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Real polynomials and flip relation

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Abstract

Let M_n be the space of all affine conjugacy classes of polynomial maps of degree n. We define a map Ψ_n from M_n to $\mathbb{C}^{n-1}$ via the elementary symmetric functions of multipliers of the fixed points. The image of M_n under Ψ_n is denoted by $\Sigma(n)$. In this paper, we define the algebraic variety $\mathcal{G}_4$ that indicates essential property of M_4, and introduce "flip relation" to conjugacy classes. First, we show that the image of the real moduli space $\mathrm{M}_4(\mathbb{R})$ under Ψ_4 is the real section of $\Sigma(4)$ with $\mathrm{discr}\{\mathrm{G}_4\} \geq 0$, where $\mathrm{G}_4 = 0$ is the defining equation of $\mathcal{G}_4$. Next, we show connection between flip relation and sign of the discriminant of G_4.

1 Introduction

Let Poly_n be the space of all polynomial maps of degree n, and M_n be the space of all affine conjugacy classes of these polynomials. We define a projection Ψ_n from M_n to $\mathbb{C}^{n-1}$ via the elementary symmetric functions of the multipliers of the fixed points. In [3], we show the projection is not surjective for every $n \geq 4$. The image of M_n under Ψ_n is denoted by $\Sigma(n)$ and the complement $\mathbb{C}^{n-1} \setminus \Sigma(n)$ called the exceptional set, denoted by $\mathcal{E}(n)$. We remark that for $n = 2$ or 3 the exceptional set is empty and the real section of the space $\Sigma(n)$ coincides with the real moduli space $\mathrm{M}_n(\mathbb{R})$, the space of all conjugacy classes of polynomial maps of degree n with real coefficients (see [8]).

In this paper, we introduce "flip relation" and investigate relations between the real section of the space $\Sigma(4)$ and the real moduli space $\mathrm{M}_4(\mathbb{R})$.

In section 2, we define the map Ψ_4 and the algebraic variety $\mathcal{G}_4$ that indicates essential property of Poly_4. In section 3, we introduce flip relation to conjugacy classes. First, we show $\Psi_4(\mathrm{M}_4(\mathbb{R})) = \mathbb{R}^3 \cap \Sigma(4) \cap \{\mathrm{G}_4(s_1, s_2, s_4) = 0$ has a real root$\}$, where $\mathrm{G}_4(s_1, s_2, s_4) = 0$ is the defining equation of $\mathcal{G}_4$, as Theorem 1. And next, we show connection between flip relation and sign of the discriminant of the defining equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$ of $\mathcal{G}_4$ as Theorem 2.

2 The moduli space M_4

Let Poly_4 be the space of all polynomials of the form:

$$p(z) = a_4 z^4 + a_3 z^3 + a_2 z^3 + a_1 z + a_0 \quad (a_4 \neq 0).$$

Two maps p_1, $p_2 \in \mathrm{Poly}_4$ are *holomorphically conjugate*, denoted by $p_1 \sim p_2$ if and only if there exists $g \in \mathfrak{A}$ with $g \circ p_1 \circ g^{-1} = p_2$, where $\mathfrak{A}$ is the group of all affine transformations.

The space, $\mathrm{Poly}_4/_\sim$, of holomorphic conjugacy classes $\langle p \rangle$ of quartic polynomials is denoted by M_4.

For each $p(z) \in \mathrm{Poly}_4$, let $z_1, \cdots, z_4, z_5 = \infty$ be the fixed points of p, and $\mu_1, \cdots, \mu_4, \mu_5 = 0$ the multipliers of z_j, i.e. $\mu_j = p'(z_j)$. Let $\sigma_{4,1}, \sigma_{4,2}, \cdots, \sigma_{4,5}$ be the elementary symmetric functions of these multipliers

$$
\begin{aligned}
\sigma_{4,1} &= \mu_1 + \mu_2 + \mu_3 + \mu_4, \\
\sigma_{4,2} &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\
\sigma_{4,3} &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4, \\
\sigma_{4,4} &= \mu_1\mu_2\mu_3\mu_4, \\
\sigma_{4,5} &= 0.
\end{aligned}
$$

These multipliers are *invariant* under taking an affine conjugate.

