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Simultaneous linearization and its application

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Abstract

This note gives a proof of Ueda's simultaneous linearization theorem with real multipliers and its simple application to quadratic dynamics. This note is based on my talk at RIMS on 5 October 2006, titled "A proof of simultaneous linearization with a polylog estimate."

1 Simultaneous linearization

Here we give an alternative proof of Ueda's simultaneous linearization in a simplified setting. For $R \geq 0$, let E_R denote the region $\{z \in \mathbb{C} : \mathrm{Re}\, z \geq R\}$.

Theorem 1.1 (Simultaneous Linearization) *For $\epsilon \in [0,1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on $\{|z| \geq R > 0\}$ such that*

$$\begin{aligned} f_\epsilon(z) &= \tau_\epsilon z + 1 + O(1/z) \\ \longrightarrow f_0(z) &= z + 1 + O(1/z) \end{aligned}$$

uniformly as $\epsilon \to 0$ where $\tau_\epsilon = 1 + \epsilon$. If $R \gg 0$, then for any $\epsilon \in [0,1]$ there exists a holomorphic map $u_\epsilon : E_R \to \bar{\mathbb{C}}$ such that

$$u_\epsilon(f_\epsilon(z)) = \tau_\epsilon u_\epsilon(z) + 1$$

and $u_\epsilon \to u_0$ uniformly on compact sets of E_R.

Indeed, a similar theorem holds for any radial (= non-tangential) convergence $\tau_\epsilon \to 1$ outside the unit disk. See Ueda's original proof ([Ue1], [Ue2]). Moreover, the error term $O(1/z)$ can be replaced by $O(|z|^{-\sigma})$ with $0 < \sigma \leq 1$. (See [Ka2].) Here we present a simplified proof only for real $\tau_\epsilon \to 1$ based on the argument of [Mi, Lemma 10.10]. The idea can be traced back at least to Leau's work on the Abel equation [L]. We first check:

Lemma 1.2 *If $R \gg 0$, there exists $M > 0$ such that $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/|z|$ on $\{|z| \geq R\}$ and $\operatorname{Re} f_\epsilon(z) \geq \operatorname{Re} z + 1/2$ on E_R for any $\epsilon \in [0,1]$.*

Proof. The first inequality and the existence of M is obvious. By replacing R by larger one, we have $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/R < 1/2$ on E_R. Then

$$\operatorname{Re} f_\epsilon(z) \ \geq \ \operatorname{Re}(\tau_\epsilon z + 1) - 1/2 \ \geq \ \operatorname{Re} z + 1/2.$$

■

Let us fix such an $R \gg 0$. Next we show:

Lemma 1.3 *For any $\epsilon \in [0,1]$ and $z_1,\ z_2 \in E_{2S}$ with $S > R$, we have:*

$$\left| \frac{f_\epsilon(z_1) - f_\epsilon(z_2)}{z_1 - z_2} - \tau_\epsilon \right| \ \leq \ \frac{M}{S^2}.$$

Proof. Set $g_\epsilon(z) := f_\epsilon(z) - (\tau_\epsilon z + 1)$. For any $|z| \geq 2S$ and $w \in B(z,S)$, we have $|w| > S$. This implies $|g_\epsilon(w)| \leq M/|w| < M/S$ and thus g_ϵ maps $B(z,S)$ into $B(0,M/S)$. By the Cauchy integral formula or the Schwarz lemma, it follows that $|g'_\epsilon(z)| \leq (M/S)/S = M/S^2$ on $\{|z| \geq S\}$. Now the inequality easily follows by:

$$|g_\epsilon(z_1) - g_\epsilon(z_2)| \ = \ \left| \int_{[z_2,z_1]} g'_\epsilon(z) dz \right| \ \leq \ \int_{[z_2,z_1]} |g'_\epsilon(z)||dz| \ \leq \ \frac{M}{S^2}|z_1 - z_2|.$$

(Note that the segment $[z_2, z_1]$ is contained in $E_{2S} \subset \{|z| \geq 2S\}$.) ■

Proof of Theorem 1.1. Set $z_n := f_\epsilon^n(z)$ for $z \in E_{2R}$. Note that such z_n satisfies

$$|z_n| \ \geq \ \operatorname{Re} z_n \ \geq \ \operatorname{Re} z + \frac{n}{2} \ \geq \ 2R + \frac{n}{2}$$

by Lemma 1.2. Now we fix $a \in E_{2R}$ and define $u_{n,\epsilon} = u_n : E_{2R} \to \mathbb{C}$ $(n \geq 0)$ by

$$u_n(z) \ := \ \frac{z_n - a_n}{\tau_\epsilon^n}.$$

Then we have

$$\left|\frac{u_{n+1}(z)}{u_n(z)}-1\right| = \left|\frac{z_{n+1}-a_{n+1}}{\tau_\epsilon(z_n-a_n)}-1\right| = \frac{1}{\tau_\epsilon}\cdot\left|\frac{f_\epsilon(z_n)-f_\epsilon(a_n)}{z_n-a_n}-\tau_\epsilon\right|.$$

