Simultaneous linearization and its application

Tomoki Kawahira (川平 友規)
Graduate School of Mathematics
Nagoya University

Abstract

This note gives a proof of Ueda's simultaneous linearization theorem with real multipliers and its simple application to quadratic dynamics. This note is based on my talk at RIMS on 5 October 2006, titled "A proof of simultaneous linearization with a polylog estimate."

1 Simultaneous linearization

Here we give an alternative proof of Ueda's simultaneous linearization in a simplified setting. For $R \geq 0$, let $E_R$ denote the region $\{z \in \mathbb{C} : \text{Re } z \geq R\}$.

Theorem 1.1 (Simultaneous Linearization) For $\epsilon \in [0,1]$, let $\{f_\epsilon\}$ be a family of holomorphic maps on $\{|z| \geq R > 0\}$ such that

$$f_\epsilon(z) = \tau_\epsilon z + 1 + O(1/z)$$
$$\rightarrow f_0(z) = z + 1 + O(1/z)$$

uniformly as $\epsilon \to 0$ where $\tau_\epsilon = 1 + \epsilon$. If $R \gg 0$, then for any $\epsilon \in [0,1]$ there exists a holomorphic map $u_\epsilon : E_R \rightarrow \mathbb{C}$ such that

$$u_\epsilon(f_\epsilon(z)) = \tau_\epsilon u_\epsilon(z) + 1$$

and $u_\epsilon \to u_0$ uniformly on compact sets of $E_R$. 
Indeed, a similar theorem holds for any radial (= non-tangential) convergence $\tau_\epsilon \to 1$ outside the unit disk. See Ueda's original proof ([Ue1], [Ue2]). Moreover, the error term $O(1/z)$ can be replaced by $O(|z|^{-\sigma})$ with $0 < \sigma \leq 1$. (See [Ka2].) Here we present a simplified proof only for real $\tau_\epsilon \to 1$ based on the argument of [Mi, Lemma 10.10]. The idea can be traced back at least to Leau's work on the Abel equation [L]. We first check:

**Lemma 1.2** If $R \gg 0$, there exists $M > 0$ such that $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/|z|$ on $\{|z| \geq R\}$ and $\text{Re} f_\epsilon(z) \geq \text{Re} z + 1/2$ on $E_R$ for any $\epsilon \in [0, 1]$.

**Proof.** The first inequality and the existence of $M$ is obvious. By replacing $R$ by larger one, we have $|f_\epsilon(z) - (\tau_\epsilon z + 1)| \leq M/R < 1/2$ on $E_R$. Then

$$\text{Re} f_\epsilon(z) \geq \text{Re} (\tau_\epsilon z + 1) - 1/2 \geq \text{Re} z + 1/2.$$

Let us fix such an $R \gg 0$. Next we show:

**Lemma 1.3** For any $\epsilon \in [0, 1]$ and $z_1, z_2 \in E_{2S}$ with $S > R$, we have:

$$\frac{f_\epsilon(z_1) - f_\epsilon(z_2)}{z_1 - z_2} - \tau_\epsilon \leq \frac{M}{S^2}.$$

**Proof.** Set $g_\epsilon(z) := f_\epsilon(z) - (\tau_\epsilon z + 1)$. For any $|z| \geq 2S$ and $w \in B(z, S)$, we have $|w| > S$. This implies $|g_\epsilon(w)| \leq M/|w| < M/S$ and thus $g_\epsilon$ maps $B(z, S)$ into $B(0, M/S)$. By the Cauchy integral formula or the Schwarz lemma, it follows that $|g_\epsilon'(z)| \leq (M/S)/S = M/S^2$ on $\{|z| \geq S\}$. Now the inequality easily follows by:

$$|g_\epsilon(z_1) - g_\epsilon(z_2)| = \left| \int_{[z_2, z_1]} g_\epsilon'(z)dz \right| \leq \int_{[z_2, z_1]} |g_\epsilon'(z)||dz| \leq \frac{M}{S^2}|z_1 - z_2|.$$

(Note that the segment $[z_2, z_1]$ is contained in $E_{2S} \subset \{|z| \geq 2S\}$.)

