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SURFACE KNOTS AND THEIR GENERIC PLANAR PROJECTIONS

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1. INTRODUCTION

In classical knot theory, we often use regular planar projections of knots and links in 3-space. Regular projections do not have triple points and the double points are given 3-dimensional informations, that is, upper-lower informations. Therefore, we can recover the original classical knot from the projection up to isotopy. In fact, many knot invariants can be constructed through regular planar projections: for example, crossing number, skein polynomials, Khovanov homology etc.

By a *surface knot*, we mean a closed connected smoothly embedded surface in $\mathbf{R}^4$. In surface knot theory, we often use generic projections into 3-space. By putting 4-dimensional informations to the singularities of a generic projection into 3-space, we get a so-called *broken surface diagram* [4]. Note that we can recover the original surface knot from its broken surface diagram. A generic projection into 3-space of a surface knot has double points, triple points, and branch points as its singularities. On a broken surface diagram the projection image is cut along its double curves.

In this report, we use generic projections into the plane for studying surface knots. Such planar projections were used, for example, in [2], [6], [7], [10]. Planar projections of surface knots have fold points and cusps as their singularities. Cusps appear as discrete points and fold points appear as 1-dimensional submanifolds. The set consisting of the cusps and the fold points in the surface is called the *singular set*.

Given a planar projection of a surface knot with no additional information, we cannot recover the original surface knot. In this report, we introduce the notion of a *braids with a band* and show that a planar projection together with a banded braid attached to the image of the singular set can recover the original surface knot. A banded braid is a braid in the usual sense in which one of the strings is replaced by a (possibly twisted) band. The band part corresponds to a neighborhood of the fold curve in the surface.

Finally we prove the Whitney congruence as an application. Whitney congruence is the congruence concerning normal Euler number which is isotopy invanriant of surface knot. First, the congruence was showed by Whitney and he used the tool of differential topology for showing the proof. However, the method was not geometric. The latter, Carter-Saito showed the congruence by using geometric method which was generic projection into 3-space of surface knot. On the other hand, we will show the congruence by using generic planar projection which is geometric method.

The report is organized as follows. In Section 2 we introduce how to make the braided diagram which is analogy of broken surface diagram and we will show the diagram is determined uniquely for a surface knot. In Section 3 we give two explicit

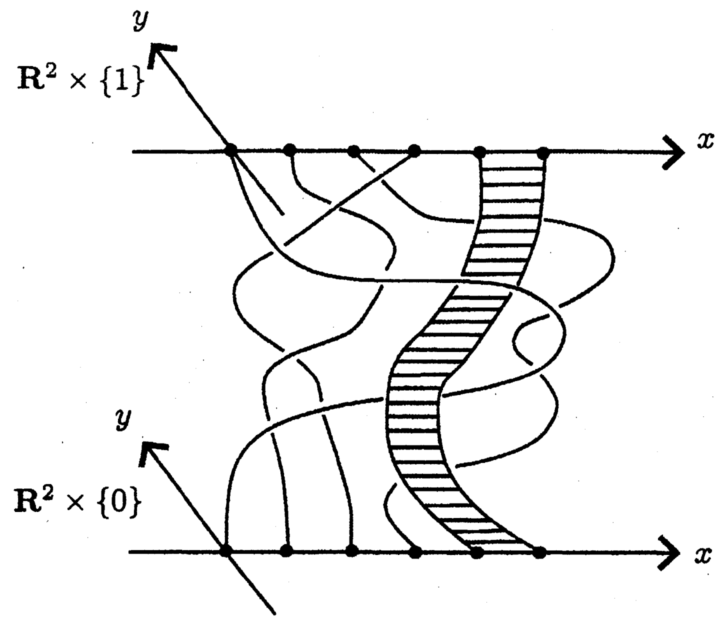

FIGURE 1. A banded braid

examples. In Section 4 we explain how to calculate the normal Euler number by using a planar projection and prove the Whitney congruence as an application.

Throughout the report, we work in the smooth category.

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2. SURFACE KNOTS AND THEIR BRAIDED DIAGRAMS

In this section we introduce how to make braided diagram from planar projected surface knot and show the braided diagram is determind uniquely for a surface knot.

2.1. Braided diagram associated with a given surface knot. Let us show how to construct a planar projection with additional informations from a given embedded surface in $\mathbf{R}^4$. For this, we introduce the notion of a *banded braid.*

Definition 2.1. We call an embedding $\varphi\colon I_1\cup\cdots\cup I_r\cup(J_1\times J_2)\to\mathbf{R}^2\times(I=[0,1])$ (or $\varphi(I_1\cup\cdots\cup I_r\cup(J_1\times J_2))$) a *braid with a band* if φ satisfies following:

(i) $I_1=\cdots=I_r=[0,1],\ J_1=J_2=[0,1]$,

(ii-1) $p\circ\varphi|_{I_j}\colon I_j\to I, j=1,2,\cdots,r$, is the identity mapping, where $p\colon\mathbf{R}^2\times I\to I$ is the projection to the second factor,

(ii-2) $p\circ\varphi|_{J_1\times J_2}\to I$ is the projection to the first factor,

(iii-1) for $0\in I_j$, we have $\varphi(0)\in(\mathbf{R}\times\{0\})\times\{0\}$, $j=1,2,\cdots,r$,

(iii-2) for $0\in J_1$, we have $\varphi(0\times J_2)\subset(\mathbf{R}\times\{0\})\times\{0\}$,

(iii-3) for $1\in I_j$, we have $\varphi(1)\in(\mathbf{R}\times\{0\})\times\{1\}$, $j=1,2,\cdots,r$,

(iii-4) for $1\in J_1$, we have $\varphi(1\times J_2)\subset(\mathbf{R}\times\{0\})\times\{1\}$.

