数理解析研究所講究録
第 1540 巻 2007 年 43-55

COBORDISMS OF FOLD MAPS

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ABSTRACT. We summerize and extend some of our existing results about cobordisms of fold maps. We establish a relation between fold maps and immersions and obtain geometrical invariants of cobordism classes of fold maps in terms of immersions with prescribed normal bundles. These invariants are complete invariants of the cobordism classes of simple fold maps of oriented $(n+1)$-dimensional manifolds into an n-dimensional manifold and detect stable homotopy groups as direct summands of the cobordism group of fold maps of $(n+q)$-dimensional manifolds into n-dimensional manifolds. We give a Pontryagin-Thom type construction for -1 codimensional fold maps, and also study the cobordism classes of source manifolds of fold maps giving estimations about the cobordism classes of manifolds which have fold maps into stably parallelisable manifolds.

1. INTRODUCTION

Fold maps of $(n+q)$-dimensional manifolds into n-dimensional manifolds have the formula $f(x_1,\ldots,x_{n+q}) = (x_1,\ldots,x_{n-1},\pm x_n^2 \pm \cdots \pm x_{n+q}^2)$ as a local form around each singular point, and the subset of the singular points in the source manifold is a $(q+1)$-codimensional submanifold (for results about fold maps, see, for example, [1, 2, 3, 5, 6, 14, 27, 30]). If we restrict a fold map to the set of its singular points, then we obtain a codimension one immersion into the target manifold of the fold map. This immersion together with more detailed informations about the neighbourhood of the set of singular points in the source manifold can be used as a geometrical invariant (see Section 3) of fold cobordism classes (see Definition 2.1) of fold maps (for results about cobordisms of singular maps, see, for example, [3, 4, 8, 9, 11, 12, 14, 16, 17, 21, 26] and the works of A. Szűcs in References). In this way we obtain a geometrical relation between fold maps and immersions with prescribed normal bundles via cobordisms. In [15] we showed that these invariants describe completely the cobordisms of simple fold maps of $(n+1)$-dimensional

2000 *Mathematics Subject Classification.* Primary 57R45; Secondary 57R75, 57R42, 55Q45.
Key words and phrases. Fold singularity, fold map, immersion, cobordism, stable homotopy group.
The author is supported by Canon Foundation in Europe.

manifolds into n-dimensional manifolds and in [14] we showed that these invariants detect direct summands of the cobordism group of fold maps, namely stable homotopy groups of spheres. In this paper we extend the results of [14] and show that these invariants also detect stable homotopy groups of the classifying spaces $BO(k)$.

The paper is organized as follows. In Section 2 we give basic notations and definitions, in Section 3 we define cobordism invariants of fold maps and summerize our already existing results concerning these invariants and study the cobordism classes of manifolds which have fold maps into stably parallelisable manifolds. In Section 4 we extend the results of [14].

1.1. **Notations.** In this paper the symbol "$\amalg$" denotes the disjoint union, for any number x the symbol "$\lfloor x \rfloor$" denotes the greatest integer i such that $i \leq x$, γ^1 denotes the universal line bundle over $\mathbb{R}P^\infty$, ε^1_X (shortly ε^1) denotes the trivial line bundle over the space X, and the symbols ξ^k, η^k, etc. usually denote k-dimensional real vector bundles. The symbols $\det\xi^k$ and $T\xi^k$ denote the determinant line bundle and the Thom space of the bundle ξ^k, respectively. The symbol $\mathrm{Imm}^{\xi^k}_N(n-k,k)$ denotes the cobordism group of k-codimensional immersions into an n-dimensional manifold N whose normal bundles can be induced from ξ^k (this group is isomorphic to the group $\{\dot{N}, T\xi^k\}$, where $\dot{N}$ denotes the one point compactification of the manifold N and the symbol $\{X,Y\}$ denotes the group of stable homotopy classes of continuous maps from the space X to the space Y. The symbol $\mathrm{Imm}^{\xi^k}(n-k,k)$ denotes the cobordism group of k-codimensional immersions into $\mathbb{R}^n$ whose normal bundles can be induced from ξ^k (this group is isomorphic to $\pi^s_n(T\xi^k)$). The symbol $\mathrm{Imm}_N(n-k,k)$ denotes the cobordism group $\mathrm{Imm}^{\gamma^k}_N(n-k,k)$ where γ^k is the universal bundle for k-dimensional real vector bundles and N is an n-dimensional manifold. The symbol $\pi^s_n(X)$ (π^s_n) denotes the nth stable homotopy group of the space X (resp. spheres). The symbol "id_A" denotes the identity map of the space A. The symbol ε denotes a small positive number. All manifolds and maps are smooth of class C^∞.

