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Some Families of Analytic Functions of Complex Order

H. M. Srivastava

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia V8W 3P4, Canada
E-Mail: harimsri@math.uvic.ca

Abstract

Our *main* objective in this lecture is to present some interesting recent developments concerning inclusion relationships, coefficient bounds, and neighborhood properties associated with certain families of univalent and p-valent analytic functions of *complex* order. Some of the various analytic and multivalent function classes, which are considered in this lecture, are defined by means of the familiar Ruscheweyh derivative operator and its suitably *extended* version applicable to p-valently analytic functions. Several corollaries and consequences of the main results, including relationships with known results, will also be considered briefly.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{T}(n,p)$ denote the class of (*normalized*) functions f of the form:

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k \, z^k \tag{1.1}$$

$$(a_k \geqq 0;\ k \in \mathbb{N} \setminus \{1, \cdots, n+p-1\};\ n,p \in \mathbb{N};\ \mathbb{N} := \{1,2,3,\cdots\}),$$

which are *analytic* and *p-valent* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Throughout this presentation, we shall make use of the following *simplified* notations:

$$\mathcal{T}(n,1) =: \mathcal{T}(n),\ \mathcal{T}(1,p) =: \mathcal{T}_p \quad \text{and} \quad \mathcal{T}(1,1) = \mathcal{T}_1 = \mathcal{T}(1) =: \mathcal{T}.$$

Following the earlier investigations by Goodman [10] and Ruscheweyh [18], we *first* define the (n,δ)-neighborhood of a function $f \in \mathcal{T}(n)$ by (see also [2], [3], [4], and [20])

$$N_{n,\delta}(f) := \left\{ g : g \in \mathcal{T}(n),\ g(z) = z - \sum_{k=n+1}^{\infty} b_k \, z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leqq \delta \right\}. \tag{1.2}$$

In particular, for the *identify* function

$$e(z) = z, \tag{1.3}$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g : g \in \mathcal{T}(n),\ g(z) = z - \sum_{k=n+1}^{\infty} b_k \, z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leqq \delta \right\}. \tag{1.4}$$

The above concept of (n,δ)-neighborhoods was extended and applied recently to families of *analytically* multivalent functions by Altintaş *et al.* [6] and to families of *meromorphically* multivalent functions by Liu and Srivastava ([12] and [13]) (see also the more recent works [17] and [23]). Thus, more generally, we can also define the (n,δ)-neighborhood of a function $f(z) \in \mathcal{T}(n,p)$ $(p \in \mathbb{N})$ by means of the following equation:

$$\mathcal{N}_{n,\delta}(f;p) := \left\{ g : g \in \mathcal{T}(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k \, z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k\,|a_k - b_k| \leqq \delta \right\}, \tag{1.5}$$

so that, obviously,

$$\mathcal{N}_{n,\delta}(h;p) := \left\{ g : g \in \mathcal{T}(n,p) : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k \, z^k \quad \text{and} \quad \sum_{k=n+p}^{\infty} k\,|b_k| \leqq \delta \right\}, \tag{1.6}$$

where [*cf.* Equation (1.3) above]

$$h(z) = z^p \qquad (p \in \mathbb{N}) \tag{1.7}$$

denotes the corresponding *identity* function.

In Sections 2 and 3 of this presentation, we propose to investigate the (n,δ)-neighborhoods of several subclasses of the class $\mathcal{T}(n)$ of *normalized* analytic and univalent functions in $\mathbb{U}$ with negative and missing coefficients, which are introduced here by making use of the Ruscheweyh derivative operator defined by (1.14) or (1.15) below. The rest of this paper deals mainly with the coefficient bounds and inclusion relationships involving the (n,δ)-neighborhoods $N_{n,\delta}(h;p)$ and $N_{n,\delta}(f;p)$ for two other subclasses of the function class $\mathcal{T}(n,p)$, which are introduced in Section 4 of this presentation.

First of all, we say that a function $f \in \mathcal{T}(n)$ is *starlike of complex order* γ $(\gamma \in \mathbb{C}\setminus\{0\})$, that is, $f \in \mathcal{S}_n^\star(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\right) > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C}\setminus\{0\}). \tag{1.8}$$

Furthermore, a function $f \in \mathcal{T}(n)$ is said to be *convex of complex order* γ $(\gamma \in \mathbb{C}\setminus\{0\})$, that is, $f \in \mathcal{C}_n(\gamma)$, if it also satisfies the following inequality:

$$\Re\left(1+\frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right) > 0 \quad (z \in \mathbb{U};\ \gamma \in \mathbb{C}\setminus\{0\}). \tag{1.9}$$

The classes $\mathcal{S}_n^\star(\gamma)$ and $\mathcal{C}_n(\gamma)$ stem essentially from the classes of starlike and convex functions of *complex* order, which were considered earlier by Nasr and Aouf [15] and Wiatrowski [24], respectively (see also [5], [7], [8], [14] and [16] *and* the relevant other citations in *each* of these works).