For example, see Fig. 1.

In the following, two banded braids are considered to be the same if they are isotopic through banded braids.

Let us prepare several notions from singularity theory.

Definition 2.2. Let F be a closed connected surface. Denote by $C^\infty(F, \mathbf{R}^2)$ the set of all smooth mappings from F to $\mathbf{R}^2$, endowed with the Whitney C^∞ topology [5]. Let g and h be elements of $C^\infty(F, \mathbf{R}^2)$. Then g is *equivalent* to h if there exist diffeomorphisms $p\colon F \to F$ and $q\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $q \circ g = h \circ p$.

Definition 2.3. Let g be an element of $C^\infty(F, \mathbf{R}^2)$. Then g is said to be C^∞ *stable* if there exists a neighborhood N_g of g in $C^\infty(F, \mathbf{R}^2)$ such that each h in N_g is equivalent to g.

Definition 2.4. Let $g\colon F \to \mathbf{R}^2$ be a smooth mapping from F to $\mathbf{R}^2$. Then $q \in F$ is called a *fold point* if we can choose local coordinates (x, y) centered at q and (U, V) centered at $g(q)$ such that g, in a neighborhood of q, is of the form:

$$U = x, \ V = y^2.$$

Moreover, $q \in F$ is called a *cusp* if we can choose local coordinates (x, y) centered at q and (U, V) centered at $g(q)$ such that g, in a neighborhood of q, is of the form:

$$U = x, \ V = xy + y^3.$$

We denote by $S_1(g)$ the set of fold points and cusps, and by $S_1^2(g)$ the set of cusps.

Note that $S_1(g)$ is a regular 1-dimensional submanifold of F and $S_1^2(g)$ is a discrete set.

Recall the following well-known characterization of C^∞ stable mappings in $C^\infty(F, \mathbf{R}^2)$.

Proposition 2.5. *Let $g\colon F \to \mathbf{R}^2$ be a smooth mapping from a closed connected surface F to $\mathbf{R}^2$. Then g is C^∞ stable if and only if g has only fold points and cusps as its singularities, its restriction to the set of fold points is an immersion with normal crossings, and for each cusp q, we have:*

$$g^{-1}(g(q)) \cap S_1(g) = \{q\}.$$

Let F be a closed connected surface. For a smooth map $g\colon F \to \mathbf{R}^2$, we set

$$S(g) = \{x \in F \mid \operatorname{rank} dg_x < 2\},$$

which is called the *singular point set* of g. If g is C^∞ stable, then we clearly have $S_1(g) = S(g)$.

Definition 2.6. Let $f\colon F \to \mathbf{R}^4$ be an embedding of a closed connected surface. Let $\pi\colon \mathbf{R}^4 \to \mathbf{R}^2$ be an orthogonal projection. Then we say that π is *generic* with respect to f (or with respect to $f(F)$) if $\pi \circ f$ is C^∞ stable.

Definition 2.7. Let $f\colon F \to \mathbf{R}^4$ be an embedding of a closed connected surface and $\pi\colon \mathbf{R}^4 \to \mathbf{R}^2$ an orthogonal projection which is generic with respect to f. For $x \in S(\pi \circ f)$, the 1-dimensional subspace $df_x(\operatorname{Ker}(d(\pi \circ f)_x\colon T_xF \to T_{\pi\circ f(x)}\mathbf{R}^2))$ of $df_x(T_xF)$ is called the *kernel line* at $f(x) \in f(F)$.

We often regard the kernel line as included in $\pi^{-1}(\pi \circ f(x)) \cong \mathbf{R}^2$.

The kernel line at a fold point and that at a cusp are depicted in Figs. 2 and 3 respectively.

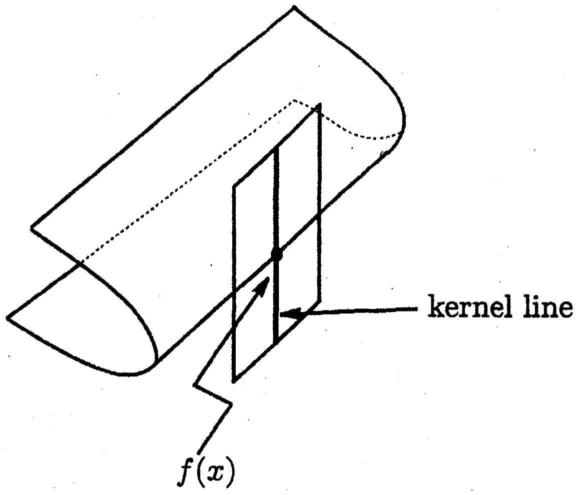

FIGURE 2. The kernel line at a fold point

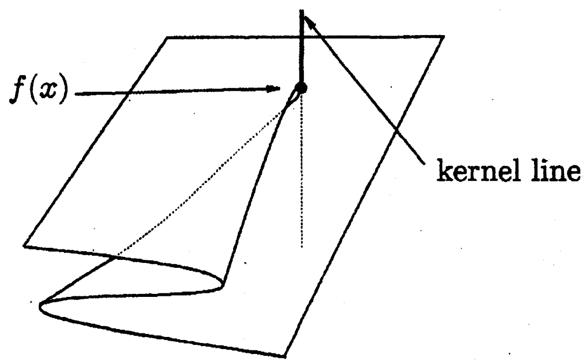

FIGURE 3. The kernel line at a cusp

We construct a braided diagram from a given embedded surface in $\mathbf{R}^4$ like follows.