**Proof of Theorem 1.1.** Set $z_n := f_\epsilon^n(z)$ for $z \in E_{2R}$. Note that such $z_n$ satisfies

$$|z_n| \geq \text{Re} z_n \geq \text{Re} z + \frac{n}{2} \geq 2R + \frac{n}{2}$$

by Lemma 1.2. Now we fix $a \in E_{2R}$ and define $u_{n, \epsilon} = u_n : E_{2R} \to \mathbb{C}$ ($n \geq 0$) by

$$u_n(z) := \frac{z_n - a_n}{\tau_\epsilon^n}.$$
Then we have
\[ \left| \frac{u_{n+1}(z)}{u_n(z)} - 1 \right| = \left| \frac{z_{n+1} - a_{n+1}}{\tau_{\epsilon}(z_n - a_n)} - 1 \right| = \frac{1}{\tau_{\epsilon}} \left| \frac{f_{\epsilon}(z_n) - f_{\epsilon}(a_n)}{z_n - a_n} - \tau_{\epsilon} \right|. \]

We apply Lemma 1.3 with $2S = 2R + n/2$. Then
\[ \left| \frac{u_{n+1}(z)}{u_n(z)} - 1 \right| \leq \frac{M}{\tau_{\epsilon}(R + n/4)^2} \leq \frac{C}{(n+1)^2}, \]
where $C = 16M$ and we may assume $R > 1/4$. Set $P := \prod_{n\geq 1}(1 + C/n^2)$. Since $|u_{n+1}(z)/u_n(z)| \leq 1 + C/(n+1)^2$, we have
\[ |u_n(z)| = \left| \frac{u_n(z)}{u_{n-1}(z)} \right| \cdots \left| \frac{u_1(z)}{u_0(z)} \right| \cdot |u_0(z)| \leq P|z-a|. \]
Hence
\[ |u_{n+1}(z) - u_n(z)| = \left| \frac{u_{n+1}(z)}{u_n(z)} - 1 \right| \cdot |u_n(z)| \leq \frac{CP}{(n+1)^2} \cdot |z-a|. \]

This implies that $u_{\epsilon} = u_0 + (u_1 - u_0) + \cdots = \lim u_n$ converges uniformly on compact subsets of $E_{2R}$ and for all $\epsilon \in [0,1]$. The univalence of $u_{\epsilon}$ is shown in the same way as [Mi, Lemma 10.10].

Next we check that $u_{\epsilon}(f_{\epsilon}(z)) = \tau_{\epsilon} u_n(z) + C_{\epsilon}$ with $C_{\epsilon} \to 1$ as $\epsilon \to 0$. One can easily check that $u_n(f_{\epsilon}(z)) = \tau_{\epsilon} u_{n+1}(z) + C_n$ where
\[ C_n = \frac{a_{n+1} - a_n}{\tau_{\epsilon}^n} = \frac{(\tau_{\epsilon} - 1)a_n}{\tau_{\epsilon}^n} + \frac{1 + g_{\epsilon}(a_n)}{\tau_{\epsilon}^n}. \]
When $\tau_{\epsilon} = 1$, $C_n$ tends to 1 since $|g_{\epsilon}(a_n)| \leq M/|a_n| \leq M/(2R + n/2)$. When $\tau_{\epsilon} > 1$, the last term of the equation above tends to 0. For $n \geq 1$, we have
\[ a_n = \tau_{\epsilon}^n a + \frac{\tau_{\epsilon}^n - 1}{\tau_{\epsilon} - 1} + \sum_{k=0}^{n-1} \tau_{\epsilon}^{n-1-k} g_{\epsilon}(a_k). \]
Thus
\[ \frac{(\tau_{\epsilon} - 1)a_n}{\tau_{\epsilon}^n} = (\tau_{\epsilon} - 1) \left( a + \frac{g_{\epsilon}(a)}{\tau_{\epsilon}} + \sum_{k=1}^{n-1} \frac{g_{\epsilon}(a_k)}{\tau_{\epsilon}^{k+1}} \right) + 1 - \frac{1}{\tau_{\epsilon}^n}. \]

Since $|g_{\epsilon}(a_k)| \leq M/(2R + k/2) \leq 2M/k$, we have
\[ \left| \sum_{k=1}^{n-1} \frac{g_{\epsilon}(a_k)}{\tau_{\epsilon}^{k+1}} \right| \leq \frac{2M}{\tau_{\epsilon}} \sum_{k=1}^{n-1} \frac{1}{k \tau_{\epsilon}^k} \leq -2M \log(1 - \frac{1}{\tau_{\epsilon}}). \]
and this implies that the sums above converge as $n \to \infty$. Hence $C_n \to C_\epsilon = 1 + O(\epsilon \log \epsilon)$.

