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A differentiable structure on a mapping space quotient and its application to the moduli problem

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1 Introduction.

The aim of this note is to provide an announcement of the results of the paper [9]. All results announced in this note are proved with additional descriptions in the full paper [9].

In differential geometry and singularity theory of differentiable mappings, we often encounter the classification problem inducing several moduli. We treat a class of mappings which forms an infinite dimensional space and, as a result of the classification, we obtain the quotient space which is of finite or infinite dimension. In this paper we introduce the general method to give an differentiable structures on such a quotient space.

The method to provide a "differentiable structure" to a mapping space quotient (a moduli space) should be not unique [2][16]. For instance, consider the problem how to define a differentiable structure on a mapping space $C^\infty(N, M)$ itself for C^∞ manifolds N and M. Then one of the standard methods seems to define, first, Fréchet differentiable functions on the Banach manifolds $C^r(N, M)$, for each finite r, and regard $C^\infty(N, M)$ as the inverse limit of $C^r(N, M)$ to define the structure sheaf of differentiable functions on

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it. However we apply another method in [9]: We regard Fréchet differential of a functional as a kind of "total differential". Then we could consider, instead, "partial differentials". Namely, to define C^∞ functions on $C^\infty(N, M)$, first we define the notion of finite dimensional C^∞ families in $C^\infty(N, M)$ by the very classical and natural manner. Then we call a function on $C^\infty(N, M)$ of class C^∞ if its restriction to any finite dimensional family in $C^\infty(N, M)$ is of class C^∞ in the ordinary sense.

First we describe our idea in §2 and §3. Then we explain the case of quotients of finite dimensional manifolds in §4. In 5, we give the general definition of differentiable structures on mapping space quotients. In §6, we apply our method to treat the classification problem of plane curves by symplectomorphisms.

2 What are structures?

Let $\{X_\nu\}$ be a family of sets. The family X_ν is supposed to consist of quotients of subspaces of a topological space, in particular a mapping space $C^\infty(N, M)$ for manifolds N, M.

To define a "differentiable structure" on each X_ν from $\{X_\nu\}$, it is sufficient to give a criterion, for each pair $X_\nu, X_{\nu'}$, X_ν and $X_{\nu'}$ are "diffeomorphic". For that it is sufficient to give a criterion that a mapping $\Phi : X_\nu \to X_{\nu'}$ is "differentiable" or not.

Then, for example, how should we define that a given mapping $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ (L, W are manifolds) is "differentiable"?

Let $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ be a mapping. Then, for each differentiable mapping $f \in C^\infty(N, M)$, there corresponds a differentiable mapping $\Phi(f) \in C^\infty(L, W)$. Now we propose to call Φ differentiable if, for any "differentiable" family $h_\lambda \in C^\infty(N, M)$, $\Phi(h_\lambda) \in C^\infty(L, W)$ is "differentiable", where the "parameter" λ runs over a finite dimensional manifold Λ. In fact moreover we demand that Φ is continuous. As an ordinary term in global analysis and differential topology, we call $h_\lambda : N \to M$, $(\lambda \in \Lambda)$ is a *differentiable family* if there exists a differentiable mapping $H : \Lambda \times N \to M$ which satisfies $h_\lambda(x) = H(\lambda, x)$ for each $(\lambda, x) \in \Lambda \times N$. Then the mapping $h : \Lambda \to C^\infty(N, M)$ defined by $h(\lambda) = h_\lambda$ is called *differentiable* naturally.

Then for $\Phi(h_\lambda) \in C^\infty(L, W)$, we can take a differentiable mapping $G : \Lambda \times L \to W$ with $\Phi(h_\lambda)(x') = G(\lambda, x')$, $(\lambda, x') \in L \times W$. Therefore we can take the derivative of $\Phi(h_\lambda)$ with respect to λ.

