

# Note on certain analytic functions

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## Abstract

Let  $\mathcal{A}$  be the class of all analytic functions  $f(z)$  in the open unit disk  $\mathbb{U}$ . For  $f(z) \in \mathcal{A}$ , a subclass  $\mathcal{B}_k(\alpha, \beta, \gamma)$  of  $\mathcal{A}$  is introduced. The object of the present paper is to discuss some properties of functions  $f(z)$  belonging to the class  $\mathcal{B}_k(\alpha, \beta, \gamma)$ .

## 1 Introduction

Let  $\mathcal{A}$  be the class of functions  $f(z)$  of form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ . A function  $f(z) \in \mathcal{A}$  is said to be a member of the subclass  $\mathcal{B}_k(\alpha, \beta, \gamma)$  of  $\mathcal{A}$  if it satisfies

$$(1.2) \quad \operatorname{Re}\{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z)\} > \gamma \quad (k \in \mathbb{N} = \{1, 2, 3, \dots\}; z \in \mathbb{U})$$

for some  $a_j \in \mathbb{R}$  ( $j = 2, 3, 4, \dots, k$ ),  $\alpha \in \mathbb{R}$ ,  $\beta \in \mathbb{R}$  ( $\beta \neq 0$ ), and  $\gamma \in \mathbb{R}$  ( $0 \leq \gamma < k! \alpha a_k$ ;  $a_1 = 1$ ). We consider some properties for functions  $f(z)$  belonging to the class  $\mathcal{B}_k(\alpha, \beta, \gamma)$ .

**Remark 1.**  $\mathcal{B}_k(\alpha, \beta, \gamma)$  is convex.

Because, for  $f(z) \in \mathcal{B}_k(\alpha, \beta, \gamma)$  and  $g(z) \in \mathcal{B}_k(\alpha, \beta, \gamma)$ , we define

$$F(z) = (1-t)f(z) + tg(z) \quad (0 \leq t \leq 1).$$

Then

$$\begin{aligned} & \operatorname{Re}\{\alpha F^{(k)}(z) + \beta F^{(k+1)}(z)\} \\ &= \operatorname{Re}\{\alpha(1-t)f^{(k)}(z) + \alpha t g^{(k)}(z) + \beta(1-t)z f^{(k+1)}(z) + \beta t z g^{(k+1)}(z)\} \\ &= (1-t)\operatorname{Re}\{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z)\} + t\operatorname{Re}\{\alpha g^{(k)}(z) + \beta z g^{(k+1)}(z)\} \\ &> (1-t)\gamma + t\gamma = \gamma. \end{aligned}$$

Therefore  $F(z) \in \mathcal{B}_k(\alpha, \beta, \gamma)$ , that is,  $\mathcal{B}_k(\alpha, \beta, \gamma)$  is convex.

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In the present paper, we consider some properties of functions  $f(z)$  belonging to the class  $\mathcal{B}_k(\alpha, \beta, \gamma)$ .

## 2 Properties of the class $\mathcal{B}_1(\alpha, \beta, \gamma)$ and $\mathcal{B}_2(\alpha, \beta, \gamma)$

We begin with the statement and the proof of the following result.  
For cases  $k = 1$ , we obtain

**Theorem 1.** *A function  $f(z) \in \mathcal{A}$  is in the class of  $\mathcal{B}_1(\alpha, \beta, \gamma)$  if and only if*

$$(2.1) \quad f(z) = z + 2(\alpha - \gamma) \int_{|x|=1} \left( \sum_{n=2}^{\infty} \frac{1}{n((n-1)\beta + \alpha)} x^{n-1} z^n \right) d\mu(x)$$

where  $\mu(x)$  is the probability measure on  $X = \{x \in \mathbb{C} : |x| = 1\}$ .

