal{L}_s$ satisfying

$$\lambda \mathcal{L}_s h = h \quad \text{and} \quad E(R) \text{ is infinite},$$

then $\lambda [\mathcal{L}_s h_k]_{R_{k+1}(c)} = h_{k+1}$ and $\lambda \sum_{k=0}^{\infty} [\mathcal{L}_s h_k]_c = h_0$.

~~Proof~~ By a straightforward calculation, we have

$$\mathcal{L}_s h_k = [\mathcal{L}_s h_k]_{R_{k+1}(c)} + [\mathcal{L}_s h_k]_c .$$

<u>Theorem</u> 7.2 The eigenvalue problem $\lambda \mathcal{L}_s h = h$ of our transfer operator $\mathcal{L}_s : \mathcal{M}_{E(R)} \to \mathcal{M}_{E(R)}$ reduces to an "eigenvalue" problem of an integral operator

$$T_s : \mathcal{O}_0(\gamma_c) \longrightarrow \mathcal{O}_0(\gamma_c)$$

defined by

$$(T_s[\varphi])(u) = (\Delta_c \Psi_s)(u) \cdot \frac{1}{2\pi i} \int_{\gamma_c} H_s(u, t; \lambda)\, \varphi(t)\, dt$$

where

$$H_s(u,t;\lambda) = -\sum_{k=0}^{\infty} \lambda^k \Psi_{s,k}(t)\, R'_{k+1}(t)\, \chi_{R_{k+1}(t)}(u),$$

with $\lambda T_s h_0 = h_0$, $h_0 = \Delta_c h$.

~~Proof~~ As $h_{k+1} = \lambda [\mathcal{L}_s h_k]_{R_{k+1}(c)} = \lambda [\Psi_s \cdot G_k \circ Q_{k+1}]_{R_{k+1}(c)}$

$= \lambda [ g_k \circ Q_{k+1} \cdot \Psi_s \circ R_{k+1} \circ Q_{k+1} \cdot \Psi_{s,k} \circ Q_{k+1} ]_{R_{k+1}(c)}$

$= \lambda [ g_k \circ Q_{k+1} \cdot \Psi_{s,k+1} \circ Q_{k+1} ]_{R_{k+1}(c)}$

and $h_{k+1} = [G_{k+1} \circ Q_{k+1}]_{R_{k+1}(c)} = [g_{k+1} \circ Q_{k+1} \cdot \Psi_{s,k+1} \circ Q_{k+1}]_{R_{k+1}(c)}$

we have $g_{k+1} = \lambda g_k$ for $k \geq 0$.

Hence $g_k = \lambda^k h_0$, which implies

$$h_k = [G_k \circ Q_k]_{R_k(c)} = [\lambda^k h_0 \circ Q_k \cdot \Psi_{s,k} \circ Q_k]_{R_k(c)}$$

and $h_0 = \lambda \sum_{k=0}^{\infty} [\mathcal{L}_s h_k]_c = \lambda \sum_{k=0}^{\infty} [\psi_s \cdot [G_k \circ Q_{k+1}]_{R_{k+1}(\cdot)}]_c$

$= \lambda \sum_{k=0}^{\infty} [\psi_s \cdot \lambda^k [h_0 \circ Q_{k+1} \cdot \psi_{s,k} \circ Q_{k+1}]_{R_{k+1}(\cdot)}]_c$

$= \lambda \sum_{k=0}^{\infty} \left[\psi_s \cdot \lambda^k \frac{-1}{2\pi i} \int_{\gamma_{R_{k+1}(c)}} (\Delta_{R_{k+1}(c)} [h_0 \circ Q_{k+1}])(\sigma) \cdot \psi_{s,k} \circ Q_{k+1}(\sigma) \chi_\sigma d\sigma\right]_c$

$= \lambda \sum_{k=0}^{\infty} \left[\psi_s \cdot \lambda^k \frac{-1}{2\pi i} \int_{\gamma_c} (\Delta_c h_0)(\tau)\, \psi_{s,k}(\tau) \cdot R'_{k+1}(\tau) \chi_{R_{k+1}(\tau)} d\tau\right]_c$

(here we changed variables $\tau = Q_{k+1}(\sigma)$ and $d\sigma = R'_{k+1}(\tau) d\tau$)

$= \frac{-\lambda}{2\pi i} \int_{\gamma_c} (\Delta_c \psi_s)(u) \left(\frac{-1}{2\pi i} \int_{\gamma_c} \sum_{k=0}^{\infty} \lambda^k \psi_{s,k}(t) \cdot R'_{k+1}(t) \chi_{R_{k+1}(t)}(u)\, (\Delta h_0)(t)\, dt\right) \chi_u du$

This yields an integral equation for $h_0 \in \mathcal{M}_c$:

$$(\Delta h_0)(u) = \lambda (\Delta_c \psi_s)(u) \frac{1}{2\pi i} \int_{\gamma_c} H_s(u, t; \lambda)(\Delta_c h_0)(t)\, dt$$

more biefly

$$h_0 = \lambda \left[\psi_s \cdot \frac{1}{2\pi i} \int_{\gamma_c} H_s(u, t; \lambda)(\Delta_c h_0)(t)\, dt\right]_c,$$

by setting $H_s(u, t; \lambda) = -\sum_{k=0}^{\infty} \lambda^k \psi_{s,k}(t)\, R'_{k+1}(t) \chi_{R_{k+1}(t)}(u)$.