3 Differentiability along finite dimensional directions.

Consider another example. How to define the differentiability of a functional $\Psi : C^\infty(L, W) \to \mathbb{R}$? The real value $\Psi(g)$ is determined for each mapping $g \in C^\infty(L, W)$. The function $\Psi(g_\lambda)$ of variable λ is determined for finite dimensional differentiable family $g_\lambda \in C^\infty(L, W), \lambda \in \Lambda$. Then we call a mapping $\Psi : C^\infty(L, W) \to \mathbb{R}$ *differentiable* if the function $\Psi(g_\lambda)$ is differentiable on λ. We regard each $g_\lambda \in C^\infty(L, W)$ as a point in the space $C^\infty(L, W)$. Then the family of mapping $g_\lambda \in C^\infty(L, W)$ is regarded as a finite dimensional subspace in $C^\infty(L, W)$. The family $\Psi(g_\lambda)$ is the restriction of Ψ to there, and we look at the differentiability of $\Psi(g_\lambda)$ in the ordinary sense. The differentiability we are going to define may be called the *differentiability along finite dimensional directions.*

If $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ and $\Psi : C^\infty(L, W) \to \mathbb{R}$ are differentiable then the composition $\Psi \circ \Phi : C^\infty(N, M) \to \mathbb{R}$ is differentiable. If fact, for any differentiable family $h_\lambda \in C^\infty(N, M)$, we have $(\Psi \circ \Phi)(h_\lambda) = \Psi(\Phi(h_\lambda))$ and $\Phi : C^\infty(N, M) \to C^\infty(L, W)$ is differentiable, we see $\Phi(h_\lambda)$ is differentiable on λ. Since Ψ is differentiable, $\Psi(\Phi(h_\lambda))$ is differentiable, so is $(\Psi \circ \Phi)(h_\lambda)$ on λ.

We have defined that $\Psi : C^\infty(L, W) \to \mathbb{R}$ is differentiable. On the other hand, since $\mathbb{R}$ is identified with $C^\infty(\{\mathrm{pt}\}, \mathbb{R})$, we can regard $\Psi : C^\infty(L, W) \to C^\infty(\mathrm{pt}, \mathbb{R})$. Then Ψ is differentiable in the sense of the first definition. In fact, for any differentiable family $g_\lambda \in C^\infty(L, W)$, $\Psi(g_\lambda)$ is differentiable on λ. If we define $H : \Lambda \times \{\mathrm{pt}\} \to \mathbb{R}$ by $H(\lambda, \mathrm{pt}) = \Psi(g_\lambda)$, then H is differentiable. By definition, $\Psi : C^\infty(L, W) \to C^\infty(\{\mathrm{pt}\}, \mathbb{R})$ is differentiable.

4 Differential structure of manifold quotients.

First we start with the case that the mapping space is a subset of a finite dimensional manifold N which will be identified with the space $C^\infty(\{\mathrm{pt}\}, N)$.

Let N be a differentiable manifold, S a subset of N, and $\sim$ a equivalence relation on S. Assume Λ, M and Q are also differentiable manifolds which play a role of "test spaces".

Then the differentiability is introduced inductively as follows:

(1) We call a mapping $h : \Lambda \to S$ from a manifold to a subset of a manifold *differentiable* if the composed mapping $h : \Lambda \to S \hookrightarrow N$ is a differentiable

mapping from the manifold Λ to the manifold N.

(2) We call a mapping $k : S \to Q$ from a subset of a manifold to a manifold *differentiable* if k is continuos, and, for any differentiable mapping $h : \Lambda \to S$ in the sense of (1), the composed mapping $k \circ h : \Lambda \to Q$ is a differentiable mapping from the manifold Λ to the manifold Q.

(3) We call a mapping $\ell : S/\sim \to Q$ from a quotient of a subset of a manifold to a manifold *differentiable* if the composed mapping $\ell \circ \pi : S \to S/\sim \to Q$ is differentiable in the sense of (2).

(4) We call a mapping $m : \Lambda \to S/\sim$ from a manifold to a quotient of a subset of a manifold *differentiable* if, for any differentiable mapping $\ell : U \to Q$ in the sense of (3), from an open subset $\subseteq S/\sim$, the composed mapping $\ell \circ m : m^{-1}(U) \to Q$ is a differentiable mapping from the manifold $m^{-1}(U)$ to the manifold Q.

