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Some topics on Fatou maps in higher dimensional complex dynamics

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This is the abstract of my talk in the conference held at RIMS, October 2-6 2006. The results obtained in [M] and recent related results will be explained.

We study *Fatou maps* for a holomorphic map in a compact complex manifold. Fatou maps were first introduced by Ueda in his research on dynamics in the complex projective space $\mathbb{P}^k$. (Fornæss & Sibony also considered such a notion in an implicit way.)

Let M be a compact complex manifold of dimension $k \geq 1$ and let f be a holomorphic self-map of M.

Definition 0.1. (Fatou maps) Let N be a complex manifold and let $\psi : N \to M$ be a holomorphic map such that $\{f^n \circ \psi\}_{n\geq 0}$ is a normal family in N. We call such ψ a Fatou map. Particularly, in case when ψ is a holomorphic disc, we call it a Fatou disc. We say that a map $\phi : N \to M$ is a limit map of $\{f^n \circ \psi\}_{n\geq 0}$ if there is a subsequence of $\{f^n \circ \psi\}_{n\geq 0}$ which converges to ϕ locally uniformly in N.

We treat two topics on Fatou maps as follows.

1 Stable dynamics in the whole space

Let M be a compact complex manifold of dimension $k \geq 1$ and let f be a holomorphic self-map of M. Since M is compact, the Remmert proper mapping theorem implies that $f^n(M)$ is an analytic subset of M for all $n \geq 0$ and there exists a number $m_0 \geq 0$ such that

$$f^{m_0}(M) = f^{m_0+1}(M) = \cdots .$$

We put $S := f^{m_0}(M)$ and call it *the minimal image*. Denoting by g the restriction of f on S, the map g is a surjective holomorphic self-map of S.

In this section, we treat the case when $\{f^n\}_{n\geq 1}$ is a normal family in M, i.e. the case when the identity id_M is a Fatou map. By using Bochner-Montgomery theorem, we can obtain the following criterion.

Theorem 1.1. *([M])* $\{f^n\}_{n\geq 1}$ *is a normal family in* M *if and only if* $\{f^n\}_{n\geq 1}$ *has at least one subsequence which converges uniformly in* M.

By showing that S is a holomorphic retract, the next proposition follows.

Proposition 1.2. *([M]) Suppose that* $\{f^n\}_{n\geq 1}$ *is a normal family in* M. *Then,* S *has no singular points, i.e.* S *is a complex submanifold in* M.

Next, we consider the number of periodic points of f. We denote by $\mathrm{Fix}(f^n)$ the set of fixed points of f^n and put

$$\mathrm{Per}(f) := \bigcup_{n\geq 1} \mathrm{Fix}(f^n).$$

The following theorem shows that the total number of periodic points of f is independent of f and it is regulated by the Euler characteristic $\chi(M)$.

Theorem 1.3. *Let* f *be a holomorphic automorphism of* M. *Suppose* $\{f^n\}_{n\geq 1}$ *is a normal family in* M *and* $\sharp\mathrm{Fix}(f^n) < +\infty$ *for all* $n \geq 1$. *Then,*

$$\sharp\mathrm{Per}(f) = \chi(M).$$

Example 1.4. *We regard* $f(x,y) = (e^{\beta}y, e^{\alpha}x)$ *as a holomorphic self-map of* $\mathbb{P}^1 \times \mathbb{P}^1$. *Suppose* $\frac{\alpha+\beta}{2\pi i} \in \mathbb{R} \setminus \mathbb{Q}$. *Then,* $(0,0), (\infty,\infty)$ *are fixed points,* $(0,\infty), (\infty,0)$ *are period 2 points and there are no other periodic points. Hence,* $\sharp\mathrm{Per}(f) = 4 = \chi(\mathbb{P}^1 \times \mathbb{P}^1)$.

2 Semi-repellers outside the post-critical set

In this section, we describe semi-repelling property of forward invariant compact sets which are outside the closure of the post-critical set in terms of repelling points and non-contracting Fatou discs.

Let M be a compact complex manifold of dimension $k \geq 1$ with a hermitian metric $|\cdot|$. Let $f : M \to M$ be a surjective holomorphic map. We denote by C the set of critical points of f and put

$$D := \overline{\bigcup_{n\geq 1} f^n(C)}.$$

Definition 2.1. (Non-contracting Fatou discs) Let ψ be a Fatou disc for f. We say that ψ is non-contracting if no limit map of $\{f^n \circ \psi\}_{n\geq 0}$ is constant.

Definition 2.2. (Repelling points) Let $p \in M$. Denote by $\mathbf{T}_p$ the holomorphic tangent space at p. We say that p is repelling for f if $\min_{v\in \mathbf{T}_p, |v|=1} |D(f^j)(v)| \to +\infty$ as $j \to +\infty$.

Let Δ denote the unit disc.

Theorem 2.3. *([M]) Let E be a compact subset in M such that $f(E) \subset E$ and $E \cap D = \emptyset$. Suppose that each connected component of $M \setminus D$ which meets E is hyperbolically embedded in M. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties;*

(i) $E^u \cup E^c = E$, $E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u$, $f(E^c) \subset E^c$;

(iii) Each point in E^u is repelling;

(iv) For each $p \in E^c$, there is a non-contracting Fatou disc $\psi : \Delta \to M$ such that ψ is an embedding and $\psi(0) = p$.

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then E is a repeller with respect to the hermitian metric.

Remark 2.4. In Theorem 2.3, the hyperbolicity condition can not be removed.

In case when f is a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2, we can remove the hyperbolicity condition in Theorem 2.3, thanks to Ueda's normality criterion.

Theorem 2.5. *([M]) Let f be a holomorphic self-map of $\mathbb{P}^k$ of degree at least 2. Let E be a compact subset in M such that $f(E) \subset E$ and $E \cap D = \emptyset$. Then, there are two subsets $E^u, E^c \subset E$ which have the following properties;*

(i) $E^u \cup E^c = E$, $E^u \cap E^c = \emptyset$;

(ii) $f(E^u) \subset E^u$, $f(E^c) \subset E^c$;

(iii) Each point in E^u is repelling;

(iv) For each $p \in E^c$, there is a non-contracting Fatou disc $\psi : \Delta \to M$ such that ψ is an embedding and $\psi(0) = p$.

Moreover, if $f(E) = E$ and $E^c = \emptyset$, then E is a repeller with respect to the Fubini-Study metric.

Here we can find an interesting question.

Question. Let f, E be the same as in Theorem 2.5. When E is the support of the Green measure, E^c is empty?

This is still unsolved at present.

References

[M] K.MAEGAWA, On Fatou maps into compact complex manifolds, *Ergod. Th. & Dynam. Sys.,* **25** , 2005, 1551-1560.

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