Let $\mathcal{S}_n(\gamma,\alpha,\mu,\beta)$ denote the subclass of the function class $\mathcal{T}(n)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\left|\frac{1}{\gamma}\left(\frac{\alpha\mu\, z^3 f'''(z)+(2\alpha\mu+\alpha-\mu)z^2 f''(z)+zf'(z)}{\alpha\mu\, z^2 f''(z)+(\alpha-\mu)zf'(z)+(1-\alpha+\mu)f(z)}-1\right)\right| < \beta \tag{1.10}$$

$$(z \in \mathbb{U};\ \gamma \in \mathbb{C}\setminus\{0\};\ 0 \leqq \mu \leqq \alpha;\ 0 < \beta \leqq 1).$$

Suppose also that $\mathcal{R}_n(\gamma,\alpha,\mu,\beta)$ denotes the subclass of the function class $\mathcal{T}(n)$ consisting of functions $f(z)$ which satisfy the the following inequality:

$$\left|\frac{1}{\gamma}\left(\alpha\mu\, z^2 f'''(z)+(2\alpha\mu+\alpha-\mu)zf''(z)+f'(z)-1\right)\right| < \beta \tag{1.11}$$

$$(z \in \mathbb{U};\ \gamma \in \mathbb{C}\setminus\{0\};\ 0 \leqq \mu \leqq \alpha;\ 0 < \beta \leqq 1).$$

The classes $\mathcal{S}_n(\gamma,\alpha,\mu,\beta)$ and $\mathcal{R}_n(\gamma,\alpha,\mu,\beta)$ were studied recently by Orhan and Kamali [16].

Next, for the functions $f_j(z)$ $(j=1,2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j}\, z^k \quad (j=1,2), \tag{1.12}$$

we denote by $(f_1 \star f_2)(z)$ the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 \star f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1}\, a_{k,2}\, z^k =: (f_2 \star f_1)(z). \tag{1.13}$$

Thus the Ruscheweyh derivative operator

$$D^\lambda : \mathcal{T} \to \mathcal{T} \quad (\mathcal{T} := \mathcal{T}(1) = \mathcal{T}_1 = \mathcal{T}(1,1))$$

is defined by

$$D^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} \star f(z) \quad (\lambda > -1;\ f \in \mathcal{T}). \tag{1.14}$$

or, equivalently, by

$$D^\lambda f(z) := z - \sum_{k=2}^{\infty} \binom{\lambda + k - 1}{k-1} a_k\, z^k \quad (\lambda > -1;\ f \in \mathcal{T}) \tag{1.15}$$

for a function $f \in \mathcal{T}$ of the form (1.1). Here, and in what follows, we make use of the following standard notation for a binomial coefficient:

$$\binom{\kappa}{n} := \frac{\kappa(\kappa-1)\cdots(\kappa-n+1)}{n!} \quad (\kappa \in \mathbf{C};\ n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}). \tag{1.16}$$

In particular, we have

$$D^n f(z) = \frac{z\big(z^{n-1} f(z)\big)^{(n)}}{n!} \quad (n \in \mathbf{N}_0). \tag{1.17}$$

Finally, in terms of the Ruscheweyh derivative operator D^λ $(\lambda > -1)$ defined by (1.14) or (1.15) above, let $\mathcal{S}_n(\gamma, \lambda, \alpha, \mu, \beta)$ denote the subclass of the function class $\mathcal{T}(n)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\left| \frac{1}{\gamma} \left(\frac{\alpha\mu z^3 \big(D^\lambda f(z)\big)''' + (2\alpha\mu + \alpha - \mu) z^2 \big(D^\lambda f(z)\big)'' + z\big(D^\lambda f(z)\big)'}{\alpha\mu z^2 \big(D^\lambda f(z)\big)'' + (\alpha - \mu) z \big(D^\lambda f(z)\big)' + (1 - \alpha + \mu) D^\lambda f(z)} - 1 \right| < \beta \tag{1.18}$$

$$(z \in \mathbf{U};\ \gamma \in \mathbf{C} \setminus \{0\};\ \lambda > -1;\ 0 < \beta \leqq 1;\ 0 \leqq \mu \leqq \alpha).$$