More generally:

(5) We call a mapping $\varphi : S/\sim \to T/\approx \leftarrow T \subseteq M$ from a quotient of a subset of a manifold to another quotient of a subset of a manifold *differentiable* if φ is continuous and, for any differentiable mapping $\ell : U(\subseteq T/\approx) \to Q$ in the sense of (3), the composed mapping $\ell \circ \varphi : \varphi^{-1}(U) \to Q$ is differentiable in the sense of (3).

(6) A mapping $\varphi : S/\sim \to T/\approx$ is called a *diffeomorphism* if φ is differentiable in the sense of (5), bijective, and the inverse mapping $\varphi^{-1} : T/\approx \to S/\sim$ is differentiable in the sense of (5).

(7) The quitient spaces $S/\sim$ and $T/\approx$ are called *diffeomorphic* if there exists a diffeomorphism $\varphi : S/\sim \to T/\approx$.

Remark 4.1 There is a different definition for the stage (2) (cf. [15]): A mapping $k : S \to Q$ is called differentiable if there exists an open neighborhood U in N and a differentiable mapping $\bar{k} : U \to Q$ satisfying $\bar{k}|_S = k$. Compared with this definition which is based on extensions of mappings on S, our definition is based on parametrizations of S and may be called a "parametric-minded" definition.

Example 4.2 (Differentiable structure on orbifolds). Let G be a finite subgroup of $\mathrm{GL}(n, \mathbb{R})$ which acts on $\mathbb{R}^n$ naturally.

By the above general theory, we can endow with the "orbifold" $\mathbb{R}^n/G$ the ordinary differentiable structure.

Example 4.3 The quotient space $\mathbb{R}/\sim$ is diffeomorphic to $\mathbb{R}_{\geq 0}$, where $\sim$ is an equivalence relation on $\mathbb{R}$ defined by that $x \sim x'$ if and only if $x' = \pm x$.

In fact $\varphi : \mathbb{R}/\sim \to \mathbb{R}_{\geq 0}$, $\varphi([x]) = x^2$ is a diffeomorphism. For, $\varphi \circ \pi : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $(\varphi \circ \pi)(x) = x^2$ is a continuous differentiable mapping by (1), we see φ is a differentiable mapping by (3). The inverse mapping is given by $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}/\sim$, $\psi(y) = [\sqrt{y}]$. To see ψ is differentiable, we check, based on (5), for any differentiable mapping $\ell : \mathbb{R}/\sim \to Q$, that $\ell \circ \psi : \mathbb{R}_{\geq 0} \to Q$ is differentiable. By (3), $\ell \circ \pi : \mathbb{R} \to Q$ is differentiable. Since $(\ell \circ \pi)(x) = (\ell \circ \pi)(-x)$, we see there exists a differentiable mapping $\rho : \mathbb{R} \to Q$ with $(\ell \circ \pi)(x) = \rho(x^2)$. Then $(\ell \circ \psi)(y) = \ell([\sqrt{y}]) = (\ell \circ \pi)(\sqrt{y}) = \rho(y)$. Thus $\ell \circ \psi$ is differentiable. $\square$

Example 4.4 We give the equivalence relation $\sim$ on $\mathbb{R}^2$ by that $(x, y) \sim (x', y')$ if and only if $(x', y') = \pm(x, y)$. Then we see $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{R}^2$ but $\mathbb{R}^2/\sim$ is not diffeomorphic to $\mathbb{R}^2$.