## §8. Dual spaces and Cauchy's transformations.

Let $\mathcal{O}(J)$ denote the space of germs of holomorphic function along the Julia set $J = J(R)$. Each element of $\mathcal{O}(J)$ has a reprecentative $f: J \to \mathbb{C}$ which can be extended to a holomorphic function in a neighborhood of $J$. As $J$ is a perfect set, the analytic continuation is uniquely determined by $f$. Topology in $\mathcal{O}(J)$ is given by the uniform convergence on $J$.

Linear functional $G^J: \mathcal{O}(J) \to \mathbb{C}$ is said to be holomorphic if for any holomorphic family $f_\lambda: \Lambda \to \mathcal{O}(J)$, $G^J[f_\lambda]: \Lambda \to \mathbb{C}$ is holomorphic. We require the continuity of $G^J$ with respect to the sup norms on $N_\varepsilon(J)$. The space of continuous holomorphic linear functionals $G^J: \mathcal{O}(J) \to \mathbb{C}$ will be denoted by $\mathcal{O}^*(J)$. As in the previous sections, $\mathcal{O}_0(F)$ denotes the space of holomorphic functions in the Fatou set $F$ vanishing at the infinity. If $p \in F$ then $\chi_p = \frac{1}{z-p}$ belongs to $\mathcal{O}(J)$. For holomorphic linear functional $G^J$ in $\mathcal{O}^*(J)$, define a holomorphic function $g^J \in \mathcal{O}_0(F)$ by $g^J(z) = G^J[-\chi_z]$. Then family

of holomorphic functions $F \to \mathcal{O}(J)$ defined by $z \mapsto -\chi_z$ is a holomorphic family, $g^J : F \to \mathbb{C}$ is a holomorphic function, since the functional $G^J$ is holomorphic. By the continuity of $G^J$, we see immediately $g^J(\infty) = 0$ and hence $g^J \in \mathcal{O}_0(F)$.
This correspondence between $\mathcal{O}^*(J)$ and $\mathcal{O}_0(F)$ is called the Cauchy transformation.

<u>Proposition</u> 8.1. For $f_J \in \mathcal{O}(J)$, $G^J[f_J]$ can be expressed in a integration form

$$G^J[f_J] = \frac{1}{2\pi i}\int_{\gamma_F} f_J(\tau)\, g^J(\tau)\, d\tau,$$

where the integration path $\gamma_F$ goes around the Julia set in the clockwise direction.

<u>Proof</u> As $f_J$ is holomorphic near $J$, for $z$ in a neighborhood of $J$,

$$f_J(z) = \frac{1}{2\pi i}\int_{\gamma_J} f_J(\tau)\chi_z(\tau)\, d\tau = \frac{-1}{2\pi i}\int_{\gamma_J} f_J(\tau)\chi_\tau(z)\, d\tau,$$

where $\gamma_J$ runs around the Julia set in the counter-clockwise direction. The right hand side of this equality gives an expression of $f_J(z)$ as a "linear combination" of unit poles. We have

$$G^J[f_J] = \frac{-1}{2\pi i}\int_{\gamma_J} f_J(\tau)\, G^J[\chi_\tau]\, d\tau = \frac{-1}{2\pi i}\int_{\gamma_J} f_J(\tau)\, g^J(\tau)\, d\tau$$
$$= \frac{1}{2\pi i}\int_{\gamma_F} f_J(\tau)\, g^J(\tau)\, d\tau.$$

In the following, we shall denote such pairings of functions as

$$\langle g^J, f_J \rangle_F = \frac{1}{2\pi i}\int_{\gamma_F} f_J(\tau)\, g^J(\tau)\, d\tau.$$

<u>Proposition</u> 8.2 $\qquad \mathcal{O}^*(J) \approx \mathcal{O}_0(F)$

<u>Proof</u> The Cauchy transformation defines a complex linear map from $\mathcal{O}^*(J)$ to $\mathcal{O}_0(F)$, and the pairing along $\gamma_F$ defines a complex linear map from $\mathcal{O}_0(F)$ to $\mathcal{O}^*(J)$. These two transformation are mutually inverse.

Let $\mathcal{O}_0^*(F)$ denote the space of holomorphic linear and continuous functional $G^F : \mathcal{O}_0(F) \to \mathbb{C}$.

<u>Proposition</u> 8.3 $\quad \mathcal{O}(J) \subset \mathcal{O}_0^*(F)$.

Proof. If $z \in J$ then $-\chi_z \in \mathcal{O}_0(F)$. For $G^F \in \mathcal{O}_0^*(F)$, let $g^F(z) = G^F[-\chi_z]$. Then $g^F : J \to \mathbb{C}$ is a continuous function.