Also let $\mathcal{R}_n(\gamma, \lambda, \alpha, \mu, \beta)$ denote the subclass of the function class $\mathcal{T}(n)$ consisting of functions $f(z)$ which satisfy the following inequality:

$$\left| \frac{1}{\gamma} \left(\alpha\mu z^2 \big(D^\lambda f(z)\big)''' + (2\alpha\mu + \alpha - \mu) z \big(D^\lambda f(z)\big)'' + \big(D^\lambda f(z)\big)' - 1 \right| < \beta \tag{1.19}$$

$$(z \in \mathbf{U};\ \gamma \in \mathbf{C} \setminus \{0\};\ \lambda > -1;\ 0 < \beta \leqq 1;\ 0 \leqq \mu \leqq \alpha).$$

Various *further* subclasses of the function class $\mathcal{S}_n(\gamma, \lambda, \alpha, \mu, \beta)$ with

$$\gamma = 1 \quad \text{and} \quad \alpha = \mu = 0 \tag{1.20}$$

were studied in many earlier works (*cf.*, *e.g.*, [9], [11], [21] and [22]; see also the references cited in each of these earlier works). Clearly, in these cases of (for example) the class $\mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta)$, we have the following relationships:

$$\mathcal{S}_n(\gamma,0,0,0,1) \subset \mathcal{S}_n^*(\gamma) \quad \text{and} \quad \mathcal{S}_n(\gamma,0,1,0,1) \subset \mathcal{C}_n(\gamma) \qquad (1.21)$$
$$(n \in \mathbb{N};\ \gamma \in \mathbb{C} \setminus \{0\}).$$

2. Inclusion Relationships Involving the (n,δ)-Neighborhood $N_{n,\delta}(e)$

In our investigation of the inclusion relationships involving the (n,δ)-neighborhood $N_{n,\delta}(e)$ defined by (1.4), we shall require the following lemmas.

Lemma 1. *Let $f \in \mathcal{T}(n)$ be defined by* (1.1) (*with $p=1$*). *Then f is in the class $\mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} \binom{\lambda+k-1}{k-1} \eta(k)\, a_k \leqq \beta|\gamma|, \qquad (2.1)$$

where

$$\eta = \eta(k) := \Big(\alpha\mu k^3 + (\alpha-\mu-2\alpha\mu+\alpha\mu\beta|\gamma|)k^2 + \big(\alpha\mu-2\alpha-2\mu+1+(\alpha-\mu-\alpha\mu)\beta|\gamma|\big)k + (1-\alpha+\mu)(\beta|\gamma|-1)\Big).$$

Proof. We first suppose that $f \in \mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta)$. Then, by appealing to the condition (1.18), we readily find that

$$\Re\left(\frac{\alpha\mu z^3\big(D^\lambda f(z)\big)''' + (2\alpha\mu+\alpha-\mu)z^2\big(D^\lambda f(z)\big)'' + z\big(D^\lambda f(z)\big)'}{\alpha\mu z^2\big(D^\lambda f(z)\big)'' + (\alpha-\mu)z\big(D^\lambda f(z)\big)' + (1-\alpha+\mu)D^\lambda f(z)} - 1\right) > -\beta|\gamma| \qquad (z \in \mathbb{U}) \qquad (2.2)$$

or, equivalently, that

$$\Re\left(\frac{-\sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}\left[\alpha\mu k^3 + (\alpha-\mu-2\alpha\mu)k^2 + (\alpha\mu-2\alpha+2\mu+1)k - (1-\alpha+\mu)\right] a_k\, z^k}{z - \sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}\left[\alpha\mu k^2 + (\alpha-\mu-\alpha\mu)k + (1-\alpha+\mu)\right] a_k\, z^k}\right) > -\beta|\gamma| \qquad (z \in \mathbb{U}), \qquad (2.3)$$

where we have made use of the explicit representation (1.15) and the definition (1.1) (with $p=1$). We now choose values of z on the real axis and let $z \to 1-$ through *real* values. Then the inequality (2.3) immediately yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find that

$$\left|\frac{\alpha\mu z^3(D^\lambda f(z))''' + (2\alpha\mu+\alpha-\mu)z^2(D^\lambda f(z))''' + z(D^\lambda f(z))'}{\alpha\mu z^2(D^\lambda f(z))'' + (\alpha-\mu)z(D^\lambda f(z))' + (1-\alpha+\mu)D^\lambda f(z)} - 1\right|$$