The mapping $s : \mathbb{R}^2/\sim \to \mathbb{R}^2$, $s([(x,y)]) = (x^2 - y^2, 2xy)$ is a homeomorphism. However s is not a diffeomorphism. Moreover we see that there exists no diffeomorphism between $\mathbb{R}^2/\sim$ and $\mathbb{R}^2$. To see that, suppose that there exist a differentiable mapping $\psi : \mathbb{R}^2 \to \mathbb{R}^2/\sim$ and a differentiable mapping $\varphi : \mathbb{R}^2/\sim \to \mathbb{R}^2$ satisfying $\psi \circ \varphi = \mathrm{id}, \varphi \circ \psi = \mathrm{id}$. Since $\varphi \circ \pi : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable and invariant under the transformation $(x, y) \mapsto (-x, -y)$, there exists a differentiable mapping $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying $(\varphi \circ \pi)(x, y) = \rho(x^2, xy, y^2)$. Therefore there exists a differentiable mapping $\Phi : \mathbb{R}^2/\sim \to \mathbb{R}^3$ with $(\Phi \circ \pi)(x, y) = (x^2, xy, y^2)$ so with $\varphi \circ \pi = \rho \circ \Phi \circ \pi$. Since π is a surjective, we have $\varphi = \rho \circ \Phi$. Therefore $\mathrm{id} = \varphi \circ \psi = \rho \circ (\Phi \circ \psi) : \mathbb{R}^2 \to \mathbb{R}^2$. However the image of $\Psi := \Phi \circ \psi : \mathbb{R}^2 \to \mathbb{R}^3$ is contained in $\{(x^2, xy, y^2) \mid (x, y) \in \mathbb{R}^2\} = \{(X, Y, Z) \in \mathbb{R}^3 \mid XZ - Y^2 = 0\}$ and thus $\mathrm{rank}_0 \Psi \leq 1$. This leads a contradiction. $\square$

5 Differentiable structure on a mapping space quotient.

We denote by $C^\infty(N, M)$ the space of C^∞ mappings from a (finite dimensional) C^∞ manifold N to a (finite dimensional) C^∞ manifold M. In this section, also P, Q, L, K always designate (finite dimensional) C^∞ manifolds respectively.

Let $X \subseteq C^\infty(N, M)$ be a subset. Then, such a set X is a *mapping space*. Let $X/\sim$ be any quotient of X under an equivalence relation $\sim$ on X. We give on the quotient space $X/\sim$ the quotient topology of X with the relative topology of the C^∞ topology on $C^\infty(N, M)$, not Whitney (fine) C^∞ topology. Such space is called a *mapping space quotient*. Then we will endow, in the following seven steps, a differentiable structure with the mapping space quotient X, depending just on the representation $X/\sim \leftarrow X \subseteq C^\infty(N, M)$. We note that the notion of differentiable structures can provided just by defining the notion of diffeomorphisms. Therefore our goal is to define the notion of "diffeomorphisms".

(i) We call a mapping $h : P \to X$ *differentiable* (or C^∞) if there exists a C^∞ mapping (between manifolds) $H : P \times N \to M$ satisfying $H(p, x) = h(p)(x) \in M$, $(p \in P, x \in N)$.

(ii) We call a mapping $k : X \to Q$ *differentiable* if k is a continuous mapping and, for any differentiable mapping $h : P \to X$ in the sense of (i), the composition $k \circ h : P \to Q$ is a C^∞ mapping between manifolds.

Now, if $\sim$ is an equivalence relation on a mapping space X, then we get the quotient space $X/\sim$. Then the canonical projection $\pi = \pi_X : X \to X/\sim$ is defined by $\pi(x) = [x]$ (the equivalence class of x).

(iii) We call a mapping $\ell : X/\sim \to Q$ *differentiable* if the composition $\ell \circ \pi : X \to Q$ with the projection π is differentiable in the sense of (ii).

(iv) We call a mapping $\varphi : X/\sim \to Y/\approx$ from a mapping space quotient $X/\sim$ to another mapping space quotient $Y/\approx \leftarrow Y \subseteq C^\infty(L, K)$ *differentiable* if φ is a continuous mapping and, for any open subset $U \subseteq Y/\approx$ $(\leftarrow \pi_Y^{-1}(U) \subseteq C^\infty(L, K))$ and for any differentiable mapping $\ell : U \to Q$ in the sense of (iii), the composition $\ell \circ \varphi : \varphi^{-1}(U)(\leftarrow \pi_X^{-1}(\varphi^{-1}(U)) \subseteq C^\infty(N, M)) \to Q$ is differentiable in the sense of (iii).

We call a mapping $\varphi : X/\sim \to Y/\approx$ a *diffeomorphism* if φ is differentiable in the sense of (iv), φ is a bijection and the inverse mapping $\varphi^{-1} : Y/\approx \to X/\sim$ is also differentiable in the sense of (iv). Moreover we call two mapping space quotients $X/\sim$ and $Y/\approx$ *diffeomorphic* if there exists a diffeomorphism $\varphi : X/\sim \to Y/\approx$ in the sense of (iv).