Let $f\colon F \longrightarrow \mathbf{R}^4$ be an embedding of a closed connected surface and $\pi\colon \mathbf{R}^4 \longrightarrow \mathbf{R}^2$ an orthogonal projection which is generic with respect to f. The C^∞ stable map $\pi \circ f$ has fold points and cusps as its singularities. Let $q \in \mathbf{R}^2$ be a fold crossing which is normal crossing of restriction of $\pi \circ f$ to fold points set. Then we can regard $\pi^{-1}(q)$ as a plane. Note that $\pi^{-1}(q) \cong \mathbf{R}^2$ contains exactly two fold points. Moreover, by isotoping f if necessary, we may assume that the points $\pi^{-1}(q) \cap f(F)$ and the kernel lines at the fold points all lie in $\mathbf{R} \times \{0\} \subset \mathbf{R}^2 = \pi^{-1}(q)$ as depicted in Fig. 4 for all fold crossings q. Furthermore, for the image $q \in \mathbf{R}^2$ of each cusp, we arrange $f(F)$ by isotopy so that the points $\pi^{-1}(q) \cap f(F)$ and the kernel line at the cusp point all lie in $\mathbf{R} \times \{0\} \subset \mathbf{R}^2 = \pi^{-1}(q)$. See Fig. 5.

Let R be a bounded region of $\mathbf{R}^2 \setminus \pi \circ f(S(\pi \circ f))$. If R is an open disk, then we do nothing for R. If R is not an open disk, then we take disjointly embedded arcs $a_1, a_2, \cdots, a_k$ in $\widetilde{R} = R \cup (\pi \circ f(S(\pi \circ f)))$ such that $(a_1 \cup a_2 \cup \cdots \cup a_k) \cap (\pi \circ f(S(\pi \circ f))) = \partial(a_1 \cup a_2 \cup \cdots \cup a_k)$ and each component of $R \setminus (a_1 \cup \cdots \cup a_k)$ is an open disk (see Fig. 6). Furthermore, we take the arcs $a_1, a_2, \cdots, a_k$ so that their end points are not fold crossing nor the image of cusps. We call the arcs $a_1, a_2, \cdots, a_k$ *additional arcs*. For each non-disk region of $\mathbf{R}^2 \setminus \pi \circ f(S(\pi \circ f))$ we take additional