$$= \left|\frac{\sum\limits_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}[\alpha\mu k^3 + (\alpha-\mu-2\alpha\mu)k^2 + (\alpha\mu-2\alpha+2\mu+1)k - (1-\alpha+\mu)]\,a_k\,z^k}{1-\sum\limits_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}[\alpha\mu k^2 + (\alpha-\mu-\alpha\mu)k + (1-\alpha+\mu)]\,a_k\,z^k}\right|$$

$$\leqq \frac{\beta|r|\left[1-\sum\limits_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}[\alpha\mu k^2 + (\alpha-\mu-\alpha\mu)k + (1-\alpha+\mu)]\,a_k\right]}{1-\sum\limits_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}[\alpha\mu k^2 + (\alpha-\mu-\alpha\mu)k + (1-\alpha+\mu)]\,a_k}$$

$$\leqq \beta|\gamma|. \tag{2.4}$$

Hence, by the *maximum modulus principle*, we have

$$f \in \mathcal{S}_n(\gamma, \lambda, \alpha, \mu, \beta),$$

which evidently completes the proof of Lemma 1.

Similarly, we can prove the following result.

Lemma 2. *Let the function* $f \in \mathcal{T}(n)$ *be defined by* (1.1) (*with* $p = 1$). *Then* f *is in the class* $\mathcal{R}_n(\gamma, \lambda, \alpha, \mu, \beta)$ *if only if*

$$\sum_{k=n+1}^{\infty}\binom{\lambda+k-1}{k-1}\left[\alpha\mu k^3 + (\alpha-\mu-\alpha\mu)k^2 + (1-\alpha+\mu)k\right]a_k \leqq \beta|\gamma|. \tag{2.5}$$

Remark 1. A special case of Lemma 1 when

$$n = 1,\ \mu = \alpha = 0,\ \gamma = 1 \quad \text{and} \quad \beta = 1 - c \quad (0 \leqq c < 1)$$

was given by Ahuja [1]. Furthermore, in Lemma 1 with

$$n = 1,\ \mu = \alpha = 0,\ \gamma = 1 \quad \text{and} \quad \beta = 1 - c \quad (0 \leqq c < 1),$$

if we set

$$\lambda = 0 \quad \text{and} \quad \lambda = 1.$$

we obtain the relatively more familiar results of Silverman [19].

Our first main result is given by Theorem 1 below.

Theorem 1. *If*

$$\delta := \frac{(n+1)\beta|\gamma|}{\binom{\lambda+n}{n}\rho}, \tag{2.6}$$

then

$$\mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta) \subset N_{n,\delta}(e), \tag{2.7}$$

where

$$\begin{aligned}\rho := \Big[&\alpha\mu(n+1)^3 + (\alpha\mu\beta|\gamma| + \alpha - \mu - 2\alpha\mu)(n+1)^2 \\ &+ \big((\alpha-\mu-\alpha\mu)\beta|\gamma| + 1 - 2\alpha + 2\mu + \alpha\mu\big)(n+1) \\ &+ (1-\alpha+\mu)(\beta|\gamma|-1)\Big].\end{aligned} \tag{2.8}$$

Proof. For a function $f \in \mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta)$ of the form (1.1) (with $p=1$) and for ρ defined already by (2.8), Lemma 1 immediately yields

$$\binom{\lambda+n}{n}\rho\sum_{k=n+1}^{\infty} a_k \leqq \beta|\gamma|,$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{\beta|\gamma|}{\binom{\lambda+n}{n}\rho}. \tag{2.9}$$

On the other hand, we also find from (2.1) that

$$\binom{\lambda+n}{n}\tau\sum_{k=n+1}^{\infty} k a_k \leqq \beta|\gamma|,$$

where

$$\begin{aligned}\tau = \Big[&\alpha\mu(n+1)^2 + (\alpha\mu\beta|\gamma| + \alpha - \mu - 2\alpha\mu)(n+1) \\ &+ \big((\alpha-\mu-\alpha\mu)\beta|\gamma| + 1 - 2\alpha + 2\mu + \alpha\mu\big) \\ &+ \left(\frac{(1-\alpha+\mu)(\beta|\gamma|-1)}{n+1}\right)\Big],\end{aligned} \tag{2.10}$$

that is, that

$$\sum_{k=n+1}^{\infty} k a_k \leqq \frac{\beta|\gamma|(n+1)}{\binom{\lambda+n}{n}\rho} := \delta, \tag{2.11}$$

which, in view of the definition (1.4), proves Theorem 1.

In a similar manner, by applying Lemma 2 instead of Lemma 1, we can prove Theorem 2 below.