If $g^F \in \mathcal{O}(J)$, then for $f_F \in \mathcal{O}_0(F)$ with

$$f_F(z) = \frac{1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,\chi_z(\tau)\,d\tau = \frac{-1}{2\pi i}\int_{\gamma_F} f_F(\tau)\chi_\tau(z)\,d\tau,$$

we have

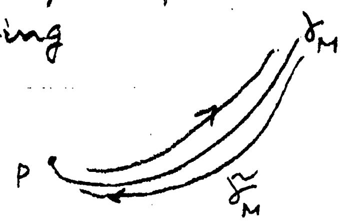
$$G^F[f_F] = \frac{-1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,G^F[\chi_\tau]\,d\tau = \frac{-1}{2\pi i}\int_{\gamma_F} f_F(\tau)\,g^F(\tau)\,d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_J} f_F(\tau)\,g^F(\tau)\,d\tau = \langle g^F, f_F\rangle_J,$$

where the integration path $\gamma_J$ is given by $g^F$ which is holomorphic in a neighborhood of $J$. This pairing will be denoted as $\langle g^F, f_F\rangle_J$.

We define the third pairings $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{\tilde{M}}$ related to the external rays and pre-microfunctions. Let $\mathcal{M}$ denote the space of pre-microfunctions and let $\gamma_M$ denote the "sum" of external rays supporting the pre-microfunctions. We use symbol $M$ to indicate the object is related to the pre-microfunction component. When we apply operations $\Delta_M$, $I_M$ etc. we take the "sum" of the objects over external rays. The dual space $\mathcal{M}^*$ is the space of holomorphic linear and continuous functionals $G^M : \mathcal{M} \to \mathbb{C}$.

For $\gamma_M$, we denote by $\tilde{\gamma}_M$ the integration path passing along the external ray both sides of $\gamma_M$ coming from the infinity to $p$ on the negative side of $\gamma_M$ and coming back from $p$ to the infinity on the positive side of $\gamma_M$. If $g^M \in \mathcal{O}_0(\gamma_M)$, that is, $g^M$ is a holomorphic function in a neighborhood of $\gamma_M$ with regular singularities at the infinity and each landing points.



For $f_M \in \mathcal{M}$, we can rewrite it in the following form.

$$f_M(z) = \frac{1}{2\pi i}\int_{\gamma_M} (\Delta_M f_M)(\tau)\chi_z(\tau)\,d\tau = \frac{-1}{2\pi i}\int_{\gamma_M} (\Delta_M f_M)(\tau)\chi_\tau(z)\,d\tau$$

For $g^M \in \mathcal{O}(\gamma_M)$ define a holomorphic functional $G^M \in \mathcal{M}^*$ by

$$G^M[f_M] = \langle g^M, \Delta_M f_M\rangle_M = \langle g^M, f_M\rangle_{\tilde{M}}.$$

$G^M[f_M]$ is defined if $\Delta_M f_M \cdot g^M \in L_1(\gamma_M)$. Note that

$$G^M[-\chi_z] = \langle g^M, -\chi_z\rangle_{\tilde{M}} = g^M(z)$$

by Cauchy's integration formula. Note that if $g^M \in \mathcal{O}_0(\gamma_M)$ and $g^M = I_M[g^M]$, then for $\zeta \in \mathbb{C}\setminus\gamma_M$

$$g^M(\zeta) = \langle g^M, \chi_\zeta\rangle_{\tilde{M}} = \langle g^M, \chi_\zeta\rangle_M$$

holds. If $f_M \in \mathcal{M}$, then

$$G^M[f_M] = G^M\left[\frac{1}{2\pi i}\int_{\gamma_M}(\Delta_M f_M)(\tau)(-\chi_\tau)\cdot d\tau\right] = \frac{1}{2\pi i}\int_{\gamma_M}(\Delta_M f_M)(\tau)G^M[-\chi_\tau]d\tau$$

$$= \frac{1}{2\pi i}\int_{\gamma_M}(\Delta_M f_M)(\tau)g^M(\tau)\,d\tau = \langle g^M, \Delta_M f_M\rangle_M.$$

We have a splitting of pre-microfunctions,

$$f = f_J \oplus f_M \oplus f_F \in \mathcal{O}_0(J)\oplus\mathcal{M}\oplus\mathcal{O}_0(F)$$

and a splitting of its dual space

$$G = G^J \oplus G^M \oplus G^F \in \mathcal{O}_0^*(J)\oplus\mathcal{M}^*\oplus\mathcal{O}_0^*(F),$$

with Cauchy's transforms given by

$$\begin{aligned}
G^J[-\chi_z] &= g^J(z), & z \in F, \quad & g^J \in \mathcal{O}_0(F)\\
G^M[-\chi_z] &= g^M(z), & z \in \gamma_M, \quad & g^M \in \mathcal{O}(\gamma_M)\\
G^M[\chi_\zeta] &= \tilde{g}^M(\zeta), & \zeta \in \mathbb{C}\setminus\gamma_M, \quad & \tilde{g}^M \in \widehat{\mathcal{M}}\\
G^F[-\chi_z] &= g^F(z), & z \in J, \quad & g^F \in \mathcal{O}(J).
\end{aligned}$$

The pairing of $f$ an $G$ is defined by

$$\begin{aligned}
G[f] &= G^J[f_J] + G^M[f_M] + G^F[f_F]\\
&= \langle g^J, f_J\rangle_J + \langle g^M, \Delta_M f_M\rangle_M + \langle g^F, f_F\rangle_F.
\end{aligned}$$

Projections of $\mathcal{H} = \mathcal{O}_0(J)\oplus\mathcal{M}\oplus\mathcal{O}_0(F)$ to components are denoted by $f \longmapsto [f]_J$, $f \longmapsto [f]_M$, $f \longmapsto [f]_F$ respectively.

