数理解析研究所講究録
第 1540 巻 2007 年 145-151

TOWARDS A CLASSIFICATION OF BLOW-NASH TYPES

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ABSTRACT. We present a simplified proof for the invariance of the corank and index of Nash function germs under blow-Nash equivalence. We address also the question of the blow-Nash types of simple singularities.

In order to address the question of a classification of the singularities of Nash function germs, that is analytic and semi-algebraic germs, one need to consider a relevant equivalence relation between such germs. Whereas in the complex case the topological classification make sense, over the reals the situation is much more complicated. In this paper we study the blow-Nash equivalence (see [2, 4]) which is a Nash version of the blow-analytic equivalence between real analytic function germs proposed by Kuo [7]. To give an idea, this means that we consider as equivalent germs such germs that become Nash equivalent after resolution of their singularities (for a precise statement see definition 1.1).

For this blow-Nash equivalence we know invariants called zeta functions [2]. These invariants take into account the geometry of polynomial arcs passing through a germ with a given order. We recalled their construction is section 1.2. Using these invariants we proved in [3] that the corank and index of Nash function germs are preserved by blow-Nash equivalence. This establishes a first step in the classification of the singularities of Nash function germs with respect to the blow-Nash equivalence.

In this paper, we address the two following issues. First, we present in section 2 a simplified proof for one crucial point in the proof of the invariance of the corank and index. The point is to compute the virtual Poincaré polynomial of real algebraic sets defined by quadratic polynomials. Second, we deal with the question of the classification of simple Nash germs in section 3. In particular we announce that their classification under blow-Nash equivalence coincide with their classification under analytic equivalence. We prove moreover the particular case of E_6, E_7, E_8-singularities in order to give an idea of the general proof.

Acknowledgements. The author wish to thank T. Fukui for motivating discussions and also for its nice stay at Saitama University. He is also embed to the Japan Society for the Promotion of Science for its financial support during this stay at Saitama University where this paper has been written.

1. BLOW-NASH EQUIVALENCE

1.1. **Definition.** The definition of blow-Nash equivalence comes from an adaptation of the definition of the blow-analytic equivalence of Kuo (see [7]) for Nash function germs. It states roughly speaking that two germs are blow-analytically equivalent if they become analytically equivalent after resolution of their singularities. Similarly to the blow-analytic case, several slightly different definitions exist, and to find the appropriate definition is still a work in progress (see [5]). We adopt in this paper

1991 *Mathematics Subject Classification.* 14B05, 14P20, 14P25, 32S15.

the strong definition of blow-Nash equivalence for which, in particular, we require a Nash isomorphism between the exceptional spaces of the resolutions (see [3, 4]).

Definition 1.1.

(1) A Nash modification of a Nash function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ is a proper surjective Nash map $\sigma_f : \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0)$, between semi-algebraic neighbourhoods of 0 in $\mathbb{R}^d$ and $\sigma_f^{-1}(0)$ in M_f, whose complexification is an isomorphism except on some thin subset of $\mathbb{R}^d$ and for which $f \circ \sigma$ is in normal crossing.

(2) Let $f, g : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ be Nash function germs. They are said to be blow-Nash equivalent if there exist two Nash modifications

$$\sigma_f : \big(M_f, \sigma_f^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0) \text{ and } \sigma_g : \big(M_g, \sigma_g^{-1}(0)\big) \longrightarrow (\mathbb{R}^d, 0),$$

such that $f \circ \sigma_f$ and $\operatorname{jac} \sigma_f$ (respectively $g \circ \sigma_g$ and $\operatorname{jac} \sigma_g$) have only normal crossings simultaneously, and a Nash isomorphism (i.e. a semi-algebraic map which is an analytic isomorphism) Φ between semi-algebraic neighbourhoods $\big(M_f, \sigma_f^{-1}(0)\big)$ and $\big(M_g, \sigma_g^{-1}(0)\big)$ which preserves the multiplicities of the Jacobian determinants of σ_f and σ_g along the components of the exceptional divisors, and which induces a homeomorphism $\phi : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}^d, 0)$ such that $f = g \circ \phi$, as illustrated by the commutative diagram:

$$\begin{tikzcd}
\big(M_f, \sigma_f^{-1}(0)\big) \arrow[rr, "\Phi"] \arrow[d, "\sigma_f"'] & & \big(M_g, \sigma_g^{-1}(0)\big) \arrow[d, "\sigma_g"] \\
(\mathbb{R}^d, 0) \arrow[rr, "\phi"] \arrow[dr, "f"'] & & (\mathbb{R}^d, 0) \arrow[dl, "g"] \\
& (\mathbb{R}, 0) &
\end{tikzcd}$$

We refer to [2, 3, 4] for an overview of the properties of the blow-Nash equivalence.