Finally, by taking additional linear coordinate change by $z \mapsto z/C_\epsilon$, $u_\epsilon$ gives a desired holomorphic map. 

Notes.

- One can check that $u_\epsilon(z) = z(C_\epsilon^{-1} + o(1))$ ($\Re z \to \infty$).

- There is a quasiconformal version of linearization theorem by McMullen. [Mc, §8].

## 2 Applications.

This section is devoted for a worked out example to explain my personal motivation to consider the simultaneous linearization theorem.

**Cauliflower.** In the family of quadratic maps, the simplest parabolic fixed point is given by $g(z) = z + z^2$. Now we consider its perturbation of the form $f(z) = \lambda z + z^2$ with $\lambda \nearrow 1$. According to [Mi, §8 and §10], we have the following fact:

**Proposition 2.1** (Königs and Fatou coordinates) Let $K_f$ and $K_g$ be the filled Julia sets of $f$ and $g$. Then we have the following:

1. There exists a unique holomorphic branched covering map $\phi_f : K_f^\circ \to \mathbb{C}$ satisfying the Schröder equation $\phi_f(f(z)) = \lambda \phi_f(z)$ and $\phi_f(0) = \phi_f(-\lambda/2) - 1 = 0$. $\phi_f$ is univalent near $z = 0$.

2. There exists a unique holomorphic branched covering map $\phi_g : K_g^\circ \to \mathbb{C}$ satisfying the Abel equation $\phi_g(g(z)) = \phi_g(z) + 1$ and $\phi_g(-1/2) = 0$. $\phi_g$ is univalent on a disk $|z + r| < r$ with small $r > 0$.

Note that $-\lambda/2$ and $-1/2$ are the critical points of $f$ and $g$ respectively.

**Observation.** Set $w = \phi_f(z)$. Now the proposition above asserts that the action of $f|_{K_f}$ is semiconjugated to $w \mapsto \lambda w$ by $\phi_f$. Let us consider a Möbius map $W = S_f(W) = \lambda(w - 1)/(\lambda - 1)w$ that sends $\{0, 1, \lambda\}$ to $\{\infty, 0, 1\}$ respectively. By taking
the conjugation by $S_f$, the action of $w \mapsto \lambda w$ is viewed as $W \mapsto W/\lambda + 1$. Let us set $W = \Phi_f(z) := S_f \circ \phi_f(z)$. Now we have

$$\Phi_f(f(z)) = \Phi_f(z)/\lambda + 1 \text{ and } \Phi_f(-\lambda/2) = 0.$$ 

On the other hand, by setting $W = \Phi_g(z) := \phi_g(z)$, we can view the action of $g|_{\mathbb{K}_g}$ as $W \mapsto W + 1$. Thus we have

$$\Phi_g(g(z)) = \Phi_g(z) + 1 \text{ and } \Phi_g(-1/2) = 0.$$ 

If $\lambda$ tends to 1, that is, $f \to g$, the semiconjugated action in $W$-coordinate converges uniformly on compact sets. However, as one can see by referring the proof of the proposition in [Mi, §8 and §10], $\phi_f$ and $\phi_g$ are given in completely different ways thus we cannot conclude the convergence $\Phi_f \to \Phi_g$ a priori.

![Diagram](image)

**Figure 1:** Semiconjugation inside the filled Julia sets

But there is another evidence that support this observation. Figure 1 shows the equipotential curves of $\phi_f$ and $\phi_g$ in the filled Julia sets. Obviously similar patterns appear and it seems one converges to the other.

Actually, we have the following fact:
Proposition 2.2 For any compact set $E \subset K^\circ_g$,

(1) $E \subset K^\circ_g$ for all $f \approx g$; and

(2) $\Phi_f \to \Phi_g$ uniformly on $E$ as $f \to g$.

Here $f \approx g$ means that $f$ is sufficiently close to $g$, equivalently, $\lambda$ sufficiently close to 1. See [Kal, Theorem 5.5] for more general version of this proposition, which is one of the key result to show the continuity of tessellation and pinching semiconjugacies constructed in [Kal].

Proof. Let us take a general expression $f_\lambda(z) = \lambda z + z^2$ with $0 < \lambda \leq 1$ (thus $f_1 = g$).