2. Preliminaries

2.1. **Fold maps.** Let $n \geq 1$ and $q > 0$. Let Q^{n+q} and N^n be smooth manifolds of dimensions $n+q$ and n respectively. Let $p \in Q^{n+q}$ be a singular point of a smooth map $f\colon Q^{n+q} \to N^n$. The smooth map f has a *fold singularity of index* λ at the singular point p if we can write f in some local coordinates around p and $f(p)$ in the form

$$f(x_1,\dots,x_{n+q}) = (x_1,\dots,x_{n-1}, -x_n^2 - \cdots - x_{n+\lambda-1}^2 + x_{n+\lambda}^2 + \cdots + x_{n+q}^2)$$

for some λ $(0 \leq \lambda \leq q+1)$ (the index λ is well-defined if we consider that λ and $q+1-\lambda$ represent the same index).

A smooth map $f\colon Q^{n+q} \to N^n$ is called a *fold map* if f has only fold singularities.

A smooth map $f\colon Q^{n+q} \to N^n$ has a *definite fold singularity* at a fold singularity $p \in Q^{n+q}$ if $\lambda = 0$ or $\lambda = q+1$, otherwise f has an *indefinite fold singularity of index* λ at the fold singularity $p \in Q^{n+q}$.

Let $S_\lambda(f)$ denote the set of fold singularities of index λ of f in Q^{n+q}. Note that $S_\lambda(f) = S_{q+1-\lambda}(f)$. Let S_f denote the set $\bigcup_\lambda S_\lambda(f)$.

Note that the set S_f is an $(n-1)$-dimensional submanifold of the manifold Q^{n+q}.

Note that each connected component of the manifold S_f has its own index λ if we consider that λ and $q+1-\lambda$ represent the same index.

Note that for a fold map $f\colon Q^{n+q} \to \mathbb{R}^n$ and for an index λ $(0 \leq \lambda \leq \lfloor (q-1)/2 \rfloor$ or $q+1-\lfloor (q-1)/2 \rfloor \leq \lambda \leq q+1)$ the codimension one immersion $f|_{S_\lambda(f)}\colon S_\lambda(f) \to \mathbb{R}^n$ of the singular set of index λ $S_\lambda(f)$ has a canonical framing (i.e., trivialization of the normal bundle) by identifying canonically the set of fold singularities of index λ $(0 \leq \lambda \leq \lfloor (q-1)/2 \rfloor$ or $q+1-\lfloor (q-1)/2 \rfloor \leq \lambda \leq q+1)$ of the map f with the fold germ $(x_1, \ldots, x_{n+q}) \mapsto (x_1, \ldots, x_{n-1}, -x_n^2 - \cdots - x_{n+\lambda-1}^2 + x_{n+\lambda}^2 + \cdots + x_{n+q}^2)$ $(0 \leq \lambda \leq \lfloor (q-1)/2 \rfloor)$ (if we consider that λ and $q+1-\lambda$ represent the same index), see, for example, [22].

If $f\colon Q^{n+q} \to N^n$ is a fold map in general position, then the map f restricted to the singular set S_f is a general positional codimension one immersion into the target manifold N^n.

Since every fold map is in general position after a small perturbation, and we study maps under the equivalence relation *cobordism* (see Definition 2.1), in this paper we can restrict ourselves to studying fold maps which are in general position. Without mentioning we suppose that a fold map f is in general position.

2.2. Equivalence relations of fold maps.

Definition 2.1. (Cobordism) Two fold maps $f_i\colon Q_i^{n+q} \to N^n$ $(i = 0, 1)$ of closed (oriented) $(n+q)$-dimensional manifolds Q_i^{n+q} $(i = 0, 1)$ into an n-dimensional manifold N^n are *(oriented) cobordant* if

a) there exists a fold map $F\colon X^{n+q+1} \to N^n \times [0,1]$ of a compact (oriented) $(n+q+1)$-dimensional manifold X^{n+q+1},
b) $\partial X^{n+q+1} = Q_0^{n+q} \amalg (-)Q_1^{n+q}$ and
c) $F|_{Q_0^{n+q} \times [0,\varepsilon)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$ and $F|_{Q_1^{n+q} \times (1-\varepsilon,1]} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$, where $Q_0^{n+q} \times [0,\varepsilon)$ and $Q_1^{n+q} \times (1-\varepsilon, 1]$ are small collar neighbourhoods of ∂X^{n+q+1} with the identifications $Q_0^{n+q} = Q_0^{n+q} \times \{0\}$ and $Q_1^{n+q} = Q_1^{n+q} \times \{1\}$.