*Proof.* For  $f(z) \in \mathcal{A}$ , we define

$$(2.2) \quad p(z) = \frac{\alpha f'(z) + \beta z f''(z) - \gamma}{\alpha - \gamma}.$$

Then  $p(z)$  is Carathéodory function. Therefore we can write

$$(2.3) \quad \frac{\alpha f'(z) + \beta z f''(z) - \gamma}{\alpha - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x) \quad (\text{see [1]}).$$

It follows from (2.3) that

$$(2.4) \quad \begin{aligned} z^{\frac{\alpha}{\beta}-1} \left( \frac{\alpha}{\beta} f'(z) + z f''(z) \right) &= \frac{1}{\beta} z^{\frac{\alpha}{\beta}-1} \left\{ \gamma + (\alpha - \gamma) \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x) \right\} \\ &= \frac{1}{\beta} z^{\frac{\alpha}{\beta}-1} \left\{ \gamma + (\alpha - \gamma) \int_{|x|=1} (1+xz)(1+xz+x^2z^2+\dots) d\mu(x) \right\}. \end{aligned}$$

Integrating the both sides of (2.4), we know that

$$\int_0^z \zeta^{\frac{\alpha}{\beta}-1} \left( \frac{\alpha}{\beta} f'(\zeta) + \zeta f''(\zeta) \right) d\zeta = \frac{1}{\beta} \int_{|x|=1} \left\{ \int_0^z \left( \alpha \zeta^{\frac{\alpha}{\beta}-1} + 2(\alpha - \gamma) \sum_{n=1}^{\infty} x^n \zeta^{n+\frac{\alpha}{\beta}-1} \right) d\zeta \right\} d\mu(x),$$

that is, that

$$\begin{aligned} z^{\frac{\alpha}{\beta}} f'(z) &= \frac{1}{\beta} \int_{|x|=1} \left\{ \beta z^{\frac{\alpha}{\beta}} + 2(\alpha - \gamma) \left( \sum_{n=1}^{\infty} \frac{\beta}{n\beta + \alpha} x^n z^{n+\frac{\alpha}{\beta}} \right) \right\} d\mu(x) \\ &= z^{\frac{\alpha}{\beta}} + 2(\alpha - \gamma) z^{\frac{\alpha}{\beta}} \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{1}{n\beta + \alpha} x^n z^n \right) d\mu(x). \end{aligned}$$

Thus, we have

$$(2.5) \quad f'(z) = 1 + 2(\alpha - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{1}{n\beta + \alpha} x^n z^n \right) d\mu(x).$$

An integration of both sides in (2.5) gives us that

$$\int_0^z f'(\zeta) d\zeta = \int_0^z \left\{ 1 + 2(\alpha - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{1}{n\beta + \alpha} x^n \zeta^n \right) d\mu(x) \right\} d\zeta,$$

or

$$\begin{aligned} f(z) &= z + 2(\alpha - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{1}{(n+1)(n\beta + \alpha)} x^n z^{n+1} \right) d\mu(x) \\ &= z + 2(\alpha - \gamma) \int_{|x|=1} \left( \sum_{n=2}^{\infty} \frac{1}{n((n-1)\beta + \alpha)} x^{n-1} z^n \right) d\mu(x). \end{aligned}$$

This completes the proof of Theorem 1. □

**Corollary 1.** *The extreme points of  $\mathcal{B}_1(\alpha, \beta, \gamma)$  are*

$$f_x(z) = z + 2(\alpha - \gamma) \sum_{n=2}^{\infty} \frac{x^{n-1}}{n((n-1)\beta + \alpha)} z^n \quad (|x| = 1).$$

In view of Theorem 1, we have the following corollary for  $a_n$ .

**Corollary 2.** *If  $f(z) \in \mathcal{A}$  is in the class  $\mathcal{B}_1(\alpha, \beta, \gamma)$ , then*

$$|a_n| \leq \frac{2(\alpha - \gamma)}{n((n-1)\beta + \alpha)} \quad (n = 2, 3, 4, \dots).$$

*Equality holds for the function  $f(z)$  given by*

$$f(z) = z + 2(\alpha - \gamma) \sum_{n=2}^{\infty} \frac{x^{n-1}}{n((n-1)\beta + \alpha)} z^n \quad (|x| = 1).$$

Further, the following distortion inequality follows from Theorem 1.