These projections are given by

$$[f]_J(z) = \frac{1}{2\pi i}\int_{\gamma_J} f(\tau)\chi_z(\tau)\,d\tau \qquad (z \in J),$$

$$[f]_M(z) = \frac{1}{2\pi i}\int_{\gamma_M} (\Delta_M f)(\tau)\chi_z(\tau)\,d\tau \qquad (z \in \mathbb{C}\setminus\gamma_M),$$

$$[f]_F(z) = \frac{1}{2\pi i}\int_{\gamma_F} f(\tau)\chi_z(\tau)\,d\tau \qquad (z \in F).$$

And the projections of $\mathcal{H}^* = \mathcal{O}^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}_0^*(F)$ are denoted as

$$[\ ]^J : \mathcal{H}^* \longrightarrow \mathcal{O}_0(F) \subset \mathcal{O}^*(J)$$

$$[\ ]^M : \mathcal{H}^* \rightarrow \mathcal{O}(\gamma_M) \subset \mathcal{M}^*$$

$$[\ ]^F : \mathcal{H}^* \longrightarrow \mathcal{O}(J) \subset \mathcal{O}_0^*(F).$$

Let $\mathcal{L}_s^*$ denote the dual of our transfer operator $\mathcal{L}_s$. We abuse notations and confuse functionals and its Cauchy's transforms. $\mathcal{L}_s^* : \mathcal{O}_0(F) \oplus \mathcal{O}(\gamma_M) \oplus \mathcal{O}(J) \supset$ is decomposed as

$$\mathcal{L}_s^* = \begin{pmatrix} \mathcal{L}_{JJ}^* & \mathcal{L}_{JM}^* & \mathcal{L}_{JF}^* \\ \mathcal{L}_{JM}^* & \mathcal{L}_{MM}^* & \mathcal{L}_{MF}^* \\ \mathcal{L}_{FJ}^* & \mathcal{L}_{FM}^* & \mathcal{L}_{FF}^* \end{pmatrix}.$$

In the rest of this section, we compute these components more explicitly.

**Proposition 8.4.**

$$\mathcal{L}_{JJ}^* g^J = \psi_s \circ R \cdot R' \cdot g^J \circ R - [\psi_s \circ R \cdot R' \cdot g^J \circ R]_0$$

$$\mathcal{L}_{MJ}^* g^J = \Delta_0 [\psi_s \circ R \cdot R' \cdot g^J \circ R]$$

$$\mathcal{L}_{FJ}^* g^J = 0$$

**Proof** For $g^J \in \mathcal{O}_0(F)$, we compute $(\mathcal{L}_{JJ}^* g^J)(\zeta)$ for $\zeta \in F$.

$$(\mathcal{L}_{JJ}^* g^J)(\zeta) = [(\mathcal{L}_s^* G^J)[-\chi_\zeta]]^J = [G^J[-\mathcal{L}_s \chi_\zeta]]^J$$

$$= \left[G^J\left[-\psi_s(R(\zeta)) \cdot R'(\zeta)\,\chi_{R(\zeta)} - [\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c\right]\right]^J$$

$$= [G^J[-\psi_s(R(\zeta)) \cdot R'(\zeta)\chi_{R(\zeta)}]]^J \qquad (\text{since } [\ ]_c \in \mathcal{M}_c)$$

$$= [\psi_s(R(\zeta)) \cdot R'(\zeta)\, G^J[-\chi_{R(\zeta)}]]^J = [\psi_s(R(\zeta)) \cdot R'(\zeta) \cdot G^J[-\chi_{R(\zeta)}]]_F$$

$$= \psi_s(R(\zeta)) R'(\zeta)\, g^J(R(\zeta)) - [\psi_s \circ R \cdot R' \cdot g^J \circ R]_0(\zeta).$$

Next, for $\zeta \in \gamma_M$, $\mathcal{L}^*_{MJ} g^J \in \mathcal{O}(\gamma_M)$ is computed as follows.

$$(\mathcal{L}^*_{MJ} g^J)(\zeta) = [(\mathcal{L}^*_s G^J)[-\chi_\zeta]]^M$$
$$= [G^J[-\psi_s(R(\zeta))\cdot R'(\zeta)\ \chi_{R(\zeta)} - [\psi_s \cdot R'(\zeta)\cdot \chi_{R(\zeta)}]_c]]^M$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\ G^J[-\chi_{R(\zeta)}]]^M = [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^J(R(\zeta))]^M$$
$$= \Delta_0[\psi_s\circ R\cdot R'\cdot g^J\circ R](\zeta) = \Delta_c \psi_s \circ R\cdot R'\cdot g^J\circ R.$$

For $\zeta \in J$, we have

$$(\mathcal{L}^*_{FJ} g^J)(\zeta) = [(\mathcal{L}^*_s G^J)[-\chi_\zeta]]^F$$
$$= [G^J[-\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^M$$
$$= 0$$

The last equality holds since $\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} \in \mathcal{O}_0(F)$ and $[\psi_s\cdot R(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$.