We call the map F a *cobordism* between f_0 and f_1.

This clearly defines an equivalence relation on the set of fold maps of closed (oriented) $(n+q)$-dimensional manifolds into an n-dimensional manifold N^n.

We denote the set of fold (oriented) cobordism classes of fold maps of closed (oriented) $(n+q)$-dimensional manifolds into an n-dimensional manifold N^n (into the Euclidean space $\mathbb{R}^n$) by $Cob_{N,f}^{(O)}(n+q,-q)$ (by $Cob_f^{(O)}(n+q,-q)$). We note that we can define a commutative semigroup operation in the usual way on the set of cobordism classes $Cob_{N,f}^{(O)}(n+q,-q)$ by the disjoint union. In the case of $N^n = \mathbb{R}^n$ this semigroup operation is equal to the usual group operation, i.e., the far away disjoint union.

We can refine this equivalence relation by considering the singular fibers (see, for example, [19, 28, 29, 41]) of a fold map.

Definition 2.2. Let τ be a set of singular fibers. Two fold maps $f_i \colon Q_i^{n+q} \to N^n$ ($i = 0,1$) with singular fibers in the set τ of closed (oriented) $(n+q)$-dimensional manifolds Q_i^{n+q} ($i=0,1$) into an n-dimensional manifold N^n are *(oriented) τ-cobordant* if they are (oriented) cobordant in the sense of Definition 2.1 by a fold map $F \colon X^{n+q+1} \to N^n \times [0,1]$ whose singular fibers are in the set τ.

In this way we can obtain the notion of *simple fold cobordism* of *simple fold maps*, i.e., let τ be the set all the singular fibers which have at most one singular point in each of their connected components. We denote the set of simple fold cobordism classes of simple fold maps of closed (oriented) $(n+q)$-dimensional manifolds Q^{n+q} into an n-dimensional manifold N^n by $Cob_{N,s}(n+q,-q)$. For results about simple fold maps, see, for example, [15, 22, 23, 24, 25, 31, 42].

Definition 2.3. (Bordism) Two fold maps $f_i \colon Q_i^{n+q} \to N_i^n$ ($i=0,1$) from closed (oriented) $(n+q)$-dimensional manifolds Q_i^{n+1} ($i=0,1$) into closed oriented n-dimensional manifolds N_i^n ($i=0,1$) are *(oriented) bordant* if

a) there exists a fold map $F \colon X^{n+q+1} \to Y^{n+1}$ of a compact (oriented) $(n+q+1)$-dimensional manifold X^{n+q+1} to a compact oriented $(n+1)$-dimensional manifold Y^{n+1},
b) $\partial X^{n+q+1} = Q_0^{n+q} \amalg (-)Q_1^{n+q}$, $\partial Y^{n+1} = N_0^{n+1} \amalg -N_1^{n+1}$ and
c) $F|_{Q_0^{n+q} \times [0,\varepsilon)} = f_0 \times id_{[0,\varepsilon)}$ and $F|_{Q_1^{n+q} \times (1-\varepsilon,1]} = f_1 \times id_{(1-\varepsilon,1]}$, where $Q_0^{n+q} \times [0,\varepsilon)$ and $Q_1^{n+q} \times (1-\varepsilon,1]$ are small collar neighbourhoods of ∂X^{n+q+1} with the identifications $Q_0^{n+q} = Q_0^{n+q} \times \{0\}$, $Q_1^{n+q} = Q_1^{n+q} \times \{1\}$.

We call the map F a *bordism* between f_0 and f_1.

We can define a commutative group operation on the set of bordism classes by $[f_0] + [f_1] = f_0 \amalg f_1 \colon Q_0^{n+q} \amalg Q_1^{n+q} \to N_0^n \amalg N_1^n$ in the usual way.

Remark 2.4. Our results can be easily adapted to bordisms and bordism groups of fold maps even though we do not state them explicitly. In most of the cases if we replace the notion "cobordism" by "bordism", then we obtain the corresponding result about bordisms of fold maps.

3. Cobordism invariants of fold maps

3.1. **Fold germs and bundles of germs.** Let us define the fold germ $g_{\lambda,q}\colon (\mathbb{R}^{q+1},0) \to (\mathbb{R},0)$ by

$$g_{\lambda,q}(x_1,\dots,x_{q+1}) = (-x_1^2 - \cdots - x_\lambda^2 + x_{1+\lambda}^2 + \cdots + x_{1+q}^2)$$

for some $q \geq 1$ and $0 \leq \lambda \leq \lfloor (q+1)/2 \rfloor$.