1.2. **Invariants.** We recall now the definition of the zeta functions associated to a Nash function germ. To this aim, we need to introduce the virtual Poincaré polynomial defined by McCrory and Parusiński [9] for algebraic sets and extended to arc-symmetric sets [2].

Arc-symmetric sets have been introduced by Kurdyka [8]. The category of arc-symmetric sets is larger than that of real algebraic varieties. In order to recall the definition of arc-symmetric sets, we fix a compactification of $\mathbb{R}^n$, for instance $\mathbb{R}^n \subset \mathbb{P}^n$.

Definition 1.2. Let $X \subset \mathbb{P}^n$ be a semi-algebraic set. We say that X is arc-symmetric if, for every real analytic arc $\gamma :]-1, 1[\longrightarrow \mathbb{P}^n$ such that $\gamma(]-1, 0[) \subset X$, there exists $\epsilon > 0$ such that $\gamma(]0, \epsilon[) \subset X$.

One can think about arc-symmetric sets as the biggest category stable under boolean operations and containing the compact real algebraic varieties and their connected components.

We recall also that a Nash isomorphism between arc-symmetric sets X_1, X_2 is the restriction of an analytic and semi-algebraic isomorphism between compact semi-algebraic and real analytic sets Y_1, Y_2 containing X_1, X_2 respectively (see [2]).

An additive map on the category of arc-symmetric sets is a map β such that $\beta(X) = \beta(Y) + \beta(X \setminus Y)$ where Y is an arc-symmetric subset closed in X. Moreover β is called multiplicative if $\beta(X_1 \times X_2) = \beta(X_1) \cdot \beta(X_2)$ for arc-symmetric sets X_1, X_2.

Proposition 1.3. *([9, 2]) For an integer i, there exists an additive map β_i with values in $\mathbb{Z}$, defined on the category of arc-symmetric sets. It coincides with the classical Betti number $\dim H_i(\cdot, \frac{\mathbb{Z}}{2\mathbb{Z}})$ on compact nonsingular arc-symmetric sets. Moreover $\beta(\cdot) = \sum_{i \geq 0} \beta_i(\cdot) u^i$ is multiplicative, with values in $\mathbb{Z}[u]$. Finally, if X_1 and X_2 are Nash isomorphic arc-symmetric sets, then $\beta(X_1) = \beta(X_2)$.*

The invariant β_i is called the i-*th* virtual Betti number, and the polynomial β the virtual Poincaré polynomial. Note that, by evaluation of the virtual Poincaré polynomial at -1, we recover the Euler characteristic with compact support (see [9]).

Example 1.4. If $\mathbb{P}^k$ denotes the real projective space of dimension k, which is nonsingular and compact, then $\beta(\mathbb{P}^k) = 1 + u + \cdots + u^k$ since $\dim H_i(\mathbb{P}^k, \frac{\mathbb{Z}}{2\mathbb{Z}}) = 1$ for $i \in \{0, \ldots, k\}$ and $\dim H_i(\mathbb{P}^k, \frac{\mathbb{Z}}{2\mathbb{Z}}) = o$ otherwise. Now, compactify the affine line $\mathbb{A}^1_{\mathbb{R}}$ in $\mathbb{P}^1$ by adding one point at the infinity. By additivity $\beta(\mathbb{A}^1_{\mathbb{R}}) = \beta(\mathbb{P}^1) - \beta(point) = u$, and so $\beta(\mathbb{A}^k_{\mathbb{R}}) = u^k$ by multiplicativity.