<u>Proposition</u> 8.5 $\quad \mathcal{L}^*_{JF}\, g^F = 0$

$$\mathcal{L}^*_{MF}\, g^F = \Delta_0[\psi_s\circ R\cdot R'\cdot g^F\circ R]$$
$$\mathcal{L}^*_{FF}\, g^F = \psi_s\circ R\cdot R'\cdot g^F\circ R - [\psi_s\circ R\cdot R'\cdot g^F\circ R]_0$$

<u>proof</u>. For $\zeta \in F$, we have $-\chi_\zeta \in \mathcal{O}(J)$. For $g^F \in \mathcal{O}(J)$,

$$(\mathcal{L}^*_{JF} g^F)(\zeta) = [(\mathcal{L}^*_s G^F)[-\chi_\zeta]]^J = [G^F[-\mathcal{L}_s\chi_\zeta]]^J$$
$$= [G^F[-\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^J$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^F[-\chi_{R(\zeta)}]]^J = 0.$$

Here, the last equality holds since $\psi_s(R(\zeta))\cdot R'(\zeta)\,\chi_{R(\zeta)} \in \mathcal{O}(J)$, $[\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$ and $G^F[-\chi_{R(\zeta)}] = 0$.

For $\zeta \in J$, $-\chi_\zeta$ belongs to $\mathcal{O}_0(F)$. Hence

$$(\mathcal{L}^*_{FF} g^F)(\zeta) = [(\mathcal{L}^*_s G^F)[-\chi_\zeta]]^F = [G^F[-\mathcal{L}_s\chi_\zeta]]^F$$
$$= [G^F[-\psi_s(R(\zeta))\cdot R'(\zeta)\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^F$$
$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^F[-\chi_{R(\zeta)}]]^F = [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^F(R(\zeta))]^F$$
$$= \psi_s\circ R(\zeta)\cdot R'(\zeta)\cdot g^F\circ R(\zeta) - [\psi_s\circ R\cdot R'\cdot g^F\circ R]_0(\zeta).$$

We used the fact $-\psi_s(R(\zeta))\cdot R'(\zeta)\chi_{R(\zeta)} \in \mathcal{O}_0(F)$ and $[\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$.

To compute $\mathcal{L}^*_{MF} g^F$, we take $\zeta \in \gamma_M \subset F$. Then,

$$(\mathcal{L}^*_{MF} g^F)(\zeta) = [(\mathcal{L}^*_s G^F)[-\chi_\zeta]]^M$$

$$= [G^F[-\psi_s(R(\zeta))\cdot R'(\zeta)\,\chi_{R(\zeta)} - [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^M$$

$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^F[-\chi_{R(\zeta)}]]^M = [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^F(R(\zeta))]^M$$

$$= \Delta_0[\psi_s\circ R\cdot R'\cdot g^F\circ R](\zeta).$$

In the above calculations, we used the fact $[\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c \in \mathcal{M}_c$. During the computations, $\zeta$ is regarded as constant and the final result gives the formula as a function of $\zeta$.

Proposition 8.6. $\mathcal{L}^*_{JM} g^M = [\psi_s\circ R\cdot R'\cdot(I_M g^M)\circ R]_F$

$$\mathcal{L}^*_{MM} g^M = \begin{cases} [\psi_s\circ R\cdot R'\cdot g^M\circ R]^{M_p} + [I_0[g^M\circ R\cdot(\Delta_0\psi_s)\circ R\cdot R']]^{M_p} & (\zeta\in\gamma_p \text{ and } p\neq 0) \\ \left[\frac{1}{2\pi i}\int_{\gamma_0} g^M(R(\sigma))\,\psi_s(R(\sigma))\cdot R'(\sigma)\,\chi_\zeta(\sigma)\,d\sigma\right]^{M_0} & (\zeta\in\gamma_0) \end{cases}$$

$$\mathcal{L}^*_{FM} g^M = [\psi_s\circ R\cdot R'\cdot(I_M g^M)\circ R]_J$$

Proof. For $G^M \in \mathcal{M}^*$, Let $g^M(z) = G^M[-\chi_z]$, $z\in M$, $g^M\in\mathcal{O}(M)$. For $\zeta\in F$, (and $\zeta\in\mathbb{C}\setminus\gamma_M$),

$$(\mathcal{L}^*_{JM} G^M)(\zeta) = [(\mathcal{L}^* G^M)[\chi_\zeta]]^J = [G^M[\mathcal{L}_s\chi_\zeta]]^J$$