We say that a pair of diffeomorphism germs $(\alpha\colon (\mathbb{R}^{q+1},0) \to (\mathbb{R}^{q+1},0), \beta\colon (\mathbb{R},0) \to (\mathbb{R},0))$ is an *automorphism* of a fold germ $g_{\lambda,q}\colon (\mathbb{R}^{q+1},0) \to (\mathbb{R},0)$ if the equation $g_{\lambda,q} \circ \alpha = \beta \circ g_{\lambda,q}$ holds. We will work with bundles whose fibers and structure groups are germs and groups of automorphisms of germs, respectively.

If we have a fold map $f\colon Q^{n+q} \to N^n$, then for each λ $(0 \leq \lambda \leq \lfloor (q+1)/2 \rfloor)$ we have a fold germ bundle $\xi_\lambda(f)\colon E(\xi_\lambda(f)) \to S_\lambda(f)$ over the singular set of index λ $S_\lambda(f)$, i.e., the fiber of $\xi_\lambda(f)$ is the fold germ $g_{\lambda,q}$, and over the singular set $S_\lambda(f)$ we have an $(\mathbb{R}^{q+1},0)$ bundle denoted by $\xi_\lambda^{q+1}(f)\colon E(\xi_\lambda^{q+1}(f)) \to S_\lambda(f)$ and an $(\mathbb{R},0)$ bundle denoted by $\eta_\lambda^1(f)\colon E(\eta_\lambda^1(f)) \to S_\lambda(f)$ together with a fiberwise map $E(\xi_\lambda(f))\colon E(\xi_\lambda^{q+1}(f)) \to E(\eta_\lambda^1(f))$ which is equivalent on each fiber to the fold germ $g_{\lambda,q}$. The base space of the fold germ bundle $\xi_\lambda(f)$ is the singular set of index λ $S_\lambda(f)$ and the total space of this bundle $\xi_\lambda(f)$ is the fiberwise map $E(\xi_\lambda(f))\colon E(\xi_\lambda^{q+1}(f)) \to E(\eta_\lambda^1(f))$ between the total spaces of the bundles $\xi_\lambda^{q+1}(f)$ and $\eta_\lambda^1(f)$. We call the bundle $\eta_\lambda^1(f)$ the *target of the fold germ bundle* $\xi_\lambda(f)$.

By [10, 34, 39] this bundle $\xi_\lambda(f)$ is a locally trivial bundle in a sense with a fiber $g_{\lambda,q}$ and an appropriate group of automorphisms $(\alpha\colon (\mathbb{R}^{q+1},0) \to (\mathbb{R}^{q+1},0), \beta\colon (\mathbb{R},0) \to (\mathbb{R},0))$ as structure group. By [10, 39] this structure group can be reduced to a maximal compact subgroup, namely to the group $O(\lambda) \times O(q+1-\lambda)$ in the case of $0 \leq \lambda < (q+1)/2$ and the group generated by the group $O(\lambda) \times O(\lambda)$ and the transformation $T = \begin{pmatrix} 0 & I_\lambda \\ I_\lambda & 0 \end{pmatrix}$ in the case of $\lambda = (q+1)/2$, see, for example, [22]. We denote this latter group by $\langle O(\lambda) \times O(\lambda), T \rangle$.

It follows that the targets of the universal fold germ bundles of index λ $(0 \leq \lambda \leq \lfloor (q+1)/2 \rfloor)$ are the trivial line bundles $\eta_{\lambda,q}^1\colon \varepsilon^1 \to B(O(\lambda) \times O(q+1-\lambda))$ for $\lambda \neq (q+1)/2$ and the appropriate line bundle $\eta_{(q+1)/2,q}^1\colon l^1 \to B\langle O(\lambda) \times O(\lambda), T \rangle$ for q odd.

3.2. **Immersions with prescribed normal bundles.** We can construct homomorphisms

$$\xi^N_{\lambda,q}\colon Cob_{N,f}(n+q,-q) \to \mathrm{Imm}_N^{\varepsilon^1_{B(O(\lambda)\times O(q+1-\lambda))}}(n-1,1)$$

for $0 \le \lambda < (q+1)/2$ and

$$\xi^N_{(q+1)/2,q}\colon Cob_{N,f}(n+q,-q) \to \mathrm{Imm}_N^{l^1}(n-1,1)$$

for q odd by mapping a cobordism class of a fold map f into the cobordism class of the immersion of its fold singular set of index λ $S_\lambda(f)$ with normal bundle induced from the target of the universal fold germ bundle of index λ.