Then, using the virtual Poincaré polynomial, we can define the zeta functions of a Nash function germ $f : (\mathbb{R}^d, 0) \longrightarrow (\mathbb{R}, 0)$ as follows. Denote by $\mathcal{L}$ the space of arcs at the origin $0 \in \mathbb{R}^d$, that is:

$$\mathcal{L} = \mathcal{L}(\mathbb{R}^d, 0) = \{\gamma : (\mathbb{R}, 0) \longrightarrow (\mathbb{R}^d, 0) : \gamma \text{ formal}\},$$

and by $\mathcal{L}_n$ the space of arcs truncated at order $n+1$:

$$\mathcal{L}_n = \mathcal{L}_n(\mathbb{R}^d, 0) = \{\gamma \in \mathcal{L} : \gamma(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n,\ a_i \in \mathbb{R}^d\},$$

for $n \geq 0$ an integer. We define the naive zeta function $Z_f(T)$ of f as the following element of $\mathbb{Z}[u, u^{-1}][[T]]$:

$$Z_f(T) = \sum_{n \geq 1} \beta(A_n) u^{-nd} T^n,$$

where

$$A_n = \{\gamma \in \mathcal{L}_n : ord(f \circ \gamma) = n\} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = bt^n + \cdots, b \neq 0\}.$$

Similarly, we define zeta functions with sign by

$$Z_f^{+1}(T) = \sum_{n \geq 1} \beta(A_n^{+1}) u^{-nd} T^n \quad \text{and} \quad Z_f^{-1}(T) = \sum_{n \geq 1} \beta(A_n^{-1}) u^{-nd} T^n,$$

where

$$A_n^{+1} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = +t^n + \cdots\} \quad \text{and} \quad A_n^{-1} = \{\gamma \in \mathcal{L}_n : f \circ \gamma(t) = -t^n + \cdots\}.$$

Theorem 1.5. *([2]) Blow-Nash equivalent Nash function germs have the same naive zeta function and the same zeta functions with sign.*

Remark 1.6. This result is inspired by that of Koike and Parusiński [6] who proved that these zeta functions, with the Euler characteristic with compact supports in place of the virtual Poincaré polynomial, are invariant with respect to the blow-analytic equivalence.

2. Some computations of virtual Poincaré polynomials

One crucial point in order to prove the invariance of the corank and index of Nash function germs under blow-Nash equivalence is to compute the following virtual Poincaré polynomials (see [3]).

Let's denote by Q the quadratic polynomial:

$$Q(x,y) = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2$$

where $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_q)$. The virtual Poincaré polynomial of the algebraic sets

$$Y_{p,q} = \{Q(x,y) = 0\}, \quad Y_{p,q}^{\epsilon} = \{Q(x,y) = \epsilon\}$$

for $\epsilon \in \{1, -1\}$ are the following.

Proposition 2.1. *([3]) Assume $(p,q) \neq (0,0)$.*

- $\beta(Y_{p,q}) = u^{p+q-1} - u^{\max\{p,q\}-1} + u^{\min\{p,q\}}$.
- *If $p \leq q$, then* $\beta(Y_{p,q}^1) = u^{q-1}(u^p - 1)$.
- *If $p > q$, then* $\beta(Y_{p,q}^1) = u^q(u^{p-1} + 1)$.

We presented in [3] a proof using a nonsingular compactification of these algebraic sets in the projective space and computations of the homology of these compactifications. We give here a different proof. Namely, we use the additivity of the virtual Poincaré polynomial combined with a well chosen stratification (suggested by F. Sottile) of the sets $Y_{p,q}$ and $Y_{p,q}^{\epsilon}$.

Proof. We proceed by the following change of variables. Assume $p \leq q$. Then put $u_i = x_i + y_i$ and $v_i = x_i - y_i$ for $i = 1, \ldots, p$. The new expression for Q is

$$\sum_{i=1}^{p} u_i v_i - \sum_{j=p+1}^{q} y_j^2.$$

Let us compute the virtual Poincaré polynomial of $Y_{p,q}$. We stratify $Y_{p,q}$ depending on the vanishing of u_i for $i = 1, \ldots, p$. Assume $u_1 \neq 0$. Then the value of v_1 is prescribed by

$$v_1 = \frac{-1}{u_1}\left(\sum_{i=2}^{p} u_i v_i - \sum_{j=p+1}^{q} y_j^2\right)$$

and therefore $Y_{p,q} \cup \{u_1 \neq 0\}$ is isomorphic to $\mathbb{R}^* \times \mathbb{R}^{p+q-2}$, so that

$$\beta(Y_{p,q} \cup \{u_1 \neq 0\}) = (u-1)u^{p+q-2}.$$

Assume now that $u_1 = 0$. Then v_1 is free, and we may deal in the same way with u_2: if $u_2 \neq 0$ then v_2 is fixed and we obtain a contribution of