We apply Lemma 1.3 with $2S = 2R + n/2$. Then

$$\left|\frac{u_{n+1}(z)}{u_n(z)}-1\right| \le \frac{M}{\tau_\epsilon(R+n/4)^2} \le \frac{C}{(n+1)^2},$$

where $C = 16M$ and we may assume $R > 1/4$. Set $P := \prod_{n\ge1}(1 + C/n^2)$. Since $|u_{n+1}(z)/u_n(z)| \le 1 + C/(n+1)^2$, we have

$$|u_n(z)| = \left|\frac{u_n(z)}{u_{n-1}(z)}\right|\cdots\left|\frac{u_1(z)}{u_0(z)}\right|\cdot|u_0(z)| \le P|z-a|.$$

Hence

$$|u_{n+1}(z)-u_n(z)| = \left|\frac{u_{n+1}(z)}{u_n(z)}-1\right|\cdot|u_n(z)| \le \frac{CP}{(n+1)^2}\cdot|z-a|.$$

This implies that $u_\epsilon = u_0 + (u_1 - u_0) + \cdots = \lim u_n$ converges uniformly on compact subsets of E_{2R} and for all $\epsilon \in [0, 1]$. The univalence of u_ϵ is shown in the same way as [Mi, Lemma 10.10].

Next we check that $u_\epsilon(f_\epsilon(z)) = \tau_\epsilon u_\epsilon(z) + C_\epsilon$ with $C_\epsilon \to 1$ as $\epsilon \to 0$. One can easily check that $u_n(f_\epsilon(z)) = \tau_\epsilon u_{n+1}(z) + C_n$ where

$$C_n = \frac{a_{n+1}-a_n}{\tau_\epsilon^n} = \frac{(\tau_\epsilon-1)a_n}{\tau_\epsilon^n} + \frac{1+g_\epsilon(a_n)}{\tau_\epsilon^n}.$$

When $\tau_\epsilon = 1$, C_n tends to 1 since $|g_\epsilon(a_n)| \le M/|a_n| \le M/(2R + n/2)$. When $\tau_\epsilon > 1$, the last term of the equation above tends to 0. For $n \ge 1$, we have

$$a_n = \tau_\epsilon^n a + \frac{\tau_\epsilon^n - 1}{\tau_\epsilon - 1} + \sum_{k=0}^{n-1}\tau_\epsilon^{n-1-k}g_\epsilon(a_k).$$

Thus

$$\frac{(\tau_\epsilon-1)a_n}{\tau_\epsilon^n} = (\tau_\epsilon-1)\left(a+\frac{g_\epsilon(a)}{\tau_\epsilon}+\sum_{k=1}^{n-1}\frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right)+1-\frac{1}{\tau_\epsilon^n}.$$

Since $|g_\epsilon(a_k)| \le M/(2R + k/2) \le 2M/k$, we have

$$\left|\sum_{k=1}^{n-1}\frac{g_\epsilon(a_k)}{\tau_\epsilon^{k+1}}\right| \le \frac{2M}{\tau_\epsilon}\sum_{k=1}^{n-1}\frac{1}{k\tau_\epsilon^k} \le -2M\log(1-\frac{1}{\tau_\epsilon}).$$

and this implies that the sums above converge as $n \to \infty$. Hence $C_n \to C_\epsilon = 1 + O(\epsilon \log \epsilon)$.

Finally, by taking additional linear coordinate change by $z \mapsto z/C_\epsilon$, u_ϵ gives a desired holomorphic map. ■

Notes.

- One can check that $u_\epsilon(z) = z(C_\epsilon^{-1} + o(1))$ $(\mathrm{Re}\, z \to \infty)$.
- There is a quasiconformal version of linearization theorem by McMullen. [Mc, §8].

2 Applications.

This section is devoted for a worked out example to explain my personal motivation to consider the simultaneous linearization theorem.