$$= [G^M[\psi_s(R(\zeta))\cdot R'(\zeta)\cdot\chi_{R(\zeta)} + [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c]]^J$$

$$= \left[\psi_s(R(\zeta))\cdot R'(\zeta)\,G^M[\chi_{R(\zeta)}] + \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)R'(\zeta)\chi_{R(\zeta)}(\tau)\,d\tau\right]^J$$

$$= [\psi_s\circ R\cdot R'\cdot(I_M g^M)\circ R]_F(\zeta) + \frac{1}{2\pi i}\int_{\gamma_F}\frac{dz}{z-\zeta}\cdot\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)\cdot R'(z)\chi_{R(z)}(\tau)\,d\tau$$

$$= [\psi_s\circ R\cdot R'\cdot(I_M g^M)\circ R]_F(\zeta) + \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)\,d\tau\cdot\frac{1}{2\pi i}\int_{\gamma_F}\frac{R'(z)\,dz}{(z-\zeta)(\tau-R(z))}$$

$$= [\psi_s\circ R\cdot R'\cdot(I_M g^M)\circ R]_F(\zeta).$$

Here, the last equality holds since $\zeta\in F\cap(\mathbb{C}\setminus\gamma_M)$ and $z$ moves near $\partial F$ along $\gamma_F$.

Next, we compute $\mathcal{L}^*_{FM}$. For $\zeta\in J$, note that $R(\zeta)\in J$ and $[\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c\in\mathcal{M}$. Hence,

$$(\mathcal{L}^*_{FM} G^M)(\zeta) = [(\mathcal{L}^* G^M)[\chi_\zeta]]^F$$

$$= [G^M[\psi_s(R(\zeta))\cdot R'(\zeta)\cdot \chi_{R(\zeta)} + [\psi_s\cdot R'(\zeta)\cdot \chi_{R(\zeta)}]_c]]^F$$

$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot (\mathcal{I}_M g^M)(R(\zeta))]^F + [\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\cdot (\Delta_c\psi_s)(\tau)\cdot R'(\zeta)\cdot \chi_{R(\zeta)}(\tau)d\tau]_J$$

The second term of the above line is computed as follows.

$$[\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\cdot(\Delta_c\psi_s)(\tau)\cdot R'(\zeta)\cdot\chi_{R(\zeta)}(\tau)d\tau]_J$$

$$= \frac{1}{2\pi i}\int_{\gamma_J}\frac{dz}{z-\zeta}\,\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)\cdot(\Delta_c\psi_s)(\tau)R'(\tau)\,\frac{d\tau}{\tau-R(z)}$$

$$= \frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)\,d\tau\,\frac{1}{2\pi i}\int_{\gamma_J}\frac{R'(z)\,dz}{(z-\zeta)(\tau-R(z))}$$

$$= 0.$$

The last equality holds since $R'(z)$ is of degree one and the denominator $(z-\zeta)(\tau-R(z))$ is of degree three with respect to the variable of integration and the integration path $\gamma_J$ turns around the Julia set along a circle of infinitely large radius. Hence we have

$$(\mathcal{L}^*_{FM} G^M)(\zeta) = [\psi_s(R(\zeta))\cdot R'(\zeta)(\mathcal{I}_M g^M)(R(\zeta))]_J.$$

Finally, let us compute $\mathcal{L}^*_{MM}G^M \in \mathcal{O}^*(M)$.

For $\zeta\in\gamma_p$ with $p\in\mathcal{L}(R)$, the component $(\mathcal{L}^*_{MM}G^M)_p \in \mathcal{O}(\gamma_p)$ is computed as follows.

$$(\mathcal{L}^*_{MM} G^M)(\zeta) = [(\mathcal{L}^*_s G^M)[-\chi_\zeta]]^{M_p} = [G^M[-\mathcal{L}_s\chi_\zeta]]^{M_p}$$

$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot G^M[-\chi_{R(\zeta)}] + \langle g^M, [\psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}]_c\rangle_M]^{M_p}$$

$$= [\psi_s(R(\zeta))\cdot R'(\zeta)\cdot g^M(R(\zeta))]^{M_p} + [\frac{1}{2\pi i}\int_{\gamma_c} g^M(\tau)(\Delta_c\psi_s)(\tau)\cdot R'(\zeta)\chi_{R(\zeta)}(\tau)d\tau]^{M_p}$$

$$= [\psi_s\circ R\cdot R'\cdot g^M_{R(p)}\circ R]^{M_p}(\zeta) + [R'(\zeta)\cdot(\mathcal{I}_c[g^M_c\cdot\Delta_c\psi_s])\circ R]^{M_p}$$

$$= [R'\cdot(\psi_s\cdot g^M_{R(p)})\circ R]^{M_p}(\zeta) + [R'\cdot[\psi_s\cdot g^M_c]_c\circ R]^{M_p}(\zeta).$$