Since the cobordism group of k-codimensional immersions into a manifold N^n with normal bundle induced from a vector bundle ξ^k is isomorphic to the group of stable homotopy classes $\{\dot N, T\xi^k\}$ [40], the homomorphisms $\xi^N_{\lambda,q}$ for $\lambda \neq (q+1)/2$ and $\xi^N_{(q+1)/2,q}$ for q odd can be considered as homomorphisms into the groups $\{\dot N, T\varepsilon^1_{B(O(\lambda)\times O(q+1-\lambda))}\}$ and $\{\dot N, Tl^1\}$, respectively. Without mentioning we identify the cobordism group of k-codimensional immersions into a manifold N^n with normal bundle induced from a vector bundle ξ^k with the group of stable homotopy classes $\{\dot N, T\xi^k\}$.

We remark that the group $\{\dot N, T\varepsilon^1_{B(O(\lambda)\times O(q+1-\lambda))}\}$ is equal to the group $\{\dot N, S^1 \vee SB(O(\lambda)\times O(q+1-\lambda))\} \cong \{\dot N, S^1\} \oplus \{\dot N, SB(O(\lambda)\times O(q+1-\lambda))\}$. Therefore the homomorphisms $\xi^N_{\lambda,q}$ $(\lambda \neq (q+1)/2)$ can be written in the forms

$$\xi^N_{\lambda,q,1} \oplus \xi^N_{\lambda,q,2}\colon Cob_{N,f}(n+q,-q) \to \{\dot N, S^1\} \oplus \{\dot N, SB(O(\lambda)\times O(q+1-\lambda))\}$$

obviously. Note that the homomorphism $\xi^N_{\lambda,q,1}$ maps the fold cobordism class of a fold map f into the cobordism class of the framed immersion of the singular set of index λ of the fold map f $(0 \le \lambda < (q+1)/2)$.

Note that $B(O(\lambda)\times O(q+1-\lambda)) = BO(\lambda)\times BO(q+1-\lambda)$ and there exists a composition of bundle maps $\varepsilon^1_{BO(q+1-\lambda)} \to \varepsilon^1_{B(O(\lambda)\times O(q+1-\lambda))} \to \varepsilon^1_{BO(q+1-\lambda)}$ which is the identity map. Therefore the group $\{\dot N, SBO(q+1-\lambda)\}$ is a direct summand of the group $\{\dot N, SB(O(\lambda)\times O(q+1-\lambda))\}$.

Let $\varrho^N_{\lambda,q}\colon \mathrm{Imm}_N^{\varepsilon^1_{B(O(\lambda)\times O(q+1-\lambda))}}(n-1,1) \to \mathrm{Imm}_N^{\varepsilon^1_{BO(q+1-\lambda)}}(n-1,1)$ denote the natural forgetting homomorphism. Then we have weaker cobordism invariants

$$\varrho^N_{\lambda,q}\circ\xi^N_{\lambda,q}\colon Cob_{N,f}(n+q,-q) \to \{\dot N, S^1\} \oplus \{\dot N, SBO(q+1-\lambda)\}$$

$(0 \le \lambda < (q+1)/2)$.

Let $\tilde{\theta}_q^N\colon \mathrm{Imm}_N^{l^1}(n-1,1) \to \mathrm{Imm}_N(n-1,1)$ be the natural forgetting homomorphism, where $\eta^1_{(q+1)/2,q}\colon l^1 \to B\langle O(\lambda) \times O(\lambda), T\rangle$ is the target of the universal fold germ bundle of index $(q+1)/2$ for q odd.

A result about these invariants, that we obtain similarly to [14], is the following.

Theorem 3.1. *For $n \geq 1$, an n-dimensional manifold N^n and $q > 0$ the cobordism semigroup $Cob_{N,f}^{(O)}(n+q,-q)$ of fold maps of (oriented) $(n+q)$-dimensional manifolds into N^n contains the direct sum $\bigoplus_{\lambda=0}^{\lfloor (q-1)/2 \rfloor}\{\dot{N}, S^1\}$ as a direct summand. This direct sum $\bigoplus_{\lambda=0}^{\lfloor (q-1)/2 \rfloor}\{\dot{N}, S^1\}$ is detected by the homomorphisms $\xi_{\lambda,q,1}^N\colon Cob_{N,f}^{(O)}(n+q,-q) \to \{\dot{N}, S^1\}$ $(\lambda = 0, \ldots, \lfloor (q-1)/2 \rfloor)$.*

