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On Rogosinski theorem

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1 Introduction

Let $F(z)$ be analytic and univalent in the unit disc $\mathbb{E}=\{z \mid |z|<1\}$ and let $D=F(\mathbb{E})$ be the image of $\mathbb{E}$ under the mapping $w=f(z)$. Let $f(z)$ be analytic in $\mathbb{E}$, but not necessarily univalent, and $f(\mathbb{E})\subset D$. Then $f(z)$ is said to be subordinate to $F(z)$ in $\mathbb{E}$, denoted by $f(z)\prec F(z)$. It is well known that if $f(z)\prec F(z)$ in $\mathbb{E}$, then there exists a function $w(z)$, analytic in $\mathbb{E}$ and with $|w(z)|<1$, such that

$$f(z)=F(w(z)),\ z\in\mathbb{E}.$$

If $f(0)=F(0)$, then $w(0)=0$ and $|w(z)|\leq|z|$ in $\mathbb{E}$.

Rogosinski[1] proved the following theorem.
Theorem A. *Let $f(z)\prec F(z)$ in $\mathbb{E}$. Then*

$$\int_0^{2\pi}|f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi}|F(re^{i\theta})|^p d\theta$$

where $0<p$ and $0\leq r<1$.

2 Obtained results

Theorem 1. *Let $f(z)\prec F(z)$ in $\mathbb{E}$ and $F(z)\neq 0$ in $\mathbb{E}$.*
Then

$$\int_0^{2\pi}\frac{1}{|f(re^{i\theta})|^p}d\theta \leq \int_0^{2\pi}\frac{1}{|F(re^{i\theta})|^p}d\theta$$

where $0<p$ and $0\leq r<1$.

Proof. From the assumption of the Theorem, $f(z)^{-p}$ and $F(z)^{-p}$ are analytic in $\mathbb{E}$ and so, from the Poisson integral form of harmonic function theory, we have

$$\begin{aligned}\frac{1}{f(z)^p} &= \frac{1}{F(w(z))^p} \\ &= \frac{1}{2\pi}\int_{|\zeta|=R}\frac{1}{F(\zeta)^p}\left(\mathrm{Re}\frac{\zeta+w(z)}{\zeta-w(z)}\right)d\zeta\end{aligned}$$

where $z = re^{i\theta}$, $\zeta = Re^{i\varphi}$, $|z| = r < |\zeta| = R < 1$, and $|w(z)| \leq |z|$.
Since

$$\mathrm{Re}\left(\frac{\zeta+w(z)}{\zeta-w(z)}\right) > 0 \ in\ \mathbb{E},$$

it follows that

$$\begin{aligned}&\int_0^{2\pi}\frac{1}{|f(re^{i\theta})|^p}d\theta \\ \leq\ &\int_0^{2\pi}\frac{1}{2\pi}\int_{|\zeta|=R}\frac{1}{|F(\zeta)|^p}\left(\mathrm{Re}\frac{\zeta+w(z)}{\zeta-w(z)}\right)d\varphi d\theta \\ =\ &\frac{1}{2\pi}\int_{|\zeta|=R}\int_0^{2\pi}\frac{1}{|F(\zeta)|^p}\left(\mathrm{Re}\frac{\zeta+w(z)}{\zeta-w(z)}\right)d\theta d\varphi \\ =\ &\frac{1}{2\pi}\int_{|\zeta|=R}\left\{\frac{1}{|F(Re^{i\varphi})|^p}\int_{|z|=r}\left(\mathrm{Re}\frac{\zeta+w(z)}{\zeta-w(z)}\right)\frac{dz}{iz}\right\}d\varphi \\ =\ &\int_0^{2\pi}\frac{1}{|F(Re^{i\varphi})|^p}d\varphi\end{aligned}$$

Putting $R \to r$, we have

$$\int_0^{2\pi}\frac{1}{|f(re^{i\theta})|^p}d\theta \leq \int_0^{2\pi}\frac{1}{|F(re^{i\theta})|^p}d\theta.$$

□

Prof. Owa (Kinki Univ.) pointed out another proof as the following : if $f(z) \prec F(z)$ in $\mathbb{E}$ and $F(z) \neq 0$ in $\mathbb{E}$, then $\frac{1}{f(z)} \prec \frac{1}{F(z)}$ and applying Theorem A, we can obtain a proof of Theorem 1.
From Theorem A and Theorem 1, we obtain the following theorem.
Theorem 1′. *Let $f(z) \prec F(z)$ in $\mathbb{E}$ and $F(z) \neq 0$ in $\mathbb{E}$.*
Then

$$\int_0^{2\pi}|f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi}|F(re^{i\theta})|^p d\theta$$

where p is arbitrary real number and $0 \leq r < 1$.