**Corollary 3.** *If  $f(z) \in \mathcal{A}$  is in the class  $\mathcal{B}_1(\alpha, \beta, \gamma)$ , then*

$$|f(z)| \leq |z| + 2(\alpha - \gamma) \left( \sum_{n=2}^{\infty} \frac{|z|^n}{n((n-1)\beta + \alpha)} \right) \quad (z \in \mathbb{U}).$$

**Remark 2.** If  $\beta > 0$  and  $\frac{\alpha}{\beta} = j$  ( $j = 2, 3, 4, \dots$ ) in Corollary 3, then we see that

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{|z|^n}{n((n-1)\beta + \alpha)} &\leq \frac{|z|^2}{\beta} \sum_{n=2}^{\infty} \frac{1}{n(n+j-1)} \\ &= \frac{|z|^2}{\beta(j-1)} \sum_{n=2}^{\infty} \left( \frac{1}{n} - \frac{1}{n+j-1} \right) \end{aligned}$$

$$= \frac{|z|^2}{\beta(j-1)} \sum_{n=2}^j \frac{1}{n} < \frac{\log(j)}{\beta(j-1)} |z|^2.$$

Therefore, we have that

$$\begin{aligned} |f(z)| &< |z| + \frac{2(\alpha - \gamma)\log(j)}{\beta(j-1)} |z|^2 \\ &< 1 + \frac{2(\alpha - \gamma)\log(j)}{\beta(j-1)}. \end{aligned}$$

Next, for cases  $k = 2$  we show

**Theorem 2.** A function  $f(z) \in \mathcal{A}$  is in the class  $\mathcal{B}_2(\alpha, \beta, \gamma)$  if and only if

$$f(z) = z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)((n-2)\beta + \alpha)} z^n \right) d\mu(x)$$

where  $\mu(x)$  is the probability measure on  $X = \{x \in \mathbb{C} : |x| = 1\}$ .

*Proof.* For  $f(z) \in \mathcal{A}$ , we define

$$p(z) = \frac{\alpha f''(z) + \beta z f'''(z) - \gamma}{2\alpha a_2 - \gamma}.$$

Then  $p(z)$  is Carathéodory function. Hence, we can write

$$(2.6) \quad \frac{\alpha f''(z) + \beta z f'''(z) - \gamma}{2\alpha a_2 - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x).$$

In view of (2.6), we have that

$$\begin{aligned} (2.7) \quad z^{\frac{\alpha}{\beta}-1} \left( \frac{\alpha}{\beta} f''(z) + z f'''(z) \right) &= \frac{1}{\beta} z^{\frac{\alpha}{\beta}-1} \left\{ \gamma + (2\alpha a_2 - \gamma) \int_{|x|=1} \left( 1 + 2 \sum_{n=1}^{\infty} x^n z^n \right) d\mu(x) \right\} \\ &= \frac{1}{\beta} \int_{|x|=1} \left( 2\alpha a_2 z^{\frac{\alpha}{\beta}-1} + 2(2\alpha a_2 - \gamma) \sum_{n=1}^{\infty} x^n z^{n+\frac{\alpha}{\beta}-1} \right) d\mu(x). \end{aligned}$$