In the case of $p=0$, i.e., for $\zeta\in\gamma_0$, we have

$$(\mathcal{L}^*_{MM} G^M)(\zeta) = [(\mathcal{L}^* G^M)[-\chi_\zeta]]^{M_0} = [G^M[-\mathcal{L}_s\chi_\zeta]]^{M_p}$$

$$= [G^M[-\psi_s\cdot R'(\zeta)\chi_{R(\zeta)}]]^{M_0} = [\langle g^M, \psi_s\cdot R'(\zeta)\cdot\chi_{R(\zeta)}\rangle_{\gamma_c}]^{M_0}$$

$$= \left[ \frac{1}{2\pi i} \int_{\tilde{\gamma}_c} g^M(\tau)\, \psi_s(\tau) \cdot R'(\zeta) \frac{d\tau}{\tau - R(\zeta)} \right]^{M_0}$$
$$= \left[ \frac{1}{2\pi i} \int_{\gamma_0} g^M(R(\sigma))\, \psi_s(R(\sigma)) \cdot R'(\zeta) \chi_\zeta(\sigma)\, d\sigma \right]^{M_0}$$
$$= R'(\zeta) \cdot \left( \Delta_0 \left[ [g_c^M \cdot \psi_s]_c \circ R \right] \right)(\zeta).$$

## §9. Example

In this section, we compute the operator $\mathcal{L}^*_{MM}$ more precisely for $R(z) = z^2 + i$ case. In this case, the critical value $c = i$ is preperiodic and the postcritical set $P(R) = \{i, i-1, -i\}$ consists of three points.

Let us compute $\mathcal{L}^*_{MM} g_c^M$ for $g_c^M \in \mathcal{O}(\gamma_c)$, where $G_c^M \in \mathcal{M}_c^*$ and $g_c^M(\zeta) = G_c^M[-\chi_\zeta]$ for $\zeta \in \gamma_c$.

For $\zeta \in \gamma_p$ with $p \neq 0$, $p \in P(R)$
$$(\mathcal{L}^*_{MM} G^M)(\zeta) = [(\mathcal{L}_s^* G_c^M)[-\chi_\zeta]]^{M_p} = [G_c^M[-\mathcal{L}_s \chi_\zeta]]^{M_p}$$
$$= \left[ G_c^M \left[ -\psi_s(R(\zeta)) \cdot R'(\zeta)\, \chi_{R(\zeta)} - [\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c \right] \right]^{M_p}.$$

Here, as $p \neq 0$, $R(\zeta) \notin \gamma_c$, $\chi_{R(\zeta)} \notin \mathcal{M}_c$. And as $[\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c$ belongs to $\mathcal{M}_c$, we have
$$(\mathcal{L}^*_{MM} G^M)(\zeta) = [G_c^M[-[\psi_s \cdot R'(\zeta) \cdot \chi_{R(\zeta)}]_c]]^{M_p}$$
$$= \left[ \frac{1}{2\pi i} \int_{\gamma_c} g_c^M(\tau) \cdot \Delta_c \psi_s)(\tau) \cdot R'(\zeta)\, \chi_{R(\zeta)}(\tau)\, d\tau \right]^{M_p}$$
$$= \left[ R'(\zeta) \frac{1}{2\pi i} \int_{\gamma_c} g_c^M(\tau) (\Delta_c \psi_s)(\tau) \chi_{R(\zeta)}(\tau)\, d\tau \right]^{M_p}$$
$$= \left[ R'(\zeta)\, I_c[g_c^M \cdot \Delta_c \psi_s] \circ R(\zeta) \right]^{M_p}$$
$$= \left( R' \cdot I_c[g_c^M \cdot \Delta_c \psi_s] \circ R \right)(\zeta).$$

For $\zeta \in \gamma_0$,
$$(\mathcal{L}^*_{MM} G_c^M)(\zeta) = [(\mathcal{L}_s^* G_c^M)[-\chi_\zeta]]^{M_0} = [G_c^M[-\mathcal{L}_s \chi_\zeta]]^{M_0}$$
$$= [G_c^M[-\psi_s \cdot R'(\zeta) \chi_{R(\zeta)}]]^{M_0} = \left[ R'(\zeta) \frac{-1}{2\pi i} \int_{\tilde{\gamma}_c} g_c^M(\tau)\, \psi_s(\tau) \chi_{R(\zeta)}(\tau)\, d\tau \right]^{M_0}$$

$$= \left[ R(\zeta) \frac{-1}{2\pi i} \int_{\tilde{\gamma}_c} g_c^M(\tau)\, \psi_s(\tau) \frac{d\tau}{\tau - R(\zeta)} \right]^{M_0}$$

$$= \left[ R(\zeta) \{\psi_s \cdot g_c^M\}_c \circ R(\zeta) \right]^{M_0},$$

where $\{\psi_s \cdot g_c^M\}_c$ denotes the regular part of $\psi_c \cdot g_c^M$ along $\gamma_c$.

i.e.
$$\{\psi_s \cdot g_c^M\}_c = \psi_s \cdot g_c^M - I_c[\Delta_c \psi_s \cdot g_c^M]$$

and is defined as

$$\{\psi_s \cdot g_c^M\}_c(\zeta) = \frac{1}{2\pi i} \int_{\tilde{\gamma}_c} \psi_s(\tau) \cdot g_c^M(\tau) \frac{d\tau}{\tau - \zeta} \quad \text{for } \zeta \in \gamma_c$$

We have a decomposition

$$\psi_s \cdot g_c^M = [\psi_s \cdot g_c^M]_c + \{\psi_s \cdot g_c^M\}_c$$

with $[\psi_s \cdot g_c^M]_c \in \mathcal{M}_c$ and $\{\psi_s \cdot g_c^M\}_c \in \mathcal{O}(\gamma_c)$.