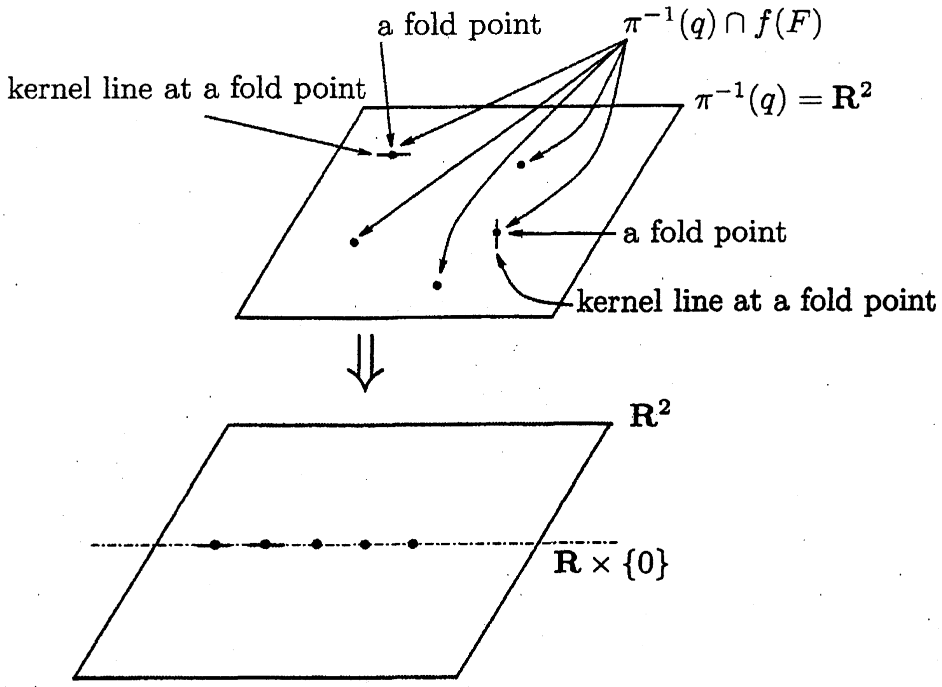

FIGURE 4. Arranging $\pi^{-1}(q) \cap f(F)$

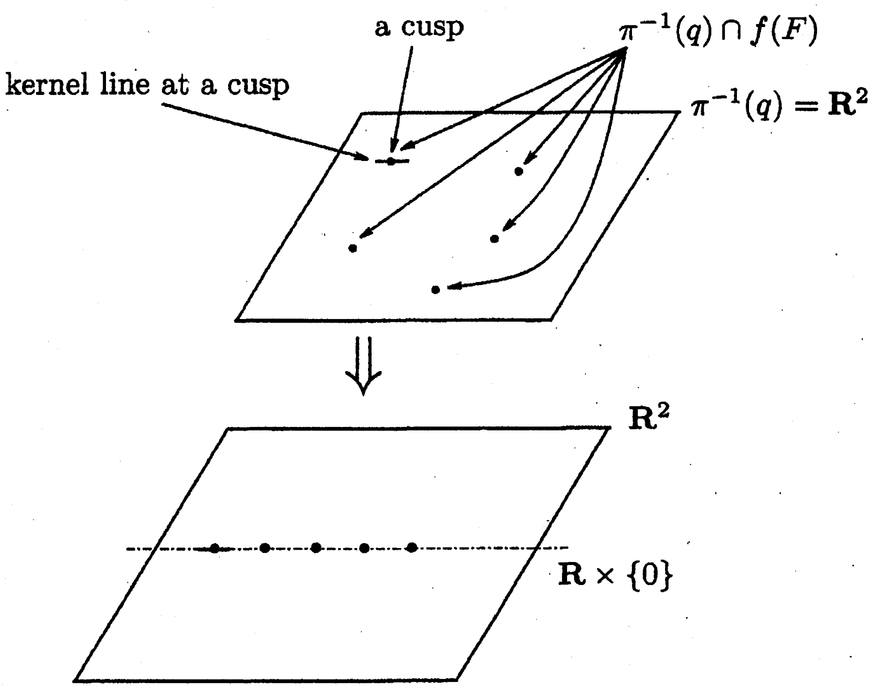

FIGURE 5. Arranging $\pi^{-1}(q) \cap f(F)$

arcs so that each bounded region of $\mathbf{R}^2 \setminus ((\pi \circ f(S(\pi \circ f))) \cup A$ is an open disk, where A is the union of all the additional arcs.

By above similar argument, we can fix terminals of additional arcs like follow. Let y be one of intersection between $\pi \circ f(S(\pi \circ f))$ and a additional arc. Then we