Now we give several related results: First, form the definition above, we immediately have that the differentiability is a local notion. Also we observe that, for any C^∞ manifold P, P is diffeomorphic to $C^\infty(\{\mathrm{pt}\}, P)$. We announce useful lemmata (shown in [9]) which follow from the definition:

Lemma 5.1 *If $h : P \to X$ is differentiable in the sense of* (i), *then $\pi \circ h : P \to X/\sim$ is differentiable in the sense of* (iv).

Lemma 5.2 *The following two conditions are equivalent to each other:*

(1) $\varphi : X/\sim \to Y/\approx$ *is differentiable in the sense of* (iv).

(2) $\varphi : X/\sim \to Y/\approx$ *is a continuous mapping and, for any differentiable mapping* $h : P \to X$ *in the sense of* (i), $\varphi \circ \pi \circ h : P \to Y/\approx$ *is differentiable in the sense of* (iv).

Lemma 5.3 *A differentiable mapping* $h : P \to X \subseteq C^\infty(N, M)$ *in the sense of* (i) *is a continuous mapping.*

Lemma 5.3 does not hold for the Whitney C^∞ topology. This is the reason we adopt the C^∞ topology.

Lemma 5.4 (1) *The identity mapping* id : $X/\sim \to X/\sim$ *is differentiable.* (2) *Let* $\varphi : X/\sim \to Y/\approx$ *and* $\psi : Y/\approx \to Z/\equiv$ *be differentiable mappings. Then the composition* $\psi \circ \varphi : X/\sim \to Z/\equiv$ *is differentiable.*

Lemma 5.5 (1) *The quotient mapping* $\pi : X \to X/\sim$ *is differentiable.* (2) *A mapping* $\varphi : X/\sim \to Y/\approx$ *is differentiable if and only if* $\varphi \circ \pi : X \to Y/\approx$ *is differentiable.*

Lemma 5.6 *If* N *and* N' *are diffeomorphic, and if* M *and* M' *are diffeomorphic, then* $C^\infty(N, M)$ *and* $C^\infty(N', M')$ *are diffeomorphic.*

6 An application to the moduli problem of plane curves on the symplectic plane.

We announce the results obtained in [9].

Let $f : S^1 \to \mathbb{R}^2$ be a generic immersion of the circle S^1 in the symplectic plane $\mathbb{R}^2$ with the standard symplectic (area) form $\omega_0 = dx \wedge dy$. Clearly the areas of domains surrounded by the curve $f(S^1)$ are invariant under symplectomorphisms. Thus, denoting the first Betti number of $f(S^1)$ by r, we see the curves isotopic to f have r-dimensional symplectic moduli.

We denote by $C^\infty(S^1, \mathbb{R}^2)$ the space of C^∞ mappings from S^1 to $\mathbb{R}^2$, which has the natural action (from "right") of the group $\mathrm{Diff}^+(S^1)$ consisting of orientation-preserving diffeomorphisms on S^1. Thus $C^\infty(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ denotes the space of *oriented* curves. The space $C^\infty(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ has the action (from "left") of the group $\mathrm{Diff}^+(\mathbb{R}^2)$ (resp. $\mathrm{Symp}(\mathbb{R}^2)$) consisting of orientation-preserving diffeomorphisms (resp. symplectomorphisms) on $\mathbb{R}^2$. For each oriented curve $f \in C^\infty(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$, we denote by $\mathrm{Diff}^+(\mathbb{R}^2)f$ the orbit through f via the action of $\mathrm{Diff}^+(\mathbb{R}^2)$. Thus $\mathrm{Diff}^+(\mathbb{R}^2)f$ consists of oriented curves of form $\tau \circ f$ for orientation preserving diffeomorphisms τ. Similarly the space $\mathrm{Diff}^+(\mathbb{R}^2)f/\mathrm{Symp}(\mathbb{R}^2)$ means the quotient space by the $\mathrm{Symp}(\mathbb{R}^2)$-action of $\mathrm{Diff}^+(\mathbb{R}^2)f$ in $C^\infty(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$.