By looking at the action of $f_\lambda$ through a new coordinate $w = \chi_\lambda(z) = -\lambda^2/z$, we have

$$\chi_\lambda \circ f_\lambda \circ \chi^{-1}_\lambda(w) = w/\lambda + 1 + O(1/w)$$

near $\infty$. Now we can set $\tau_\epsilon := 1/\lambda = 1 + \epsilon$ and $f_\epsilon := \chi_\lambda \circ f_\lambda \circ \chi^{-1}_\lambda$ to have the same setting as Theorem 1.1. We consider that $f$ and $g$ are parameterized by $\lambda$ or $\epsilon$. (It is convenient to use both parameterization.)

Let us show (1): For any compact $E \subset K^\circ_g$ and small $r > 0$, there exists $N \gg 0$ such that $g^N(E) \subset P_r = \{ |z + r| < r \}$. (For instance, one can show this fact by existence of the Fatou coordinate.) By uniform convergence, we have $f^N(E) \subset P_r$ for all $f \approx g$. To show $E \subset K^\circ_f$, it is enough to show that $f(P_r) \subset P_r$ for all $f \approx g$. Since $\chi_\lambda(P_r) = E_R$ for some $R \gg 0$, Lemma 1.2 implies that $E_R \subset f_\epsilon(E_R)$ independently of $\epsilon$. This is equivalent to $f_\lambda(P_\epsilon) \subset P_r$ in a different coordinate. Thus we have (1).

Next let us check (2): Set $\Phi_\epsilon := \Phi_f$ and $\Phi_0 := \Phi_g$. Then we have $\Phi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Phi_\epsilon(z) + 1$. On the other hand, by simultaneous linearization, we have a uniform convergence $u_\epsilon \to u_0$ on $E_R$ that satisfies $u_\epsilon(f_\epsilon(w)) = \tau_\epsilon u_\epsilon(w) + 1$. By setting $\Psi_\epsilon(z) := u_\epsilon \circ \chi_\lambda(z)$, we have $\Psi_\epsilon \to \Psi_0$ compact uniformly on $P_r$, and $\Psi_\epsilon(f_\lambda(z)) = \tau_\epsilon \Psi_\epsilon(z) + 1$.

We need to adjust the images of critical orbits mapped by $\Phi_\epsilon$ and $\Psi_\epsilon$. Since $g^n(-1/2) \to 0$ along the real axis, there is an $M \gg 0$ such that $g^M(-1/2) =: a_0 \in P_r$. By uniform convergence, we also have $f^M(-\lambda/2) =: a_\epsilon \in P_r$ and $a_\epsilon \to a_0$ as $\epsilon \to 0$.

Set $b_\epsilon := \Psi_\epsilon(a_\epsilon)$ and $c_\epsilon := \Phi_\epsilon(a_\epsilon)$ for all $\epsilon \geq 0$. Set also $\ell_\epsilon(W) = \tau_\epsilon W + 1$, then we have $c_\epsilon = \ell_\epsilon^M(0) = \tau_\epsilon^{M-1} + \cdots + \tau_\epsilon + 1$ and $c_\epsilon \to c_0 = M$ as $\epsilon \to 0$. When $\epsilon > 0$, we take an affine map $T_\epsilon$ that fixes $1/(1 - \tau_\epsilon)$ and sends $b_\epsilon$ to $c_\epsilon$. When $\epsilon = 0$, we take an affine map $T_0$ that is the translation by $b_0 - c_0$. Then one can check that $T_\epsilon \to T_0$
compact uniformly on the plane and $\tilde{\Phi}_\epsilon := T_\epsilon \circ \Psi_\epsilon$ satisfies $\tilde{\Phi}_\epsilon \to \tilde{\Phi}_0$ on any compact sets of $P_r$. Moreover, $\tilde{\Phi}_\epsilon$ still satisfies $\tilde{\Phi}_\epsilon(f_\lambda(z)) = \tau_\epsilon \tilde{\Phi}_\epsilon(z) + 1$ and the images of the critical orbit by $\Phi_\epsilon$ and $\tilde{\Phi}_\epsilon$ agree. Finally by uniqueness of $\phi_f$ and $\phi_g$, one can easily check that $\Phi_\epsilon = \tilde{\Phi}_\epsilon$ on $P_r$.

Since
\[
\Phi_f(z) = \ell_\epsilon^{-N} \circ \tilde{\Phi}_\epsilon(f^N(z)) \to \ell_0^{-N} \circ \tilde{\Phi}_0(g^N(z)) = \Phi_g(z)
\]
uniformly on $E$, we have (2).

Acknowledgement. I would like to thank T. Ueda for correspondence. This research is partially supported by Inamori Foundation and JSPS.

References