Theorem 3.2. *For $n \geq 1$, an n-dimensional manifold N^n, $q > 0$, $k \geq 1$ and $q = 2k-1$ the cobordism semigroup $Cob_{N,f}(n+q,-q)$ of fold maps of unoriented $(n+q)$-dimensional manifolds into N^n contains the direct sum $\mathrm{Imm}_N(n-1,1) \oplus \bigoplus_{\lambda=0}^{\lfloor (q-1)/2 \rfloor}\{\dot{N}, S^1\}$ as a direct summand. The direct summand $\mathrm{Imm}_N(n-1,1)$ is detected by the homomorphism $\tilde{\theta}_q^N \circ \xi_{(q+1)/2,q}^N\colon Cob_{N,f}(n+q,-q) \to \mathrm{Imm}_N(n-1,1)$, where $\tilde{\theta}_q^N \circ \xi_{(q+1)/2,q}^N$ maps a fold cobordism class $[f]$ to the cobordism class of the immersion of the singular set of index k of the fold map f.*

Remark 3.3. For q even, in Theorems 3.1 and 3.2 we could also chose the indeces $\lambda = 1, \ldots, \lfloor (q+1)/2 \rfloor$ for the homomorphisms $\xi_{\lambda,q,1}^N$ instead of the indeces $\lambda = 0, \ldots, \lfloor (q-1)/2 \rfloor$. The proof is similar to that of [14], details are left to the reader.

Another application of our invariants is the following result about simple fold maps, which we obtained in [15].

Let

$$\gamma_n^N\colon \mathrm{Imm}_N^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}_N^{\varepsilon^1 \times \gamma^1}(n-2,2) \to \mathrm{Imm}_N(n-1,1) \oplus \mathrm{Imm}_N^{\gamma^1 \times \gamma^1}(n-2,2)$$

denote the natural forgetting homomorphism, $\phi_n^N\colon Cob_{N,s}^O(n+1,-1) \to Cob_{N,f}^O(n+1,-1)$ denote the natural homomorphism which maps a simple fold cobordism class into its fold cobordism class.

Let $q = 1$ and let N^n be an n-dimensional oriented manifold. In [15] we defined a semigroup homomorphism

$$\mathcal{I}_N\colon Cob_{N,s}^O(n+1,-1) \to \mathrm{Imm}_N^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}_N^{\varepsilon^1 \times \gamma^1}(n-2,2),$$

which is just an adaptation of our invariant $\xi_{1,1}^N$ to the case of simple fold maps of oriented manifolds into oriented manifolds and their oriented simple fold cobordisms.

In [15] we showed that the target of the universal fold germ bundle of index 1 when the source and target manifolds are oriented, is the line bundle $\eta^1_{1,1}\colon \det(\gamma^1 \times \gamma^1) \to$

$\mathbb{R}P^\infty \times \mathbb{R}P^\infty$, there exists a homomorphism

$$\theta_n^N \colon \mathrm{Imm}_N^{\det(\gamma^1\times\gamma^1)}(n-1,1) \to \mathrm{Imm}_N(n-1,1) \oplus \mathrm{Imm}_N^{\gamma^1\times\gamma^1}(n-2,2)$$

such that the diagram

$$\begin{array}{ccc} Cob_{N,s}^O(n+1,-1) & \xrightarrow{\mathcal{I}_N} & \mathrm{Imm}_N^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}_N^{\varepsilon^1\times\gamma^1}(n-2,2) \\ \downarrow \phi_n^N & & \gamma_n^N \downarrow \\ Cob_{N,f}^O(n+1,-1) & \xrightarrow{\theta_n^N\circ\xi_{1,1}^N} & \mathrm{Imm}_N(n-1,1) \oplus \mathrm{Imm}_N^{\gamma^1\times\gamma^1}(n-2,2). \end{array} \tag{3.1}$$

commutes and we obtained the following.

Theorem 3.4. *Let N^n be an oriented manifold. Then, the semigroup homomorphism $\mathcal{I}_N$ is a semigroup isomorphism between the cobordism semigroup $Cob_{N,s}^O(n+1,-1)$ of simple fold maps and the group $\mathrm{Imm}_N^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}_N^{\varepsilon^1\times\gamma^1}(n-2,2)$.*

Let

$$\gamma_{n,1}^N \colon \mathrm{Imm}_N^{\varepsilon^1}(n-1,1) \to \mathrm{Imm}_N(n-1,1)$$

and

$$\gamma_{n,2}^N \colon \mathrm{Imm}_N^{\varepsilon^1\times\gamma^1}(n-2,2) \to \mathrm{Imm}_N^{\gamma^1\times\gamma^1}(n-2,2).$$

denote the natural forgetting homomorphisms.

Let $\pi_{n,2}^N \colon Cob_{N,s}^O(n+1,-1) \to \mathrm{Imm}_N^{\varepsilon^1\times\gamma^1}(n-2,2)$ denote the projection to the second factor where we identify the semigroup $Cob_{N,s}^O(n+1,-1)$ with the group $\mathrm{Imm}_N^{\varepsilon^1}(n-1,1) \oplus \mathrm{Imm}_N^{\varepsilon^1\times\gamma^1}(n-2,2)$ by the isomorphism $\mathcal{I}_N$.