We call the quotient space $\mathrm{Diff}^+(\mathbb{R}^2)f/\mathrm{Symp}(\mathbb{R}^2)$ *the symplectic moduli space* of f and denote it by $\mathcal{M}_{\mathrm{symp}}(f)$. It describes the symplectic classification of a fixed diffeomorphism class of a plane curve.

To study the moduli space minutely, we label the r-domains surrounded by the curve $f(S^1)$ as $D_1, D_2, \dots, D_r$ for a $f \in C^\infty(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$. Then, for each $\rho \in \mathrm{Diff}^+(\mathbb{R}^2)$, we can label bounded r-domains surrounded by $(\rho \circ f)(S^1)$ as $\rho(D_1), \rho(D_2), \dots, \rho(D_r)$ induced by the labelling for f. We set $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f) = \mathrm{Diff}^+(\mathbb{R}^2)/\sim_f$, where we call $\rho, \rho' \in \mathrm{Diff}^+(\mathbb{R}^2)$ are equivalent via f, and write $\rho \sim_f \rho'$, if the exists a symplectomorphism τ such that $\tau \circ \rho \circ f = \rho' \circ f$ up to $\mathrm{Diff}^+(S^1)$ and preserving the given labelling: $\tau\rho(D_j) = \rho'(D_j), 1 \leq j \leq r$. We call $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$ the *labelled symplectic moduli space* of f.

Then we have:

Theorem 6.1 *If $f \in C^\infty(S^1, \mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ is a generic labelled immersion, then the labelled symplectic moduli space $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$ is diffeomorphic to the relative cohomology space $H^2(\mathbb{R}^2, f(S^1), \mathbb{R}) \cong \mathbb{R}^r$.*

The labelled symplectic moduli space $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$ has a canonical differentiable structure. We claim in Theorem 6.1 that the labelled symplectic moduli space of f with the differentiable structure is diffeomorphic to $\mathbb{R}^r$, $r = \dim_{\mathbb{R}} H^2(\mathbb{R}^2, f(S^1), \mathbb{R})$. Actually we are going to give a diffeomorphism between $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$ and the positive cone in $H^2(\mathbb{R}^2, f(S^1), \mathbb{R})$. Note that the relative cohomology group $H^2(\mathbb{R}^2, f(S^1), \mathbb{R})$ over $\mathbb{R}$ is isomorphic to the vector space $H_2(\mathbb{R}^2, f(S^1), \mathbb{R})^* = \mathrm{Hom}_{\mathbb{R}}(H_2(\mathbb{R}^2, f(S^1), \mathbb{R}), \mathbb{R})$. The orientation of $\mathbb{R}^2$ and the labelling of the bounded domains surrounded by $f(S^1)$ give the canonical basis $[D_1], [D_2], \dots, [D_r]$ of $H_2(\mathbb{R}^2, f(S^1), \mathbb{R})$. The positive cone $H^2(\mathbb{R}^2, f(S^1), \mathbb{R})_{>0}$ of $H^2(\mathbb{R}^2, f(S^1), \mathbb{R}) \cong H_2(\mathbb{R}^2, f(S^1), \mathbb{R})^*$ is defined by

$$H^2(\mathbb{R}^2, f(S^1), \mathbb{R})_{>0} = \{\alpha \in H^2(\mathbb{R}^2, f(S^1), \mathbb{R}) \mid \alpha([D_j]) > 0, \ 1 \leq j \leq r\}.$$

The diffeomorphism of $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$ and $H^2(\mathbb{R}^2, f(S^1), \mathbb{R})_{>0}$ is given actually by the mapping

$$\varphi : \widetilde{\mathcal{M}}_{\mathrm{symp}}(f) \to H^2(\mathbb{R}^2, f(S^1), \mathbb{R})_{>0},$$

defined by

$$\varphi : [\tau] \mapsto \left([D_j] \mapsto \int_{\tau(D_j)} \omega_0 \right).$$

$1 \leq j \leq r$, $\omega_0 = dx \wedge dy$.