Integrating the both sides of (2.7), we have that

$$\begin{aligned} &\int_0^z \zeta^{\frac{\alpha}{\beta}-1} \left( \frac{\alpha}{\beta} f''(\zeta) + \zeta f'''(\zeta) \right) d\zeta \\ &= \frac{1}{\beta} \int_{|x|=1} \left\{ \int_0^z \left( 2\alpha a_2 \zeta^{\frac{\alpha}{\beta}-1} + 2(2\alpha a_2 - \gamma) \left( \sum_{n=1}^{\infty} x^n \zeta^{n+\frac{\alpha}{\beta}-1} \right) \right) d\zeta \right\} d\mu(x), \end{aligned}$$

that is, that

$$z^{\frac{\alpha}{\beta}} f''(z) = \frac{1}{\beta} \int_{|x|=1} \left\{ 2\beta a_2 z^{\frac{\alpha}{\beta}} + 2(2\alpha a_2 - \gamma) \left( \sum_{n=1}^{\infty} \frac{\beta}{n\beta + \alpha} x^n z^{n+\frac{\alpha}{\beta}} \right) \right\} d\mu(x).$$

This implies that

$$(2.8) \quad f''(z) = \int_{|x|=1} \left\{ 2a_2 + 2(2\alpha a_2 - \gamma) \left( \sum_{n=1}^{\infty} \frac{x^n}{n\beta + \alpha} z^n \right) \right\} d\mu(x).$$

An integration of both sides in (2.8) gives us that

$$\int_0^z f''(\zeta) d\zeta = \int_0^z \left\{ 2a_2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{x^n}{n\beta + \alpha} \zeta^n \right) d\mu(x) \right\} d\zeta$$

or

$$f'(z) - 1 = 2a_2 z + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{x^n}{(n+1)(n\beta + \alpha)} z^{n+1} \right) d\mu(x).$$

Therefore, we know that

$$(2.9) \quad f'(z) = 1 + 2a_2 z + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=2}^{\infty} \frac{x^{n-1}}{n((n-1)\beta + \alpha)} z^n \right) d\mu(x).$$

Applying the same method for (2.9), we see that

$$\int_0^z f'(\zeta) d\zeta = \int_0^z \left\{ 1 + 2a_2 \zeta + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=2}^{\infty} \frac{x^{n-1}}{n((n-1)\beta + \alpha)} \zeta^n \right) d\mu(x) \right\} d\zeta.$$

Thus, we obtain that

$$\begin{aligned} f(z) &= z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=2}^{\infty} \frac{x^{n-1}}{(n+1)n((n-1)\beta + \alpha)} z^{n+1} \right) d\mu(x) \\ &= z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \int_{|x|=1} \left( \sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)((n-2)\beta + \alpha)} z^n \right) d\mu(x) \end{aligned}$$

This completes the proof of Theorem 2. □

**Corollary 4.** *The extreme points of  $\mathcal{B}_2(\alpha, \beta, \gamma)$  are*

$$f_x(z) = z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \left( \sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)((n-2)\beta + \alpha)} z^n \right) \quad (|x| = 1).$$

In view of Theorem 2, we have the following corollary for  $a_n$ .

**Corollary 5.** *If  $f(z) \in \mathcal{A}$  is in the class  $\mathcal{B}_2(\alpha, \beta, \gamma)$ , then*

$$|a_n| \leq \frac{2(2\alpha a_2 - \gamma)}{n(n-1)((n-2)\beta + \alpha)} \quad (n = 3, 4, 5, \dots).$$

*Equality holds for the function  $f(z)$  given by*

$$f(z) = z + a_2 z^2 + 2(2\alpha a_2 - \gamma) \left( \sum_{n=3}^{\infty} \frac{x^{n-2}}{n(n-1)((n-2)\beta + \alpha)} z^n \right) \quad (|x| = 1).$$

Further, the following distortion inequality follows from Theorem 2.