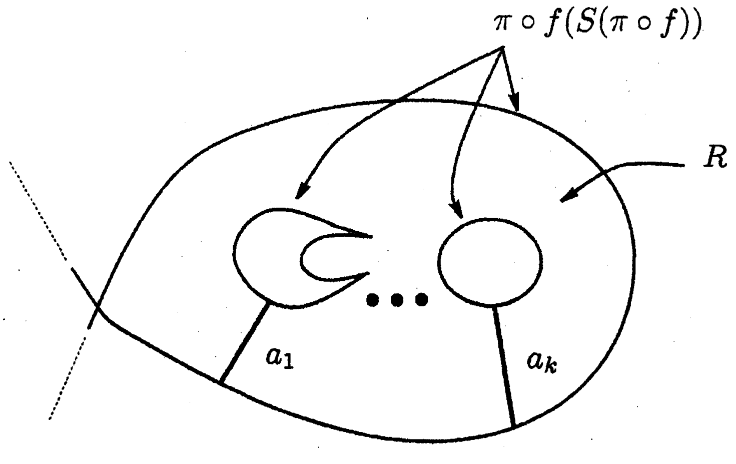

FIGURE 6. Additional arcs in a non-disk region

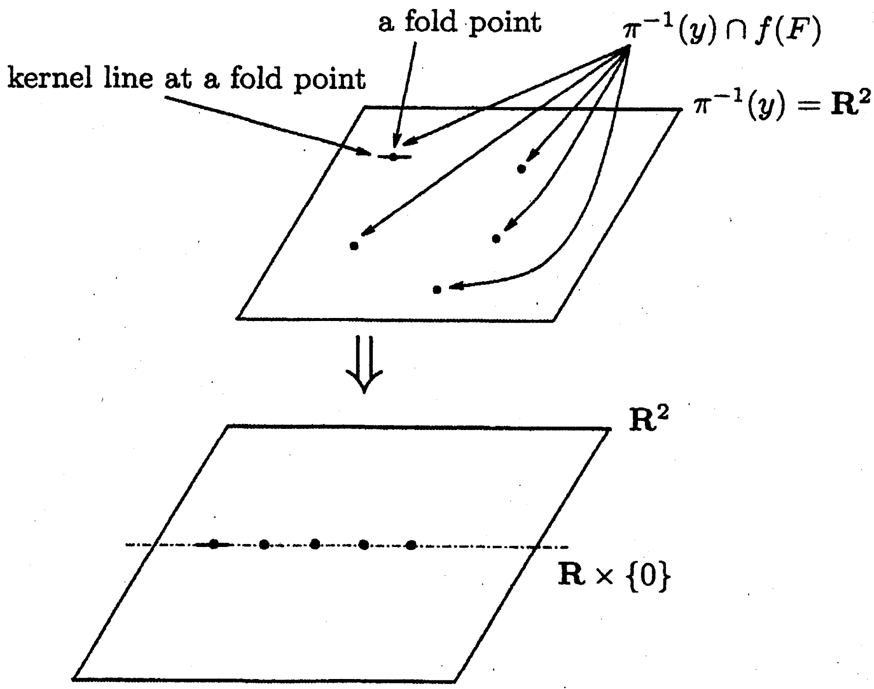

FIGURE 7. Arranging $\pi^{-1}(y) \cap f(F)$

arrange $f(F)$ by isotopy so that the points $\pi^{-1}(y) \cap f(F)$ and the kernel line at the fold point all lie in $\mathbf{R} \times \{0\} \subset \mathbf{R}^2 = \pi^{-1}(y)$ as depicted in Fig. 7.

Therefore we can regard $(\pi \circ f(S(\pi \circ f))) \cup A$ as a oriented graph which is oriented by method that the the number of pre-image of left of fold curve is larger than one of right. The vertices are fold crossings, the images of cusps, and the end points of additional arcs. Then, to every edge e of $\pi \circ f(S(\pi \circ f))$ we associate a banded braid as follows. Let e' be edge which is a little smaller than e along the e. The pre-image of e' consist of a braid of odd strings. The odd strings contain strings for fold curve. Then we make the width the string by using the kernel line segment. Therefore, we associate the banded braid to e. Furthermore, to each additional arc

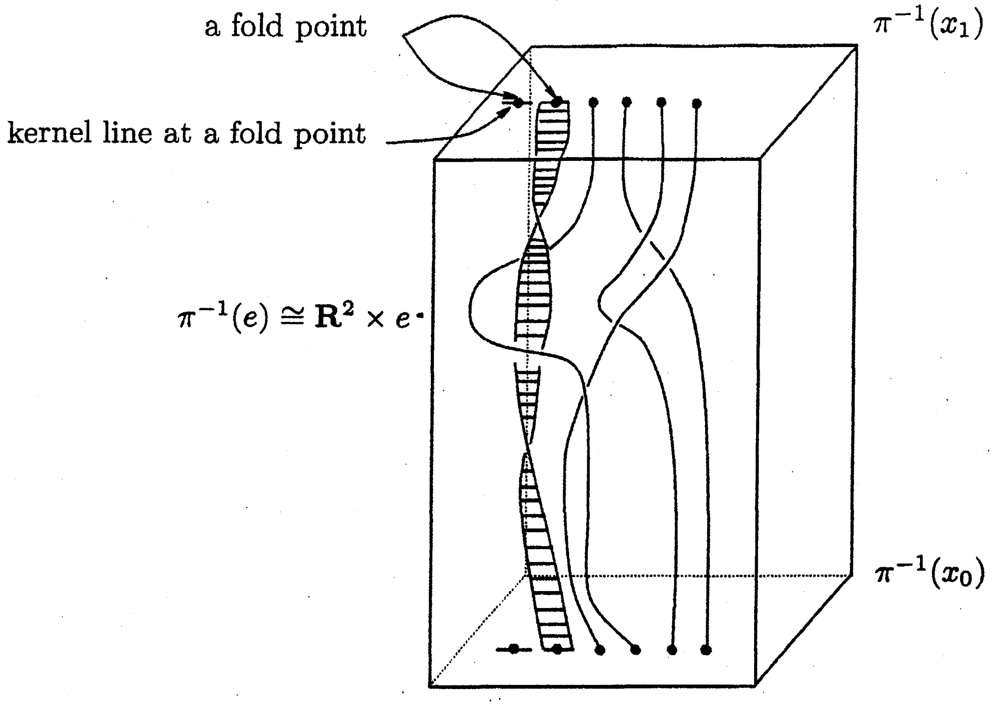

FIGURE 8. A banded braid associated with an edge

a we associate a braid in the usual sense as follows. The pre-image of a consist of a braid of odd strings. Therefore, we associate the braid in the usual sense to a. For example, let e be an edge of $\pi \circ f(S(\pi \circ f))$ with end points x_0 and x_1. Then $\pi^{-1}(e)$ is as depicted in Fig. 8. In particular, if e do not have end points, we see it is circle and we can not associate banded braid to e. Then we only attach a label which is the number of pre-image of inter of the circle, for, the inter of the circle is open disk and the banded braid associated to e is trivial closed braid. Therefore, we can define braided diagram as following.

Definition 2.8. Let $(\pi \circ f(S(\pi \circ f))) \cup A$ be oriented graph defined as above and we associate banded braid or braid in the usual sense to edge of the graph. Then we call the oriented graph with the informations (banded braid or braid in the usual sense etc) *braided diagram* of surface knot.

2.2. **Recovering embedded surface from braided diagram.** We will show recover a local embedding of surface in $\mathbf{R}^4$ from braided diagram is unique.

Definition 2.9. Let $F_{0,n}\mathbf{R}^2$ denote the subspace

$$F_{0,n}\mathbf{R}^2 = \{(p_1, \dots, p_n) \in \mathbf{R}^2 \times \dots \times \mathbf{R}^2 |\ p_i \neq p_j (i \neq j)\}.$$

Let $Q_m = \{q_1, \dots, q_m\}$ be a set of fixed distinguished points of $\mathbf{R}^2$. Then we define the *configuration space* $F_{m,n}$ of $\mathbf{R}^2$ to be the space $F_{0,n}(\mathbf{R}^2 \setminus Q_m)$.