Cauliflower. In the family of quadratic maps, the simplest parabolic fixed point is given by $g(z) = z + z^2$. Now we consider its perturbation of the form $f(z) = \lambda z + z^2$ with $\lambda \nearrow 1$. According to [Mi, §8 and §10], we have the following fact:

Proposition 2.1 (Königs and Fatou coordinates) *Let K_f and K_g be the filled Julia sets of f and g. Then we have the following:*

1. *There exists a unique holomorphic branched covering map $\phi_f : K_f^\circ \to \mathbb{C}$ satisfying the Schröder equation $\phi_f(f(z)) = \lambda\phi_f(z)$ and $\phi_f(0) = \phi_f(-\lambda/2) - 1 = 0$. ϕ_f is univalent near $z = 0$.*
2. *There exists a unique holomorphic branched covering map $\phi_g : K_g^\circ \to \mathbb{C}$ satisfying the Abel equation $\phi_g(g(z)) = \phi_g(z) + 1$ and $\phi_g(-1/2) = 0$. ϕ_g is univalent on a disk $|z + r| < r$ with small $r > 0$.*

Note that $-\lambda/2$ and $-1/2$ are the critical points of f and g respectively.

Observation. Set $w = \phi_f(z)$. Now the proposition above asserts that the action of $f|_{K_f^\circ}$ is semiconjugated to $w \mapsto \lambda w$ by ϕ_f. Let us consider a Möbius map $W = S_f(W) = \lambda(w-1)/(\lambda-1)w$ that sends $\{0, 1, \lambda\}$ to $\{\infty, 0, 1\}$ respectively. By taking the conjugation by S_f, the action of $w \mapsto \lambda w$ is viewed as $W \mapsto W/\lambda + 1$. Let us set $W = \Phi_f(z) := S_f \circ \phi_f(z)$. Now we have

$$\Phi_f(f(z)) = \Phi_f(z)/\lambda + 1 \quad \text{and} \quad \Phi_f(-\lambda/2) = 0.$$

On the other hand, by setting $W = \Phi_g(z) := \phi_g(z)$, we can view the action of $g|_{K_g^\circ}$ as $W \mapsto W + 1$. Thus we have

$$\Phi_g(g(z)) = \Phi_g(z) + 1 \quad \text{and} \quad \Phi_g(-1/2) = 0.$$

If λ tends to 1, that is, $f \to g$, the semiconjugated action in W-coordinate converges uniformly on compact sets. However, as one can see by referring the proof of the proposition in [Mi, §8 and §10], ϕ_f and ϕ_g are given in completely different ways thus we cannot conclude the convergence $\Phi_f \to \Phi_g$ a priori.

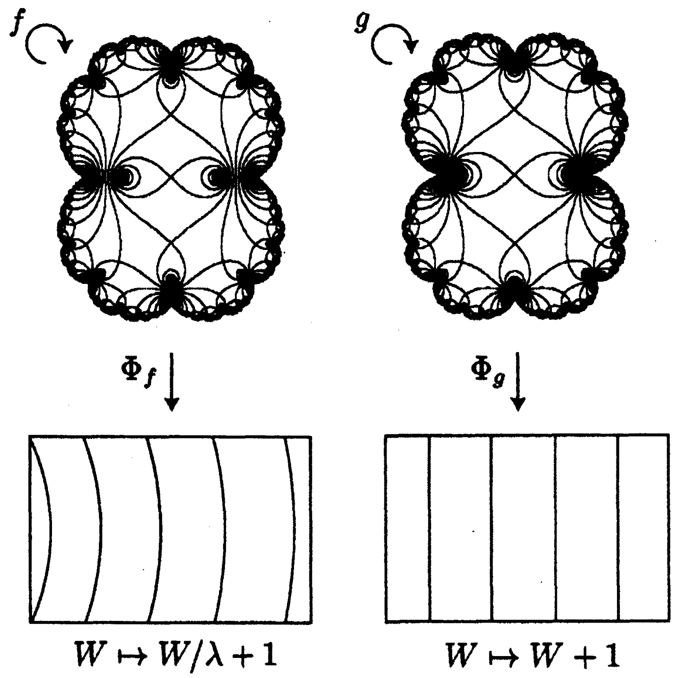

Figure 1: Semiconjugation inside the filled Julia sets

But there is another evidence that support this observation. Figure 1 shows the equipotential curves of ϕ_f and ϕ_g in the filled Julia sets. Obviously similar patterns appear and it seems one converges to the other.

Actually, we have the following fact:

Proposition 2.2 *For any compact set $E \subset K_g^\circ$,*

(1) $E \subset K_f^\circ$ for all $f \approx g$; and

(2) $\Phi_f \to \Phi_g$ uniformly on E as $f \to g$.