The symplectic moduli space $\mathcal{M}_{\mathrm{symp}}(f) = \mathrm{Diff}^+(\mathbb{R}^2)f/\mathrm{Symp}(\mathbb{R}^2)$ is obtained as a quotient of $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$. A *symmetry* of a generic immersion $f : S^1 \to \mathbb{R}^2$ is an orientation preserving diffeomorphism $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\rho \circ f = f \circ \sigma$ for some $\sigma \in \mathrm{Diff}^+(S^1)$. The group of symmetries of f induces a subgroup G_f of the permutation group S_r of the r-bounded domains of $\mathbb{R}^2 \setminus f(S^1)$. Then G_f naturally acts on $H_2(\mathbb{R}^2, f(S^1), \mathbb{R})$ and on $H_2(\mathbb{R}^2, f(S^1), \mathbb{R})^* \cong H^2(\mathbb{R}^2, f(S^1), \mathbb{R}) \cong \mathbb{R}^r$.

By Theorem 6.1, we have the following:

Corollary 6.2 *The symplectic moduli space $\mathcal{M}_{\mathrm{symp}}(f)$ is diffeomorphic to the finite quotient $\mathbb{R}^r/G_f$ of $\mathbb{R}^r$.*

Theorem 6.1 is generalised to more singular curves. To state the generalisation, first we treat the local case.

A multi-germ $f_{y_0} : (S^1, S) \to (\mathbb{R}^2, y_0)$ at a finite set $S \subset S^1$ is called *of finite codimension* (or *$\mathcal{A}$-finite*) in the sense of Mather if f_{y_0} is determined by its finite jet up to diffeomorphisms (or $\mathcal{A}$ equivalence). See [14][17]. Then the local image of f_{y_0} divides $(\mathbb{R}^2, 0)$ into several domains. We label them. Then, for any $\rho \in \mathrm{Diff}^+(\mathbb{R}^2, y_0)$, the labelling of $\rho \circ f$ is induced. Two diffeomorphism-germs $\rho, \rho' \in \mathrm{Diff}^+(\mathbb{R}^2, y_0)$ are equivalent via f_{y_0}, and write $\rho \sim_f \rho'$, if there exists a symplectomorphism-germ $\tau \in \mathrm{Symp}(\mathbb{R}^2, 0)$ such that $\tau \circ \rho \circ f_{y_0} = \rho' \circ f_{y_0}$ up to diffeomorphism-germs of (S^1, S) fixing S pointwise, and τ preserves the labelling. Thus we define the *local labelled symplectic moduli space* by

$$\widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0}) := \mathrm{Diff}^+(\mathbb{R}^2, y_0)/\sim_f .$$

Moreover we define the *local symplectic moduli space*

$$\mathcal{M}_{\mathrm{symp}}(f_{y_0}) := \mathrm{Diff}^+(\mathbb{R}^2, y_0)f/\mathrm{Symp}(\mathbb{R}^2, y_0).$$

Note that the space of map-germs

$$C^\infty((N,S),(M,y_0)) := \{f : (N,S) \to (M,y_0)\ C^\infty \text{ map-germs}\}$$

is a quotient space of $C^\infty(N,M)$, so also it has the differentiable structure. In particular $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0})$ and $\mathcal{M}_{\mathrm{symp}}(f_{y_0})$ are mapping space quotient have natural differentiable structures. Moreover there exists the canonical projection $\pi : \widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0}) \to \mathcal{M}_{\mathrm{symp}}(f_{y_0})$ defined by $\pi(\tau) = \tau \circ f$ modulo $\mathrm{Diff}^+(S^1, S)$, the diffeomorphism-germs of S^1 at S fixing S pointwise.