$$\beta(Y_{p,q} \cup \{u_1 = 0, u_2 \neq 0\}) = (u-1)u^{p+q-3}.$$

At the final step $u_1 = \cdots = u_{p-1} = 0$, if $u_p \neq 0$ then $Y_{p,q} \cup \{u_1 = \cdots = u_{p-1} = 0, u_p \neq 0\}$ is isomorphic to $\mathbb{R}^* \times \mathbb{R}^{q-1}$. If $u_p = 0$ the remaining equation

$$-\sum_{j=p+1}^{q} y_j^2 = 0$$

admits only the zero solution, hence $\beta(Y_{p,q} \cup \{u_1 = \cdots = u_p = 0\}) = u^p$ since the variables $v_1, \ldots, v_p$ are free.

Finally

$$\beta(Y_{p,q}) = (u-1)\sum_{i=1}^{p} u^{p+q-1-i} + u^p = u^{p+q-1} - u^{q-1} + u^p.$$

We proceed similarly in the case of $Y^1_{p,q}$. If $p \leq q$ the remaining equation

$$-\sum_{j=p+1}^{q} y_j^2 = 1$$

does no longer admit a solution, hence

$$\beta(Y^1_{p,q}) = (u-1)\sum_{i=1}^{p} u^{p+q-1-i} = u^{q-1}(u^p - 1).$$

In the case $p > q$ the remaining equation is that of a $p-q-1$-dimensional sphere. The virtual Poincaré polynomial of such a sphere is $1 + u^{p-q-1}$ therefore

$$\beta(Y^1_{p,q}) = (u-1)\sum_{j=1}^{q} u^{p+q-1-j} + u^q(1 + u^{p-q-1})$$

where the u^q term in front of $1 + u^{p-q-1}$ comes from the free variables $v_1, \ldots, v_q$. As a consequence

$$\beta(Y^1_{p,q}) == u^{p+q-1} - u^{p-1} + u^q + u^{p-1} = u^q(u^{p-1} + 1).$$

□

Remark 2.2. Note that we can recover p and q from $\beta(Y_{p,q})$ and $\beta(Y^1_{p,q})$. This is no longer the case if we consider the Euler characteristic with compact supports in place of the virtual Poincaré polynomial. More precisely, in the latter case we only recover the parity of p and q.

3. Blow-Nash types of simple singularities

We announce in this section some results concerning the classification of the blow-Nash types of simple singularities.

As recalled, the corank and index of a Nash function germ are invariant under blow-Nash equivalence. In order to go further in the classification of singularities, the next step is to deal with simple singularities. Considering real analytic function germs, their simple singularities have been classified [1]. A real analytic function germ with a simple singularity is analytically equivalent to a polynomial germ belonging to one of the family

$$\begin{array}{l}
A_k : x^{k+1} + \sum_{i=1}^p y_i^2 - \sum_{j=1}^q z_j^2 \quad \text{for} \quad k \geq 2 \\
D_k : x_1(\pm x_2^2 \pm x_1^{k-2}) + \sum_{i=1}^p y_i^2 - \sum_{j=1}^q z_j^2 \quad \text{for} \quad k \geq 4 \\
E_6 : x_1^3 \pm x_2^4 + \sum_{i=1}^p y_i^2 - \sum_{j=1}^q z_j^2 \\
E_7 : x_1^3 + x_1 x_2^3 + \sum_{i=1}^p y_i^2 - \sum_{j=1}^q z_j^2 \\
E_8 : x_1^3 + x_2^5 + \sum_{i=1}^p y_i^2 - \sum_{j=1}^q z_j^2
\end{array}$$

This classification holds for Nash function germs. Indeed, analytically equivalent Nash function germs are also Nash equivalent by Nash Approximation Theorem [10].

Now, the question is: are we able to distinguish the blow-Nash types of simple singularities? I claim that this is possible, using the invariance of the zeta functions under blow-Nash equivalence.

Claim. Let $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be Nash function germs. Assume f and g are simple. Then f and g are blow-Nash equivalent if and only if f and g are analytically equivalent.