Given braided diagram of surface knot, fold crossings, cusps, edges can be lift in $\mathbf{R}^4$ which is unique. However, disk regions constructed the image of singular set may not lift which is unique. To show it, we consider whether it can extend loop in configuration space which it construct n points in $\mathbf{R}^2$ to disk. In fact, following proposition [1] say it can extend to disk which is unique.

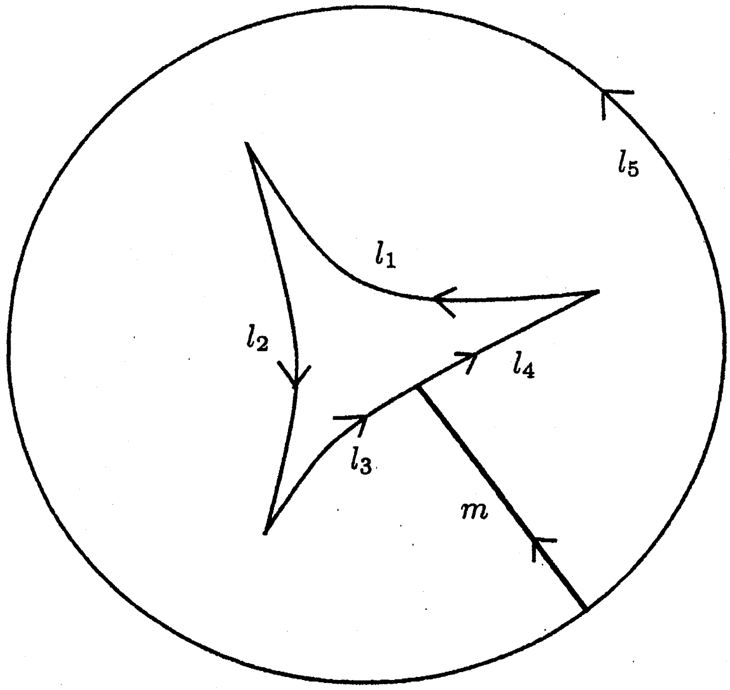

FIGURE 9. The image of singular set with three cusps

Proposition 2.10. *If* $\pi_2(\mathbf{R}^2 \setminus Q_m) = \pi_3(\mathbf{R}^2 \setminus Q_m) = 0$ *for each* $m \geq 0$, *then* $\pi_2 F_{0,n}\mathbf{R}^2 = 0$.

3. EXAMPLES OF RECOVERING

In this section we introduce an example of recovering of surface knot via planar projection. We give a local orientation to edge by method which local width (see [7]) of left of the edge is larger than the one of right. First, we consider embedding $\mathbf{RP}^2$ in $\mathbf{R}^4$ with the image of singular set as in Fig. 9. The existence of this embedding have been showed as in [10]. m is an additional arc which is not image of singular set. Then we can reconstruct original surface knot as in Fig. 10, 11. In these figure, the direction of fold l_1, l_2 etc correspond direction from top to bottom in this report. Moreover the normal Euler number of this embedding is 2 or -2 since the total of twist of band is 2 half twist (see next section).

4. NORMAL EULER NUMBER AND WHITNEY CONGRUENCE

In this section we introduce how to calculate the normal Euler number and as an application we prove the Whitney congruence.

4.1. The calculation of normal Euler number. Let $F \subset \mathbf{R}^4$ be surface knot with planar diagram. We consider edges with banded braids. Then the normal Euler number is calculated like follows. Perturb F (we will call F') by isotopy as depicted in Fig. 12. Then strings (braids) of the banded braid in $F \cap F'$ does not intersect each other since strings in $F \cap F'$ are parallel. In particular, additional arcs does not also intersect each other. However, band part of banded braid only intersect each other. Furthermore, the number of half-twist of the band correspond the number of the intersection. Therefore, the normal Euler number is the number