Theorem 2. *If*

$$\delta := \frac{\beta|\gamma|}{\binom{\lambda+n}{n}\left[\alpha\mu(n+1)^2+(\alpha-\mu-\alpha\mu)(n+1)+(1-\alpha+\mu)\right]}, \tag{2.12}$$

then

$$\mathcal{R}_n(\gamma,\lambda,\alpha,\mu,\beta) \subset N_{n,\delta}(e).$$

3. Neighborhood Properties for the Function Classes $\mathcal{S}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta)$ and $\mathcal{R}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta)$

In this section, we determine the neighborhood for each of the function classes

$$\mathcal{S}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta) \quad \text{and} \quad \mathcal{R}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta),$$

which we define here as follows.

Definition 1. A function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{S}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta)$ if there exists a function

$$g \in \mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta)$$

such that the following inequality holds true:

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-b \quad (z \in \mathbf{U};\ 0 \leqq b < 1). \tag{3.1}$$

Definition 2. A function $f \in \mathcal{T}(n)$ is said to be in the class $\mathcal{R}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta)$ if there exists a function

$$g \in \mathcal{R}_n(\gamma,\lambda,\alpha,\mu,\beta)$$

such that the inequality (3.1) holds true.

Theorem 3. *If* $g \in \mathcal{S}_n(\gamma,\lambda,\alpha,\mu,\beta)$ *and*

$$b = 1 - \frac{\binom{\lambda+n}{n}\delta\rho}{(n+1)\left[\binom{\lambda+n}{n}\rho-\beta|\gamma|\right]}, \tag{3.2}$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(b)}(\gamma,\lambda,\alpha,\mu,\beta), \tag{3.3}$$

where ρ *is given already by* (2.8).

Proof. Assuming that $f \in N_{n,\delta}(g)$, we find from the definition (1.2) that

$$\sum_{k=n+1}^{\infty} k|a_k-b_k| \leqq \delta, \tag{3.4}$$

which readily implies the following coefficient inequality:

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leqq \frac{\delta}{n+1} \quad (n \in \mathbb{N}). \tag{3.5}$$

Since $g \in \mathcal{S}_n(\gamma, \lambda, \alpha, \mu, \beta)$, we have [*cf.* Equation (2.9)].

$$\sum_{k=n+1}^{\infty} b_k = \frac{\beta|\gamma|}{\binom{\lambda+n}{n}\rho}, \tag{3.6}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=n+1}^{\infty} |a_k - b_k|}{1 - \sum_{k=n+1}^{\infty} b_k}$$

$$\leqq \frac{\delta}{n+1} \cdot \frac{\binom{\lambda+n}{n}\delta\rho}{\left[\binom{\lambda+n}{n}\rho - \beta|\gamma|\right]} =: 1 - b, \tag{3.7}$$

provided that b is given precisely by (3.2). Thus, by Definition 1, we conclude that

$$f \in \mathcal{S}_n^{(b)}(\gamma, \lambda, \alpha, \mu, \beta)$$

for b given by (3.2). This evidently completes the proof of Theorem 3.

The proof of Theorem 4 below is much akin to that of Theorem 3, and so the details involved are being omitted here.

Theorem 4. *If* $g \in \mathcal{R}_n(\gamma, \lambda, \alpha, \mu, \beta)$ *and*

$$b = 1 - \frac{\binom{\lambda+n}{n}\delta\left[\alpha\mu(n+1)^3 + (\alpha - \mu - \alpha\mu)(n+1)^2 + (1 - \alpha + \mu)(n+1)\right]}{(n+1)\left[\binom{\lambda+n}{n}\left[\alpha\mu(n+1)^3 + (\alpha - \mu - \alpha\mu)(n+1)^2 + (1 - \alpha + \mu)(n+1)\right] - \beta|\gamma|\right]}, \tag{3.8}$$

then

$$N_{n,\delta}(g) \subset \mathcal{R}_n^{(b)}(\gamma, \lambda, \alpha, \mu, \beta). \tag{3.9}$$

Remark 2. A special case of Theorem 3 when $\alpha = \mu = 0$ was proven recently by Murugusundaramoorthy and Srivastava [14].