**Corollary 6.** *If  $f(z) \in \mathcal{A}$  is in the class  $\mathcal{B}_2(\alpha, \beta, \gamma)$ , then*

$$|f(z)| \leq |z| + |a_2||z|^2 + 2(2\alpha a_2 - \gamma) \left( \sum_{n=3}^{\infty} \frac{|z|^n}{n(n-1)((n-2)\beta + \alpha)} \right) \quad (z \in \mathbb{U}).$$

### 3 Properties of the class $\mathcal{B}_k(\alpha, \beta, \gamma)$

For cases  $k$  is any natural number, we have

**Theorem 3.** *A function  $f(z) \in \mathcal{A}$  belongs to the class  $\mathcal{B}_k(\alpha, \beta, \gamma)$  if and only if*

$$f(z) = z + a_2 z^2 + \dots + a_k z^k + 2(k! \alpha a_k - \gamma) \int_{|x|=1} \left( \sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1) \dots (n-k+1)((n-k)\beta + \alpha)} \right) d\mu(x)$$

for  $k = 1, 2, 3, \dots$ , where  $\mu(x)$  is the probability measure on  $X = \{x \in \mathbb{C} : |x| = 1\}$ .

*Proof.* For  $f(z) \in \mathcal{A}$ , we define

$$p(z) = \frac{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z) - \gamma}{k! \alpha a_k - \gamma}.$$

Since  $p(z)$  is Carathéodory function, we can write that

$$(3.1) \quad \frac{\alpha f^{(k)}(z) + \beta z f^{(k+1)}(z) - \gamma}{k! \alpha a_k - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x).$$

This means that

$$(3.2) \quad z^{\frac{\alpha}{\beta}-1} \left( \frac{\alpha}{\beta} f^{(k)}(z) + z f^{(k+1)}(z) \right) = \frac{1}{\beta} z^{\frac{\alpha}{\beta}-1} \left\{ \gamma + (k! \alpha a_k - \gamma) \int_{|x|=1} \left( 1 + 2 \sum_{n=1}^{\infty} x^n z^n \right) d\mu(x) \right\} \\ = \frac{1}{\beta} \int_{|x|=1} \left( k! \alpha a_k z^{\frac{\alpha}{\beta}-1} + 2(k! \alpha a_k - \gamma) \sum_{n=1}^{\infty} x^n z^{n+\frac{\alpha}{\beta}-1} \right) d\mu(x).$$

Integrating the both sides of (3.2), we obtain that

$$\int_0^z \zeta^{\frac{\alpha}{\beta}-1} \left( \frac{\alpha}{\beta} f^{(k)}(\zeta) + \zeta f^{(k+1)}(\zeta) \right) d\zeta \\ = \frac{1}{\beta} \int_{|x|=1} \left\{ \int_0^z \left( k! \alpha a_k \zeta^{\frac{\alpha}{\beta}-1} + 2(k! \alpha a_k - \gamma) \left( \sum_{n=1}^{\infty} x^n \zeta^{n+\frac{\alpha}{\beta}-1} \right) \right) d\zeta \right\} d\mu(x),$$

that is, that

$$z^{\frac{\alpha}{\beta}} f^{(k)}(z) = \frac{1}{\beta} \int_{|x|=1} \left\{ k! \beta a_k z^{\frac{\alpha}{\beta}} + 2(k! \alpha a_k - \gamma) \left( \sum_{n=1}^{\infty} \frac{\beta}{n\beta + \alpha} x^n z^{\frac{\alpha}{\beta} - 1} \right) \right\} d\mu(x).$$

This is equivalent to

$$(3.3) \quad f^{(k)}(z) = \int_{|x|=1} \left\{ k! a_k + 2(k! \alpha a_k - \gamma) \left( \sum_{n=1}^{\infty} \frac{x^n}{n\beta + \alpha} z^n \right) \right\} d\mu(x).$$

Now, since  $f(0) = 0$ ,  $f'(0) = 1$ , and  $f^{(m)}(0) = m! a_m$  ( $m = 2, 3, 4, \dots$ ), we see that

$$\begin{aligned} \int_0^z f^{(m)}(\zeta) d\zeta &= f^{(m-1)}(z) - f^{(m-1)}(0) \\ &= f^{(m-1)}(z) - (m-1)! a_{m-1}. \end{aligned}$$