To give an idea of the proof, let us consider the case of 2-dimensional E_6, E_7, E_8-singularities. We use the notation

$$\begin{aligned} h_6^{\pm}(x,y) &= x^3 \pm y^4 \\ h_7(x,y) &= x^3 + xy^3 \\ h_8(x,y) &= x^3 + y^5 \end{aligned}$$

Proposition 3.1. *The function germs h_6^+, h_6^-, h_7, h_8 belong to different blow-Nash equivalence classes.*

In order to distinguish their blow-Nash types, we compute the virtual Poincaré polynomial of some spaces or arcs related to h_6^+, h_6^-, h_7, h_8.

Lemma 3.2. *Take $\epsilon \in \{-,+\}$.*

(1) $\beta(A_4^{\epsilon}(h_6^{\epsilon})) = 2u^6$ whereas $\beta(A_4^{-\epsilon}(h_6^{\epsilon})) = \beta(A_4(h_7)) = \beta(A_4(h_8)) = 0$.

(2) $\beta(A_5^{\epsilon}(h_7)) = (u-1)u^7$ whereas $\beta(A_5^{\epsilon}(h_8)) = u^8$.

Proof. Let us deal with point (1). We consider arcs of the form

$$\gamma(t) = (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4, b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4)$$

with $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}$. Then

$$h_6^{\epsilon}(\gamma(t)) = a_1^3 t^3 + (3a_1^2 a_2 + \epsilon b_1^4) t^4 + \cdots$$

therefore such an arc belongs to $A_4^{\epsilon}(h_6^{\epsilon})$ if and only if $a_1 = 0$ and $b_1^4 = 1$. So $A_4^{\epsilon}(h_6^{\epsilon})$ is isomorphic to the union of two 6-dimensional affine space and thus

$$\beta(A_4^{\epsilon}(h_6^{\epsilon})) = 2u^6.$$

On the other hand $A_4^{-\epsilon}(h_6^{\epsilon})$ is empty since $b_1^4 = -1$ does not admit solutions, so $\beta(A_4^{-\epsilon}(h_6^{\epsilon})) = 0$.

For the functions h_7 and h_8 the argument is even simpler because the vanishing of the t^3-coefficient of the series $h_7(\gamma(t))$ and $h_8(\gamma(t))$ implies the vanishing of the t^4-coefficient, so that $\beta(A_4(h_7)) = \beta(A_4(h_8)) = 0$.

The proof of point (2) is of the same type. Now we consider arcs of the type

$$\gamma(t) = (a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5, b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5)$$

with $a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5 \in \mathbb{R}$. Such an arc γ belongs to $A_5^{\epsilon}(h_7)$ if and only if $a_1 = 0$ and $a_2 b_1^3 = \epsilon$. The set $\{(a_2, b_1) \in \mathbb{R}^2 : a_2 b_1^3 = \epsilon\}$ is isomorphic to $\mathbb{R}^*$ therefore $A_5^{\epsilon}(h_7)$ is isomorphic to $\mathbb{R}^* \times \mathbb{R}^7$.

Finally such an arc γ belongs to $A_5^{\epsilon}(h_8)$ if and only if $a_1 = 0$ and $b_1^5 = \epsilon$ thus $A_5^{\epsilon}(h_8)$ is isomorphic to $\mathbb{R}^8$. $\square$

Now we can achieve the proof of proposition 3.1.

Proof. We prove that the function germs h_6^+, h_6^-, h_7, h_8 have different zeta functions with signs. Therefore they can not be blow-Nash equivalent by theorem 1.5.

First, note that the T^4-coefficient of the positive zeta function of h_6^+ is nonzero whereas that of h_6^-, h_7, h_8 is zero by lemma 3.2.1. Therefore h_6^+ can not be blow-Nash equivalent to h_6^-, h_7 or h_8. Similarly, considering the negative zeta function, we prove that h_6^- can not belong to the blow-Nash equivalence classes of h_7 and h_8. Finally, the T^5-coefficient of the zeta functions with sign of h_7 and h_8 differ by lemma 3.2.2, thus h_7 and h_8 also belong to different blow-Nash classes. □

Remark 3.3. Note that proposition 3.1 still holds considering the more general setting of the blow-analytic equivalence (compare with remark 2.2). Indeed, we recover the Euler characteristic with compact supports from the virtual Poincaré polynomial by evaluating it at -1. After this evaluation, we are still able to distinguish the Euler characteristics of the different spaces of arcs involved in lemma 3.2.

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