4. A Set of Coefficient Bounds for the Function Classes $\mathcal{H}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b; \mu)$

With a view to introducing the function classes

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu),$$

we begin by considering the Hadamard product (or convolution) of the function $f \in \mathcal{T}(n,p)$ given by (1.1) and the function $g \in \mathcal{T}(n,p)$ given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \quad (b_k \geqq 0;\ n, p \in \mathbf{N}), \tag{4.1}$$

which is defined (*as usual*) by

$$(f * g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{4.2}$$

We next introduce an *extended* linear derivative operator of the Ruscheweyh type given already by (1.11) or (1.12) above:

$$\mathcal{D}^{\lambda,p} : \mathcal{T}_p \to \mathcal{T}_p \quad (\mathcal{T}_p := \mathcal{T}(1,p)),$$

which is defined here by the following convolution:

$$\mathcal{D}^{\lambda,p} f(z) := \frac{z^p}{(1-z)^{\lambda+p}} * f(z) \quad (\lambda > -p;\ f \in \mathcal{T}_p). \tag{4.3}$$

In terms of the binomial coefficients in (1.16), we can rewrite (4.3) as follows:

$$\mathcal{D}^{\lambda,p} f(z) = z^p - \sum_{k=1+p}^{\infty} \binom{\lambda+k-1}{k-p} a_k z^k \quad (\lambda > -p;\ f \in \mathcal{T}_p). \tag{4.4}$$

In particular, when $\lambda = n$ $(n \in \mathbf{N})$, it is easily observed from (4.3) and (4.4) that

$$\mathcal{D}^{n,p} f(z) = \frac{z^p \left(z^{n-p} f(z)\right)^{(n)}}{n!} \quad (n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\};\ p \in \mathbf{N}), \tag{4.5}$$

so that

$$\mathcal{D}^{1,p} f(z) = (1-p) f(z) + z f'(z), \tag{4.6}$$

$$\mathcal{D}^{2,p} f(z) = \frac{(1-p)(2-p)}{2!} f(z) + (2-p) z f'(z) + \frac{z^2}{2!} f''(z), \tag{4.7}$$

and so on. In fact, by comparing the definitions (1.14) and (4.3), we readily have

$$\mathcal{D}^{\lambda,1} f(z) =: D^{\lambda} f(z) \quad (\lambda > -1;\ f \in \mathcal{T}). \tag{4.8}$$

By using this *extended* Ruscheweyh derivative operator

$$\mathcal{D}^{\lambda,p} f(z) \quad (\lambda > -p;\ p \in \mathbf{N})$$

given by (4.4), we now introduce a new subclass $\mathcal{H}^p_{n,m}(\lambda, b)$ of the p-valently analytic function class $\mathcal{T}(n,p)$, which includes functions $f(z)$ satisfying the following inequality:

$$\left|\frac{1}{b}\left(\frac{z\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m+1)}}{\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m)}} - (p-m)\right)\right| < 1 \tag{4.9}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ \lambda \in \mathbb{R};\ p > \max(m, -\lambda);\ b \in \mathbb{C} \setminus \{0\}).$$

We also denote by $\mathcal{L}^p_{n,m}(\lambda, b; \mu)$ the subclass of $\mathcal{T}(n,p)$ consisting of functions $f(z)$ which satisfy the inequality (4.10) below:

$$\left|\frac{1}{b}\left(p(1-\mu)\left(\frac{\mathcal{D}^{\lambda,p}f(z)}{z}\right)^{(m)} + \mu\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m+1)} - (p-m)\right)\right| < p-m \tag{4.10}$$

$$(z \in \mathbb{U};\ p \in \mathbb{N};\ m \in \mathbb{N}_0;\ \lambda \in \mathbb{R};\ p > \max(m, -\lambda);\ \mu \geqq 0;\ b \in \mathbb{C} \setminus \{0\}).$$

Our definitions of the function classes

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu)$$

are motivated essentially by two earlier investigations [4] and [14], in each of which further details and references to other closely-related subclasses can be found. In particular, in our definition of the function class $\mathcal{L}^p_{n,m}(\lambda, b; \mu)$ involving the inequality (1.13), we have relaxed the parametric constraint $0 \leqq \mu \leqq 1$, which was imposed earlier by Murugusundaramoorthy and Srivastava [14, p. 3, Equation (1.14)] (see also Remark 5 below).

We now prove the following results which yield the coefficient inequalities for functions in the subclasses (see also [17])

$$\mathcal{H}^p_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^p_{n,m}(\lambda, b; \mu).$$

Theorem 5. *Let $f(z) \in \mathcal{T}(n,p)$ be given by* (1.1). *Then $f(z) \in \mathcal{H}^p_{n,m}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}(k + |b| - p)\, a_k \leqq |b| \binom{p}{m}. \tag{4.11}$$

Proof. Let a function $f(z)$ of the form (1.1) belong to the class $\mathcal{H}^p_{n,m}(\lambda, b)$. Then, in view of (4.4), (4.9) yields the following inequality:

$$\Re\left(\frac{\displaystyle\sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}(p-k)z^{k-p}}{\displaystyle\binom{p}{m} - \sum_{k=n+p}^{\infty}\binom{\lambda+k-1}{k-p}\binom{k}{m}z^{k-p}}\right) > -|b| \quad (z \in \mathbb{U}). \tag{4.12}$$

Putting $z = r$ $(0 \leqq r < 1)$ in (4.12), we observe that the expression in the denominator on the left-hand side of (2.2) is positive for $r = 0$ and also for all r $(0 < r < 1)$. Thus, by letting $r \to 1-$ through *real* values, (4.12) leads us to the desired assertion (2.1) of Theorem 5.