Here $f \approx g$ means that f is sufficiently close to g, equivalently, λ sufficiently close to 1. See [Ka1, Theorem 5.5] for more general version of this proposition, which is one of the key result to show the continuity of tessellation and pinching semiconjugacies constructed in [Ka1].

Proof. Let us take a general expression $f_\lambda(z) = \lambda z + z^2$ with $0 < \lambda \le 1$ (thus $f_1 = g$). By looking at the action of f_λ through a new coordinate $w = \chi_\lambda(z) = -\lambda^2/z$, we have

$$\chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}(w) \;=\; w/\lambda + 1 + O(1/w)$$

near ∞. Now we can set $\tau_\epsilon := 1/\lambda = 1+\epsilon$ and $f_\epsilon := \chi_\lambda \circ f_\lambda \circ \chi_\lambda^{-1}$ to have the same setting as Theorem 1.1. We consider that f and g are parameterized by λ or ϵ. (It is convenient to use both parameterization.)

Let us show (1): For any compact $E \subset K_g^\circ$ and small $r > 0$, there exists $N \gg 0$ such that $g^N(E) \subset P_r = \{|z+r| < r\}$. (For instance, one can show this fact by existence of the Fatou coordinate.) By uniform convergence, we have $f^N(E) \subset P_r$ for all $f \approx g$. To show $E \subset K_f^\circ$, it is enough to show that $f(P_r) \subset P_r$ for all $f \approx g$. Since $\chi_\lambda(P_r) = E_R$ for some $R \gg 0$, Lemma 1.2 implies that $E_R \subset f_\epsilon(E_R)$ independently of ϵ. This is equivalent to $f_\lambda(P_r) \subset P_r$ in a different coordinate. Thus we have (1).

Next let us check (2): Set $\Phi_\epsilon := \Phi_f$ and $\Phi_0 := \Phi_g$. Then we have $\Phi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Phi_\epsilon(z) + 1$. On the other hand, by simultaneous linearization, we have a uniform convergence $u_\epsilon \to u_0$ on E_R that satisfies $u_\epsilon(f_\epsilon(w)) = \tau_\epsilon u_\epsilon(w) + 1$. By setting $\Psi_\epsilon(z) := u_\epsilon \circ \chi_\lambda(z)$, we have $\Psi_\epsilon \to \Psi_0$ compact uniformly on P_r, and $\Psi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Psi_\epsilon(z) + 1$.

We need to adjust the images of critical orbits mapped by Φ_ϵ and Ψ_ϵ. Since $g^n(-1/2) \to 0$ along the real axis, there is an $M \gg 0$ such that $g^M(-1/2) =: a_0 \in P_r$. By uniform convergence, we also have $f^M(-\lambda/2) =: a_\epsilon \in P_r$ and $a_\epsilon \to a_0$ as $\epsilon \to 0$. Set $b_\epsilon := \Psi_\epsilon(a_\epsilon)$ and $c_\epsilon := \Phi_\epsilon(a_\epsilon)$ for all $\epsilon \ge 0$. Set also $\ell_\epsilon(W) = \tau_\epsilon W + 1$, then we have $c_\epsilon = \ell_\epsilon^M(0) = \tau_\epsilon^{M-1} + \cdots + \tau_\epsilon + 1$ and $c_\epsilon \to c_0 = M$ as $\epsilon \to 0$. When $\epsilon > 0$, we take an affine map T_ϵ that fixes $1/(1-\tau_\epsilon)$ and sends b_ϵ to c_ϵ. When $\epsilon = 0$, we take an affine map T_0 that is the translation by $b_0 - c_0$. Then one can check that $T_\epsilon \to T_0$

compact uniformly on the plane and $\tilde{\Phi}_\epsilon := T_\epsilon \circ \Psi_\epsilon$ satisfies $\tilde{\Phi}_\epsilon \to \tilde{\Phi}_0$ on any compact sets of P_r. Moreover, $\tilde{\Phi}_\epsilon$ still satisfies $\tilde{\Phi}_\epsilon(f_\lambda(z)) = \tau_\epsilon \tilde{\Phi}_\epsilon(z) + 1$ and the images of the critical orbit by Φ_ϵ and $\tilde{\Phi}_\epsilon$ agree. Finally by uniqueness of ϕ_f and ϕ_g, one can easily check that $\Phi_\epsilon = \tilde{\Phi}_\epsilon$ on P_r.

Since

$$\Phi_f(z) \;=\; \ell_\epsilon^{-N} \circ \tilde{\Phi}_\epsilon(f^N(z)) \;\longrightarrow\; \ell_0^{-N} \circ \tilde{\Phi}_0(g^N(z)) \;=\; \Phi_g(z)$$

uniformly on E, we have (2). ■

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