Furthermore, we know that

$$\int_0^z \int_0^{\zeta_m} \dots \int_0^{\zeta_2} m! a_m d\zeta_1 d\zeta_2 \dots d\zeta_m = a_m z^m,$$

and

$$\sum_{n=1}^{\infty} \frac{x^n z^{n+k}}{(n+k)(n+k-1)\dots(n+1)(n\beta + \alpha)} = \sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\dots(n-k+1)((n-k)\beta + \alpha)}.$$

Therefore, integrating  $k$  times the both sides in (3.3), we obtain that

$$\begin{aligned} &\int_0^z \int_0^{\zeta_k} \dots \int_0^{\zeta_2} f^{(k)}(\zeta_1) d\zeta_1 d\zeta_2 \dots d\zeta_k \\ &= \int_0^z \int_0^{\zeta_k} \dots \int_0^{\zeta_2} \left\{ k! a_k + 2(k! \alpha a_k - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{x^n \zeta_1^n}{n\beta + \alpha} \right) d\mu(x) \right\} d\zeta_1 d\zeta_2 \dots d\zeta_k, \end{aligned}$$

that is, that

$$\begin{aligned} f(z) &= f(0) + \int_0^z f'(0) d\zeta_1 + \int_0^z \int_0^{\zeta_2} f''(0) d\zeta_1 d\zeta_2 + \int_0^z \int_0^{\zeta_3} \int_0^{\zeta_2} f'''(0) d\zeta_1 d\zeta_2 d\zeta_3 + \dots \\ &+ \int_0^z \int_0^{\zeta_k} \dots \int_0^{\zeta_2} \left\{ k! a_k + 2(k! \alpha a_k - \gamma) \int_{|x|=1} \left( \sum_{n=1}^{\infty} \frac{x^n \zeta_1^n}{n\beta + \alpha} \right) d\mu(x) \right\} d\zeta_1 d\zeta_2 \dots d\zeta_k. \end{aligned}$$

Thus, we conclude that

$$\begin{aligned} f(z) &= z + a_2 z^2 + a_3 z^3 + \dots + a_k z^k \\ &+ 2(k! \alpha a_k - \gamma) \int_{|x|=1} \left( \sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1)\dots(n-k+1)((n-k)\beta + \alpha)} \right) d\mu(x). \end{aligned}$$

The proof of Theorem 3 is complete. □

**Corollary 7.** *The extreme points of  $\mathcal{B}_k(\alpha, \beta, \gamma)$  are*  
 $f_x(z) = z + a_2 z^2 + a_3 z^3 + \dots + a_k z^k$

$$+2(k! \alpha a_k - \gamma) \left( \sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1) \dots (n-k+1) ((n-k)\beta + \alpha)} \right) \quad (|x| = 1).$$

In view of Theorem 3, we see that

**Corollary 8.** *If  $f(z)$  belongs to the class  $\mathcal{B}_k(\alpha, \beta, \gamma)$ , then*

$$|a_n| \leq \frac{2(k! \alpha a_k - \gamma)}{n(n-1) \dots (n-k+1) ((n-k)\beta + \alpha)} \quad (n = k+1, k+2, k+3, \dots).$$

*Equality holds for the function  $f(z)$  given by*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots + a_k z^k$$

$$+2(k! \alpha a_k - \gamma) \left( \sum_{n=k+1}^{\infty} \frac{x^{n-k} z^n}{n(n-1) \dots (n-k+1) ((n-k)\beta + \alpha)} \right) \quad (|x| = 1).$$

Further, the following distortion inequality follows from Theorem 3.

**Corollary 9.** *If  $f(z)$  belongs to the class  $\mathcal{B}_k(\alpha, \beta, \gamma)$ , then*

$$|f(z)| \leq |z| + |a_2||z|^2 + |a_3||z|^3 + \dots + |a_k||z|^k$$

$$+2(k! \alpha a_k - \gamma) \left( \sum_{n=k+1}^{\infty} \frac{|z|^n}{n(n-1) \dots (n-k+1) ((n-k)\beta + \alpha)} \right) \quad (z \in \mathbb{U}).$$

## References

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