Theorem 3.5. *If two simple fold cobordism classes $[f]$ and $[g]$ in $Cob_{N,s}^O(n+1,-1)$ are mapped into distinct elements by the natural homomorphism $\gamma_{n,2}^N \circ \pi_{n,2}^N$, then $[f]$ and $[g]$ are not fold cobordant. If $\gamma_{n,2}^N$ is injective, then so is ϕ_n^N.*

If there exists a fold map from a not null-cobordant $(n+1)$-dimensional manifold into N^n, then ϕ_n^N is not surjective.

3.3. **Pontryagin-Thom type construction.** In [16] among others we show the following, which is a negative codimensional analogue of the Pontryagin-Thom type construction for singular maps in positive codimension [21, 32, 33, 35, 36, 37].

Theorem 3.6. *There is a Pontryagin-Thom type construction for -1 codimensional fold maps, i.e.,*

(1) *there exists a universal fold map* $\xi_{-1}\colon U_{-1} \to \Gamma_{-1}$ *such that for every* -1 *codimensional fold map* $f\colon Q^{n+1} \to N^n$ *there exists a commutative diagram*

$$\begin{array}{ccc} Q^q & \longrightarrow & U_{-1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \xi_{-1}} \\ N^n & \xrightarrow{\chi_f} & \Gamma_{-1} \end{array}$$

(2) *for every positive integer* n *and* n*-dimensional manifold* N^n *there is a natural bijection*

$$\chi_*^N\colon Cob_{N^n,f}(n+1,-1) \to [\dot{N}^n, \Gamma_{-1}]$$

between the set of fold cobordism classes $Cob_{N^n,f}(n+1,-1)$ *and the set of homotopy classes* $[\dot{N}^n,\Gamma_{-1}]$. *The map* χ_*^N *maps a fold cobordism class* $[f]$ *into the homotopy class of the inducing map* $\chi_f\colon \dot{N}^n \to \Gamma_{-1}$.

By Theorem 3.6 we have a bijective cobordism invariant $\chi_*^N\colon Cob_{N^n,f}(n+1,-1) \to [\dot{N}^n, \Gamma_{-1}]$ which is a group isomorphism $\chi_*^{\mathbb{R}^n}\colon Cob_f(n+1,-1) \to \pi_n(\Gamma_{-1})$ in the case of $N^n = \mathbb{R}^n$.

By defining the singular sets of index 0 and 1 of the universal fold map $\xi_{-1}\colon U_{-1} \to \Gamma_{-1}$ in the obvious way and by inducing the immersions of these singular sets into the space Γ_{-1} we get two representatives of two stable homotopy classes in the groups $\{\Gamma_{-1}, T\varepsilon^1_{BO(2)}\}$ and $\{\Gamma_{-1}, Tl^1\}$, respectively, i.e., a map $\sigma_0\colon S^K\Gamma_{-1} \to S^K T\varepsilon^1_{BO(2)}$ and a map $\sigma_1\colon S^K\Gamma_{-1} \to S^K Tl^1$, respectively, where K is a big integer.

If we have a fold map $f\colon Q^{n+1} \to N^n$, then we have the stable homotopy class χ_f^s of the inducing map $\chi_f\colon \dot{N}^n \to \Gamma_{-1}$ in the group $\{\dot{N}^n, \Gamma_{-1}\}$. Hence we have the elements $\sigma_0 \circ \chi_f^s$ and $\sigma_1 \circ \chi_f^s$ in the groups $\{\dot{N}^n, T\varepsilon^1_{BO(2)}\}$ and $\{\dot{N}^n, Tl^1\}$, respectively, which correspond to the elements $\xi^N_{0,1}([f])$ and $\xi^N_{1,1}([f])$, respectively.

Therefore we have the following.

Proposition 3.7. *The cobordism invariants* $\xi^N_{0,1}$ *and* $\xi^N_{1,1}$ *can be induced from the stable homotopy classes* σ_0 *and* σ_1.

3.4. **Cobordism class of the source manifold of a fold map.** We have a natural homomorphism $\sigma^O_{N,q}\colon Cob^O_{N,f}(n+q,-q) \to \Omega_{n+q}$ which assigns to a class of a fold map $f\colon Q^{n+q} \to N^n$ the cobordism class $[Q^{n+q}]$ of the source manifold Q^{n+q}.

It is an easy fact that $\sigma^O_{\mathbb{R},q}$ is surjective and the image of $\sigma^O_{\mathbb{R}^2,q}$ consists of the cobordism classes of $(2+q)$-dimensional manifolds with even Euler characteristic [18].