Conversely, by applying (4.11) and setting $|z| = 1$, we find by using (4.4) that

$$\begin{aligned}
&\left| \frac{z\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m+1)}}{\left(\mathcal{D}^{\lambda,p}f(z)\right)^{(m)}} - (p-m) \right| \\
&= \left| \frac{\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}(p-k)z^{k-m}}{\binom{p}{m}z^{p-m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}z^{k-m}} \right| \\
&\leqq \frac{|b|\left[\binom{p}{m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}a_k\right]}{\binom{p}{m} - \sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k}{m}a_k} = |b|. \qquad (4.13)
\end{aligned}$$

Hence, by the *maximum modulus principle* once again, we infer that $f(z) \in \mathcal{H}_{n,m}^{p}(\lambda, b)$, which completes the proof of Theorem 5.

Remark 3. In the special case when

$$m = 0,\ p = 1, \quad \text{and} \quad b = \beta\gamma \quad (0 < \beta \leqq 1;\ \gamma \in \mathbb{C}\setminus\{0\}), \qquad (4.14)$$

Theorem 1 corresponds to a result given earlier by Murugusundaramoorthy and Srivastava [14, p. 3, Lemma 1].

By using the same arguments as in the proof of Theorem 5, we can establish Theorem 6 below.

Theorem 6. *Let* $f(z) \in \mathcal{T}(n,p)$ *be given by* (1.1). *Then* $f(z) \in \mathcal{L}_{n,m}^{p}(\lambda, b; \mu)$ *if and only if*

$$\begin{aligned}
&\sum_{k=n+p}^{\infty} \binom{\lambda+k-1}{k-p}\binom{k-1}{m}\left[\mu(k-1)+1\right]a_k \\
&\qquad \leqq (p-m)\left[\frac{|b|-1}{m!} + \binom{p}{m}\right]. \qquad (4.15)
\end{aligned}$$

Remark 4. Making use of the same parametric substitutions as mentioned above in (2.3), Theorem 2 yields another known result due to Murugusundaramoorthy and Srivastava [14, p. 4, Lemma 2].

5. Inclusion Relationships Involving the (n,δ)-Neighborhood $\mathcal{N}_{n,\delta}(h;p)$

In this section, we establish several inclusion relationships for the function classes

$$\mathcal{H}^{p}_{n,m}(\lambda,b) \quad \text{and} \quad \mathcal{L}^{p}_{n,m}(\lambda,b;\mu)$$

involving the (n,δ)-neighborhood defined by (1.6).

Theorem 7. *If*

$$\delta = \frac{(n+p)\,|b|\binom{p}{m}}{(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}} \quad (p>|b|), \tag{5.1}$$

then

$$\mathcal{H}^{p}_{n,m}(\lambda,b) \subset \mathcal{N}_{n,\delta}(h;p). \tag{5.2}$$

Proof. Let $f(z) \in \mathcal{H}^{p}_{n,m}(\lambda,b)$. Then, in view of the assertion (4.11) of Theorem 5, we have

$$(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k \leqq |b|\binom{p}{m}. \tag{5.3}$$

This yields

$$\sum_{k=n+p}^{\infty} a_k \leqq \frac{|b|\binom{p}{m}}{(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}}. \tag{5.4}$$

Applying the assertion (4.11) of Theorem 5 again, in conjunction with (5.4), we observe that

$$\begin{aligned}
&\binom{\lambda+n+p-1}{n}\binom{n+p}{m}\sum_{k=n+p}^{\infty} k a_k \\
&\leqq |b|\binom{p}{m} + (p-|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}\sum_{k=n+p}^{\infty} a_k \\
&\leqq |b|\binom{p}{m} + (p-|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m} \\
&\qquad \cdot \frac{|b|\binom{p}{m}}{(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}} \\
&= |b|\binom{p}{m}\left(\frac{n+p}{n+|b|}\right).
\end{aligned}$$

Hence we have

$$\sum_{k=n+p}^{\infty} ka_k \leqq \frac{|b|\,(n+p)\binom{p}{m}}{(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}} =: \delta \quad (p>|b|), \tag{5.5}$$

which, by virtue of (1.6), establishes the inclusion relation (5.2) of Theorem 7.