Theorem 2. *Let $f(z) \prec F(z) = z^m(a_m + a_{m+1}z + a_{m+2}z^2 + \dots)$ in $\mathbb{E}$ and let z_k, $k = 1,2,3,\dots,n$, $0 < |z_1| \leq |z_2| \leq |z_3| \leq \dots \leq |z_n|$, are the zeros of $F(z)$ in $\mathbb{E}$ which are to*

be written with their multiplicities.
Then, if $F(z) \neq 0$ *on certain circle* $|z| = r < 1$, $z = re^{i\theta}$, *we have*

$$\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta \geq \frac{2\pi}{r^{m+n}} \prod_{k=1}^{n} |z_k|$$

where $0 < p$.

Proof. Without generalization, we can choose R, $0 < R < 1$ in such a manner that $F(z) \neq 0$ on the circle $|z| = R$. Let us construct a function $B(z)$ which has the same zeros with the same multiplicities in $|z| < R < 1$ as $F(z)$ has, and so, we choose

$$B(z) = \left(\frac{z}{R}\right)^m \prod_{k=1}^{l} \frac{R(z - z_k)}{R^2 - \bar{z}_k z}, \quad l \leq n.$$

Putting

$$g(z) = \left(\frac{B(z)}{F(z)}\right)^p, \quad 0 < p \quad \textit{and} \quad z = re^{i\theta},$$

then $g(z)$ is analytic in $|z| < R$ and $g(z) \neq 0$ in $|z| < R$. From the Poisson integral form of harmonic functions, we have

$$g(z) = \frac{1}{2\pi} \int_{|\zeta|=R} g(\zeta) \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) d\varphi$$

where $|z| = r < |\zeta| = R < 1$ and $\zeta = Re^{i\varphi}$.
Then, we have

$$\begin{aligned} \left(\frac{B(w(z))}{F(w(z))}\right)^p &= \left(\frac{B(w(z))}{f(z)}\right)^p \\ &= \frac{1}{2\pi} \int_{|\zeta|=R} \left(\frac{B(\zeta)}{F(\zeta)}\right)^p \mathrm{Re}\left(\frac{\zeta + w(z)}{\zeta - w(z)}\right) d\varphi. \end{aligned}$$

Here, we have

$$\mathrm{Re}\left(\frac{\zeta + w(z)}{\zeta - w(z)}\right) > 0 \quad \textit{in} \quad |z| < R,$$

$$|B(w(z))| < 1 \quad \textit{on} \quad |z| = r < R < 1,$$

and

$$|B(\zeta)| = 1 \quad \textit{on} \quad |\zeta| = R.$$

Then, it follows that

$$\begin{aligned} \frac{1}{|f(re^{i\theta})|^p} &> \frac{|B(w(re^{i\theta}))|^p}{|f(re^{i\theta})|^p} \\ &= \left| \frac{1}{2\pi} \int_{|\zeta|=R} \left(\frac{B(\zeta)}{F(\zeta)}\right)^p \mathrm{Re}\left(\frac{\zeta + w(z)}{\zeta - w(z)}\right) d\varphi \right|. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta \\
> &\int_0^{2\pi} \left| \frac{1}{2\pi} \int_{|\zeta|=R} \left(\frac{B(\zeta)}{F(\zeta)}\right)^p \mathrm{Re}\left(\frac{\zeta + w(z)}{\zeta - w(z)}\right) d\varphi \right| d\theta \\
= &\left| \frac{1}{2\pi} \int_{|\zeta|=R} \left(\frac{B(\zeta)}{F(\zeta)}\right)^p \int_0^{2\pi} \mathrm{Re}\left(\frac{\zeta + w(z)}{\zeta - w(z)}\right) d\theta d\varphi \right| \\
= &\left| \int_{|\zeta|=R} \left(\frac{B(\zeta)}{F(\zeta)}\right)^p d\varphi \right| \\
= &\left| \int_{|\zeta|=R} \left(\frac{B(\zeta)}{F(\zeta)}\right)^p \frac{d\zeta}{i\zeta} \right| \\
= &\left| 2\pi \left(\frac{B(0)}{F(0)}\right)^p \right| \\
= &\ 2\pi \frac{\prod_{k=1}^{l} |z_k|}{R^{m+l}} > 2\pi \frac{\prod_{k=1}^{n} |z_k|}{R^{m+n}}.
\end{aligned}$$

Putting $R \to r$, we have

$$\int_0^{2\pi} \frac{1}{|f(re^{i\theta})|^p} d\theta > 2\pi \frac{\prod_{k=1}^{n} |z_k|}{r^{m+n}}.$$

This completes the proof of Theorem 2. □

References

[1] W.Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc., (2), **48**(1943), 48-82.