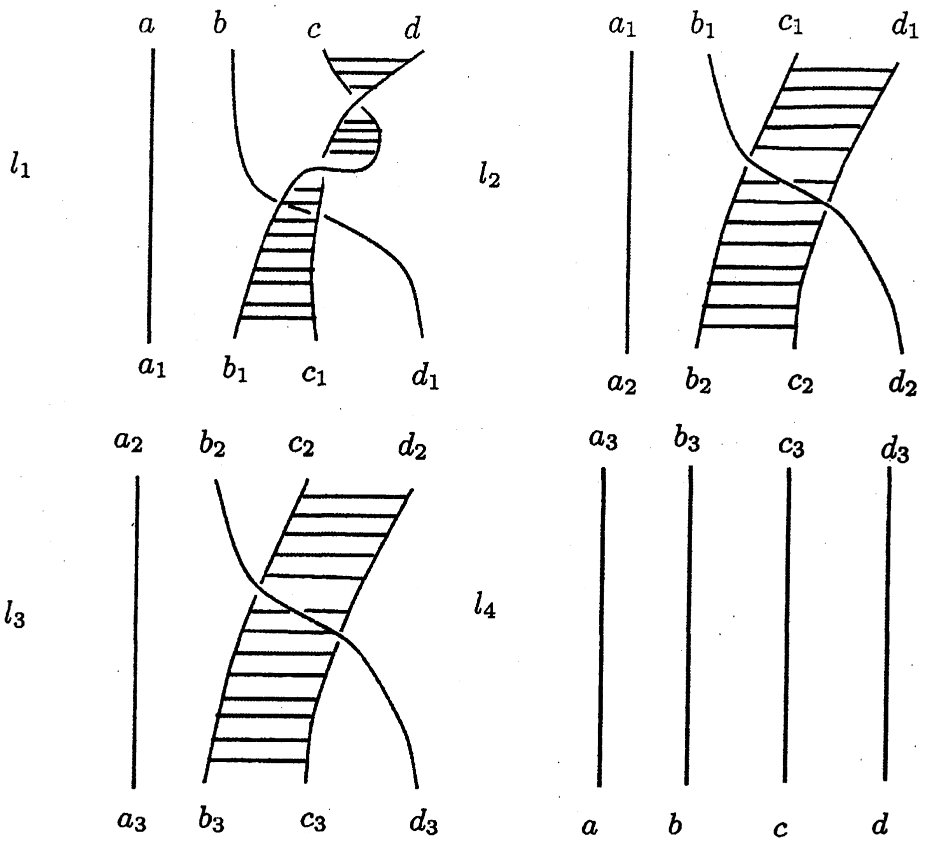

FIGURE 10. The reconstructing of region $l_1 l_2 l_3 l_4$

of the half-twist of the band. Let e be edge (without additional arc) and x_0, x_1 the terminal's vertexes. See Fig. 12.

We give a orientation to disk regions on graph made the image of singular set how it is induced from the orientation of $\mathbf{R}^2$. Moreover, we give a orientation to boundaries of disk regions how it is induced from the orientation of the disk regions (it is different from a orientation concerning the local width).

For every region which is constructed by the image of singular set, the boundary of the region is associated by trivial braid. Every two adjacent strings of the trivial braid twist 0 times. Therefore the sum of twist of every two adjacent strings for total regions is also 0. In this sum we consider the kind of half-twist of two adjacent strings.

In general, fold curve is associated by banded braid. The strings of banded braid is divided by braid part and band part. Let R_1 be a left region of fold curve and R_2 a right region of fold curve. Then the half-twist of braid part is canceled each other since the orientation of ∂R_1 and ∂R_2 are reverse direction each other. Therefore non-cancel part are the half-twist of between boundaries of band part and the strings in braid part. The former contribute normal Euler number. The latter correspond twice of the sum of twists of between core of band and the strings in braid part.

On the other hand in neighborhood of additional arc we can cancel the strings because there is no band. In neighborhood of cusps twist of banded braid is 0 times.

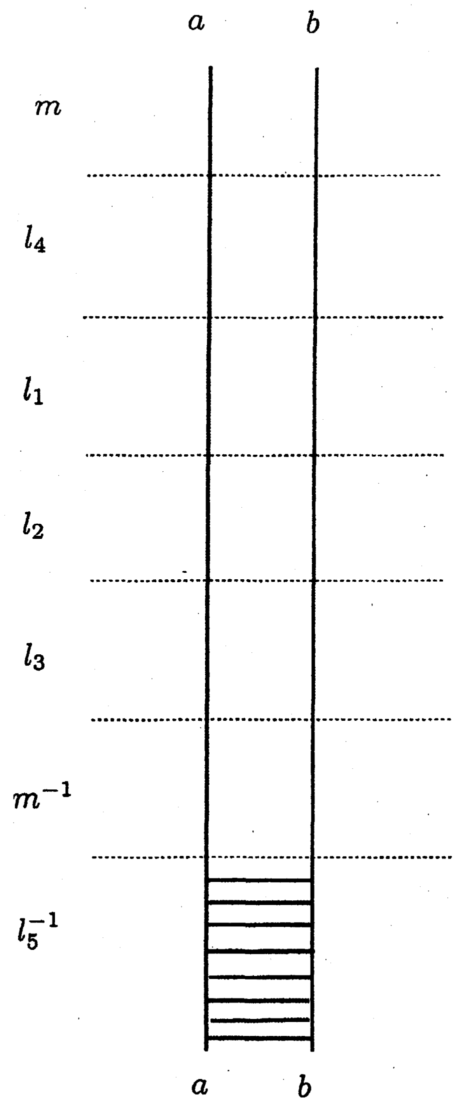

FIGURE 11. The constructing of region $ml_4l_1l_2l_3m^{-1}l_5^{-1}$

In neighborhood of fold crossing twist of banded braid also is 0 times. Therefore we can see the following Theorem.

Theorem 4.1. *We consider generic planar projection of a surface knot. Then fold curve is associated by banded braid. Let e be normal Euler number of surface knots and D be the sum of half-twist of between the core of band in banded braid and the other strings in banded braid for all regions. Then*

$$e = -2D.$$

By above theorem we can see that normal Euler number is even.

4.2. **Proof of Whitney congruence.** Whitney [9] showed relation of between normal Euler number and Euler characteristic (we call it Whitney congruence).

However Carter-Saito [3] gived the diagrammatic proof of Whitney congruence. They showed it by using generic projections into 3-space.