Proposition 3.8. *Let* N^n *be a stably parallelisable* n*-dimensional manifold, where* n *is even. Let* $f\colon Q^{n+1} \to N^n$ *be a fold map of an orientable manifold* Q^{n+1} *such that its* *singular set S_f is orientable. Then, the oriented cobordism class of the source manifold Q^{n+1} is zero.*

Remark 3.9. Proposition 3.8 generalizes the analogous result about simple fold maps [15, 22].

Proposition 3.10. *Let q be even and let N^n be a stably parallelisable manifold. Then, the rank of the image of $\sigma^O_{N,q}$ is less than or equal to the number of partitions of $(n+q)/4$ where each number in a partition is less than or equal to $(q+1)/2$. In other words, if $n > q+2$, then the homomorphism*

$$\sigma^O_{N,q} \otimes \mathbb{Q}\colon\ Cob^O_{N,f}(n+q,-q) \otimes \mathbb{Q} \to \Omega_{n+q} \otimes \mathbb{Q}$$

is not surjective.

Corollary 3.11. *Let N^n be a stably parallelisable manifold.*

(1) *The orientable $(n+2)$-dimensional manifolds which have fold map into N^n generate a subgroup with rank at most 1 of the cobordism group of $(n+2)$-dimensional manifolds.*
(2) *Let $n = 4k-2$. Let M^{4k} be a $(4k)$-dimensional oriented manifold which has a fold map into the stably parallelisable manifold N^{4k-2}. Then, the signature $\sigma(M^{4k})$ of M^{4k} is equal to $\frac{2^{2k}B_k}{(2k)!}(-1)^{k+1}\langle p_1^k(M^{4k}),[M^{4k}]\rangle$, where B_k denotes the kth Bernoulli number.*
(3) *Let $n = 4k-1$. If M^{4k} has a fold map into N^{4k-1} such that the singular set S_f is orientable, then the same holds for the signature of M^{4k} as above.*

For other results about the signatures of source manifolds of fold maps, see, for example, [27, 29, 30].

4. Subgroups of the cobordism group of fold maps

In this section we extend the results of Theorems 3.1 and 3.2.

Let $O(1,k)$ denote the subgroup of the orthogonal group $O(k+1)$ whose elements are of the form $\begin{pmatrix}1 & 0\\ 0 & M\end{pmatrix}$ where M is an element of the group $O(k)$.

Theorem 4.1. *For $q > 1$, the cobordism semigroup $Cob_{N,f}(n+q,-q)$ contains the direct sum*

$$\{\dot{N},S^1\} \oplus \{\dot{N},SB(O(1)\times O(q))\} \ominus \bigoplus_{2\le\lambda<(q+1)/2} \{\dot{N},S^1\} \oplus \{\dot{N},SBO(q+1-\lambda)\}$$

as a direct summand.

Remark 4.2. It follows that the composition

$$\xi_{j,q}^N \circ \alpha_{j,q}^N \colon \operatorname{Imm}_N^{\varepsilon^1_{B(O(1,j-1)\times O(q+1-j))}}(n-1,1) \to \operatorname{Imm}_N^{\varepsilon^1_{B(O(j)\times O(q+1-j))}}(n-1,1)$$

is equal to the natural homomorphism

$$\beta_{j,*}\colon \{\dot{N},S^1\}\oplus\{\dot{N},SB(O(1,j-1)\times O(q+1-j))\} \to \{\dot{N},S^1\}\oplus\{\dot{N},SB(O(j)\times O(q+1-j))\}$$

induced by the map $\beta_j\colon BO(1,j-1)\to BO(j)$ $(2\leq j<(q+1)/2$. Therefore if the map $\beta_{j,*}$ is injective or an isomorphism, then the cobordism semigroup $Cob_{N,f}(n+q,-q)$ contains the group $\{\dot{N},SB(O(1,j-1)\times O(q+1-j))\}$ as a subgroup or as a direct summand, respectively.

For example, when $n=2$ and $N^2=\mathbb{R}^2$, we have that the cobordism group $Cob_f(n+q,-q)$ contains the direct sum

$$\bigoplus_{1\leq j<(q+1)/2} \pi_1^s\oplus\pi_1^s(B(O(1,j-1)\times O(q+1-j))) = \begin{cases} \mathbb{Z}_2^{3q/2} & (q \text{ even}) \\ \mathbb{Z}_2^{3(q-1)/2} & (q \text{ odd}) \end{cases}$$

as a direct summand, where $O(1,0)$ denotes the orthogonal group $O(1)$.

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