In an analogous manner, by applying the assertion (4.15) of Theorem 6 instead of the assetion (4.11) of Theorem 5 to functions in the class $\mathcal{L}^p_{n,m}(\lambda, b; \mu)$, we can prove the following inclusion relationship.

Theorem 8. *If*

$$\delta = \frac{(p-m)(n+p)\left[\dfrac{|b|-1}{m!}+\binom{p}{m}\right]}{[\mu\,(n+p-1)+1]\binom{\lambda+n+p-1}{n}\binom{n+p}{m}} \quad (\mu>1), \tag{5.6}$$

then

$$\mathcal{L}^p_{n,m}(\lambda, b; \mu) \subset \mathcal{N}_{n,\delta}(h;p).$$

Remark 5. Applying the parametric substitutions listed in (4.14), Theorems 7 and 8 would yield the known results due to Murugusundaramoorthy and Srivastava [14, p. 4, Theorem 1; p. 5, Theorem 2]. Incidentally, just as we indicated in Section 4 above, the condition $\mu > 1$ is needed in the proof of one of these known results [14, p. 5, Theorem 2]. This implies that the constraint $0 \leqq \mu \leqq 1$ in [14, p. 3, Equation (1.14)] should be replaced by the less stringent constraint $\mu \geqq 0$ (see also [17, p. 5, Remark 3]).

6. Further Neighborhood Properties Involving $\mathcal{N}_{n,\delta}(f;p)$

In this last section, we determine the neighborhood properties for each of the following (*slightly modified*) function classes:

$$\mathcal{H}^{p,\alpha}_{n,m}(\lambda, b) \quad \text{and} \quad \mathcal{L}^{p,\alpha}_{n,m}(\lambda, b; \mu).$$

Here, by definition, the class $\mathcal{H}^{p,\alpha}_{n,m}(\lambda, b)$ consists of functions $f(z) \in \mathcal{T}(n,p)$ for which there exists another function

$$g(z) \in \mathcal{H}^p_{n,m}(\lambda, b)$$

such that

$$\left|\frac{f(z)}{g(z)}-1\right| < p-\alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < p). \tag{6.1}$$

Analogously, the class $\mathcal{L}^{p,\alpha}_{n,m}(\lambda, b; \mu)$ consists of functions $f(z) \in \mathcal{T}(n,p)$ for which there exists another function

$$g(z) \in \mathcal{L}^p_{n,m}(\lambda, b; \mu)$$

satisfying the inequality (6.1).

The proofs of the following results (Theorems 9 and 10) involving the neighborhood properties for the classes

$$\mathcal{H}_{n,m}^{p,\alpha}(\lambda,b) \quad \text{and} \quad \mathcal{L}_{n,m}^{p,\alpha}(\lambda,b;\mu)$$

are similar to those given already by Altintaş *et al.* [4] and, more recently, by Murugusundaramoorthy and Srivastava [14]. We, therefore, choose to skip their proofs here.

Theorem 9. *Suppose that*

$$g(z) \in \mathcal{H}_{n,m}^{p}(\lambda,b).$$

Also let

$$\alpha = p - \frac{\delta\,(n+|b|)\binom{\lambda+n+p-1}{n}\binom{n+p}{m}}{(n+p)\left[(n+|b|)\binom{\lambda+n+p-1}{n+p}\binom{n+p}{m} - |b|\binom{p}{m}\right]}. \tag{6.2}$$

Then

$$\mathcal{N}_{n,\delta}(g;p) \subset \mathcal{H}_{n,m}^{p,\alpha}(\lambda,b).$$

Theorem 10. *Suppose that*

$$g(z) \in \mathcal{L}_{n,m}^{p}(\lambda,b;\mu).$$

Also let

$$\alpha = p - \frac{\delta\,[\mu(n+p-1)+1]\binom{\lambda+n+p-1}{n}\binom{n+p-1}{m}}{(n+p)\left[[\mu(n+p-1)+1]\binom{\lambda+n+p-1}{n}\binom{n+p-1}{m} - (p-m)\left\{\frac{|b|-1}{m!} + \binom{p}{m}\right\}\right]}. \tag{6.3}$$

Then

$$\mathcal{N}_{n,\delta}(g;p) \subset \mathcal{L}_{n,m}^{p,\alpha}(\lambda,b;\mu).$$

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