On the other hand we will show it by using generic planar projections. Whitney congruence is following theorem.

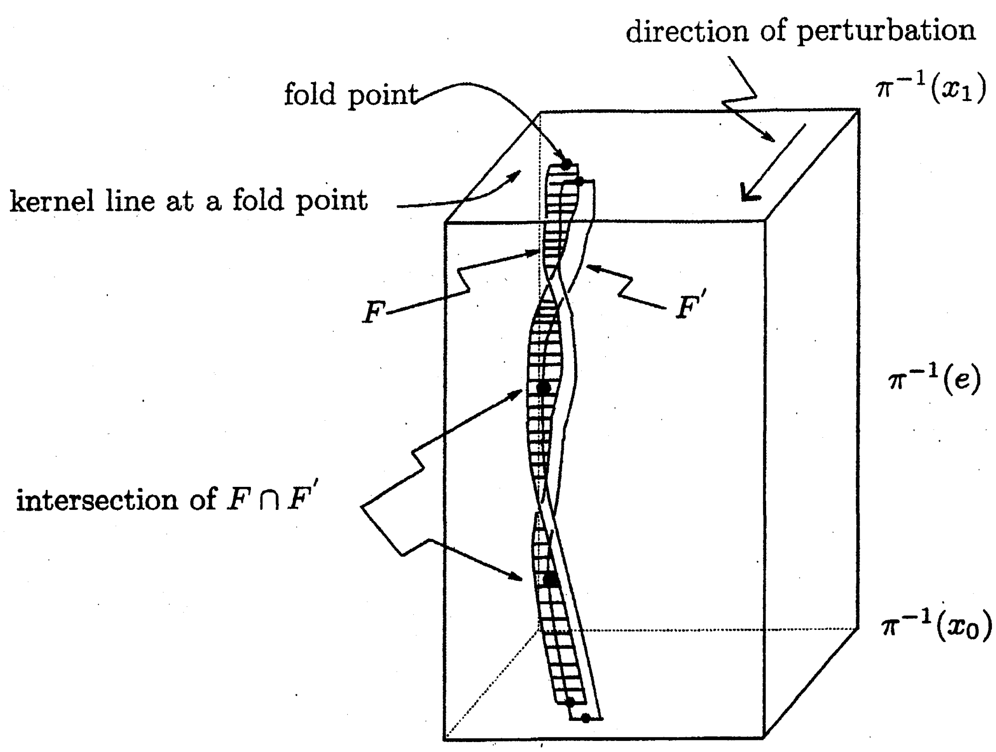

FIGURE 12. Perturbation of F

Theorem 4.2. *Let F be embedding surface into $\mathbf{R}^4$ and $e(F)$ be normal Euler number of F and $\chi(F)$ be Euler characteristic of F. Then*

$$(1/2)e(F) \equiv \chi(F) \pmod 2.$$

The following theorem is also well-known [8].

Theorem 4.3. *Let F be a closed 2-dimensional manifold, N an orientable surface, $f : F \to N$ a stable map and $C(f)$ the set of the cusp singularities of f. Then*

$$|C(f)| \equiv \chi(M) \pmod 2.$$

Therefore by Theorem 4.1, 4.3 we may show the following theorem.

Theorem 4.4. *Let Let f be embedding surface into $\mathbf{R}^4$ and π: $\mathbf{R}^4 \to \mathbf{R}^2$ be orthogonal projection and D be D in Theorem 4.1 and $|C(\pi \circ f)|$ be the number of cusps. Then*

$$D \equiv |C(\pi \circ f)| \pmod 2.$$

Any way, we have a problem, which is to prove the Whitney conjecture by using pure geometric method. Whitney conjecture was showed by Massey but the proof was not geometric. We want to prove the Whitney conjecture by using generic projection. If we use the generic planar projection, we may prove the conjecture. That is the open problem.

REFERENCES

1. J.S. Birman, *Braids, links, and mapping class groups*, Annals of Math. Studies, Vol. 82, Princeton Univ. Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974.

2. V. Carrara, J.S. Carter and M. Saito, *Singularities of the projections of surfaces in 4-space*, Pacific J. Math. **199** (2001), 21–40.
3. J.S. Carter and M. Saito, *Canceling branch points on projections of surfaces in 4-space*, Proc. Amer. Math. Soc. **116** (1992), 229–237.
4. J.S. Carter and M. Saito, *Knotted surfaces and their diagrams*, Math. Surveys and Monographs, 55, Amer. Math. Soc., Providence, RI, 1998.
5. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Grad. Texts Math., Vol. 14, Springer-Verlag, New York, 1973.
6. O. Saeki and Y. Takeda, *Canceling branch points and cusps on projections of knotted surfaces in 4-space*, Proc. Amer. Math. Soc. **132** (2004), 3097–3101.
7. Y. Takeda, *Widths of surface knots*, preprint, 2005.
8. R. Thom, *Les singularités des applications différentiables*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 43–87.
9. H. Whitney, *On the topology of differentiable manifolds*, Lectures in Topology, University of Michigan Press, 1941, pp. 101–141.
10. M. Yamamoto, *Lifting a generic map from a surface into the plane to an embedding into 4-space*, preprint, 2005.

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