## §10. Complex conformal measures.

Let $G \in \mathcal{O}_0^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}_0^*(F)$. Let $A \subset \mathbb{C}$ be an open set with smooth boundary $\partial A$ (oriented by the counter clockwise direction). The characteristic function $\chi_A(z)$ of $A$ is expressed as

$$\chi_A(z) = \frac{-1}{2\pi i} \int_{\partial A} \chi_\eta(z)\, d\eta = \frac{-1}{2\pi i} \int_{\partial A} \frac{d\eta}{z - \eta}$$

So, we can rewrite

$$\chi_A = \frac{-1}{2\pi i} \int_{\partial A} \chi_\eta \, d\eta.$$

Hence,
$$G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} G[-\chi_\eta]\, d\eta$$

defines a set function. If $G = G^J + G^M + G^F$, then

$$G[-\chi_\eta] = g^J + g^M + g^F$$

with $g^J \in \mathcal{O}_0(F)$, $g^M \in \mathcal{M}$, $g^F \in \mathcal{O}_0(J)$, and

$$G[\chi_A] = \frac{1}{2\pi i} \int_{\partial A} \left( g^J(\eta) + g^M(\eta) + g^F(\eta) \right) d\eta$$

defines an additive set function. Suppose $\lambda$ be a characteristic value of our transfer operator $\mathcal{L}_s$ and let $f \in \mathcal{H} = \mathcal{O}_0(J) \oplus \mathcal{M} \oplus \mathcal{O}_0(F)$ be an eigenfunction of $\mathcal{L}_s$ for singular value $\lambda$, i.e. $\lambda \mathcal{L}_s f = f$. And let $G \in \mathcal{H}^* = \mathcal{O}_0^*(J) \oplus \mathcal{M}^* \oplus \mathcal{O}_0^*(F)$ be the co-eigenfunctional of $\mathcal{L}_s^*$ for $\lambda$, i.e., $\lambda \mathcal{L}_s^* G = G$, with $g(z) = G[-\chi_z]$, $g \in \mathcal{O}_0(F) \oplus \mathcal{M} \oplus \mathcal{O}_0(J)$.

Define a set function $\mu_{fg}$ by

$$\mu_{fg}(A) = \frac{1}{2\pi i}\int_{\partial A} f(\tau)\, g(\tau)\, d\tau.$$

Then, we have

$$\mu_{fg}(R^{-1}(A)) = \frac{1}{2\pi i}\int_{\partial R^{-1}(A)} f(z)\, g(z)\, dz$$

$$= \frac{1}{2\pi i}\int_{\partial R^{-1}(A)} f(z) \cdot \lambda(\mathcal{L}_s^* g)(z)\, dz$$

$$= \frac{1}{2\pi i}\int_{R^{-1}(\partial A)} \lambda f(z)\, R'(z)\, \Psi_s(R(z))\, g(R(z))\, dz$$

Then by a change of variables $\zeta = R(z)$ with $d\zeta = R'(z)dz$, we have

$$\mu_{fg}(R^{-1}(A)) = \frac{1}{2\pi i}\int_{\partial A} \lambda \Psi_s(\zeta) \cdot g(\zeta) \Big( \sum_{z \in R^{-1}(\zeta)} f(z) \Big) d\zeta$$

$$= \frac{1}{2\pi i}\int_{\partial A} \lambda (\mathcal{L}_s f)(\zeta) \cdot g(\zeta) d\zeta$$

$$= \mu_{fg}(A),$$

where we used $\mathcal{L}_s f = \Psi_s \cdot R_* f$ and $\mathcal{L}_s^* g = R'(\Psi_s \circ g) \circ R$. Our set function $\mu_{fg}$ is backward invariant under $R$.

Finally, we consider the pull-back of the set function defined by the co-eigenfunction $g$. Suppose $\mathcal{L}_s^* g = g$ then, for $A$ with $R|_A : A \to R(A)$ injective, we have

$$\mu_g(R(A)) = \frac{1}{2\pi i}\int_{\partial R(A)} g(\zeta)\, d\zeta = \frac{1}{2\pi i}\int_{\partial A} g(R(z))\, R'(z)\, dz$$

$$= \frac{1}{2\pi i}\int_{\partial A} \Psi_s(R(z))^{-1} \cdot R'(z) \cdot \Psi_s(R(z)) \cdot g(R(z))\, dz$$

$$= \frac{1}{2\pi i}\int_{\partial A} \Psi_s(R(z))^{-1} g(z)\, dz.$$

This shows a kind of complex conformal property of the set function $\mu_g$ for co-invariant function $g$.