Now returning to the global case, we consider an oriented curve $f : S^1 \to \mathbb{R}^2$ up to $\mathrm{Diff}^+(S^1)$, namely, $f \in C^\infty(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$. Then we call f *of finite type* if, for some (and for any) representative $f : S^1 \to \mathbb{R}^2$ of $f \in C^\infty(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$, for some (and any) representative $f : S^1 \to \mathbb{R}^2$ of $f \in C^\infty(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$, except for a finite number of points $y_0 \in f(S^1)$, the multi-germ $f_{y_0} : (S^1, f^{-1}(y_0)) \to (\mathbb{R}^2, y_0)$ is a stable multi-germ, namely a single immersion-germ or a transversal two-immersion-germ, and, even if f_{y_0} is unstable, $f^{-1}(y_0)$ is a finite set in S^1 and f_{y_0} is of finite codimension. The condition means roughly that the $\mathrm{Diff}^+(\mathbb{R}^2)$-orbit through f in $C^\infty(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ is of finite codimension.

If $f \in C^\infty(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ is of finite type, then $f(S^1)$ divides $\mathbb{R}^2$ into a finite number of bounded domains and one unbounded domain. Then we define the the *labelling* of f as the labelling of bounded domains $D_1,\dots,D_r$ and the multiple or singular values $y_1,\dots,y_s$ of f in $\mathbb{R}^2$.

We define, similarly to the case of generic immersions, the *labelled symplectic moduli space* of plane curve f of finite type by

$$\widetilde{\mathcal{M}}_{\mathrm{symp}}(f) := \mathrm{Diff}^+(\mathbb{R}^2)/\sim_f,$$

where $\rho \sim_f \rho'$ if $\tau \circ \rho \circ f = \rho' \circ f$ for some $\tau \in \mathrm{Symp}(\mathbb{R}^2)$ preserving the labellings induced by ρ and ρ'.

Moreover we define the *symplectic moduli space* of plane curve f of finite type by

$$\mathcal{M}_{\mathrm{symp}}(f) := \mathrm{Diff}^+(\mathbb{R}^2)f/\mathrm{Symp}(\mathbb{R}^2).$$

Theorem 6.3 (Localisation Theorem) *If $f \in C^\infty(S^1,\mathbb{R}^2)/\mathrm{Diff}^+(S^1)$ is of finite type, then*

$$\widetilde{\mathcal{M}}_{\mathrm{symp}}(f) \cong_{\mathrm{diffeo.}} \prod_{y_0 \in f(S^1)} \widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0}) \times \mathbb{R}^r,$$

where $r = \dim_{\mathbb{R}} H^2(\mathbb{R}^2, f(S^1), \mathbb{R})$. *Moreover*

$$\mathcal{M}_{\mathrm{symp}}(f) \cong_{\text{diffeo.}} \left(\prod_{y_0 \in f(S^1)} \widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0}) \times \mathbb{R}^r \right) / G_f,$$

where $G_f \subset S_{r'}$ *is the group induced by the symmetry group of* f, r' *being* r *plus the number of unstable singular values of* f.

Note that we have that $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0})$ is just a point if f_{y_0} is a single immersion-germ. Therefore the product in Theorem 6.3 turns out to be a finite product.

Theorem 6.3 can be regarded as the "localisation theorem"for the global labelled moduli space of the isotopy type of a singular plane curve.

The diffeomorphism between $\widetilde{\mathcal{M}}_{\mathrm{symp}}(f)$ and the product of local symplectic moduli spaces and an open cone of $H^2(\mathbb{R}^2, f(S^1), \mathbb{R})_{>0}$ is given by the mapping

$$\Phi : \widetilde{\mathcal{M}}_{\mathrm{symp}}(f) \rightarrow \left(\prod_{y_0 \in f(S^1)} \widetilde{\mathcal{M}}_{\mathrm{symp}}(f_{y_0}) \right) \times H^2(\mathbb{R}^2, f(S^1), \mathbb{R})_{>0}$$

defined by

$$\Phi([\tau]) := \left(([\eta_{\tau(y_0)} \circ \tau])_{y_0 \in f(S^1)}, \ \varphi([\tau]) \right),$$

where $\eta_{\tau(y_0)} : (\mathbb{R}^2, \tau(y_0)) \rightarrow (\mathbb{R}^2, y_0)$ is *any* symplectomorphism-germ. Note that $[\eta_{\tau(y_0)} \circ \tau] \in \widetilde{\mathcal{M}}(f_{y_0})$ does not depend on the choice of $\eta_{\tau(y_0)}$.

The detailed proofs of all results announced here are given in the paper [9].

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