The holomorphic index of a rational function f at a fixed point $\zeta \in \mathbb{C}$ is defined to be the complex number

$$\iota(f, \zeta) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)},$$

where we integrate in a small loop in the positive direction around ζ.

The following results are well known as "Fatou's index theorem":

- If ζ is a fixed point of multiplier $\mu \neq 1$, then $\iota(f, \zeta) = \frac{1}{1-\mu}$.
- For any non-identical polynomial p,

$$\sum_{\zeta \in \mathbb{C}} \iota(p, \zeta) = 0, \tag{1}$$

  where this summation is taken over all fixed points of p.

Let $\sigma_{4,1}, \cdots, \sigma_{4,4}$ be the elementary symmetric functions of the multipliers of $p(z) \in \mathrm{Poly}_4$. The following relation holds by Fatou's index theorem.

Lemma (Theorem 1 in [4]) Among $\sigma_{4,j}'s$, there is a linear relation

$$4 - 3\sigma_{4,1} + 2\sigma_{4,2} - \sigma_{4,3} = 0.$$

Under the conjugacy of the action of $\mathfrak{A}$, it can be assumed that any quartic polynomial is "monic" and "centered", i.e. $p(z) = z^4 + c_2 z^2 + c_1 z + c_0$. There are three monic and centered polynomials in any conjugacy class $\langle p \rangle$ except for $\langle z^4 + z \rangle$, and they are transformed each other under the action of $G(3) = \{1, \omega, \omega^2\}$, where ω is the cube roots of unity.

By this lemma, for a monic and centered quartic polynomial $p(z) = z^4 + c_2z^2 + c_1z + c_0$, it is enough to consider the three invariant values $\sigma_{4,1}$, $\sigma_{4,2}$, $\sigma_{4,4}$ of $\langle p\rangle$, determined by the *transformation formula*:

$$\begin{aligned}
\sigma_{4,1} &= -8c_1 + 12, \\
\sigma_{4,2} &= 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48, \\
\sigma_{4,4} &= 16c_0c_2^4 + (-4c_1^2 + 8c_1)c_2^3 - 128c_0^2c_2^2 + (144c_0c_1^2 - 288c_0c_1 \\
&\quad +128c_0)c_2 - 27c_1^4 + 108c_1^3 - 144c_1^2 + 64c_1 + 256c_0^3.
\end{aligned} \tag{2}$$

There is a natural projection

$$\begin{array}{cccc}
\Psi_4: & \mathrm{M}_4 & \longrightarrow & \Sigma(4) \\
& \Psi & & \Psi \\
& \langle p\rangle & \longmapsto & (\sigma_{4,1},\ \sigma_{4,2},\ \sigma_{4,4}),
\end{array}$$

where $\Sigma(4)$ is the image of M_4 under Ψ_4. We call $\Sigma(4) = \{(\sigma_{4,1},\ \sigma_{4,2},\ \sigma_{4,4})\}$ *multiplier-coordinate space.* The complement $\mathbb{C}^3 \setminus \Sigma(4)$ is denoted by $\mathcal{E}(4)$, and called the *exceptional set.*

In [5] and [2], we define the following parametrized algebraic variety, by putting $c_2^3 = c$ and $s_j = \sigma_{4,j}$ $(j = 1, 2, 4)$. This variety indicates essential property of the projection Ψ_4 or of Poly_4.

Definition 1 An algebraic variety $\mathcal{G}_4$ in $\mathbb{C}^3 \cong \{(s_1, s_2, s_4)\}$ with a parameter $c \in \mathbb{C}$, is defined by

$$\begin{aligned}
\mathrm{G}_4(s_1, s_2, s_4) &= 262144(s_1 - 4)^2c^2 + 1024(27s_1^4 + (-144s_2 - 576)s_1^2 \\
&\quad +(384s_2 + 1280)s_1 + 128s_2^2 - 256s_2 - 512s_4 - 768)c \\
&\quad +(9s_1^2 + 24s_1 - 32s_2 - 48)^3 = 0.
\end{aligned}$$

The number of conjugacy classes corresponding to a point $(s_1,\ s_2,\ s_4) \in \mathbb{C}^3$ equals to the number of possible parameter values c on $\mathcal{G}_4$. The following result is obtained by counting the number of solution c of the defining equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$.

The exceptional set $\mathcal{E}(4)$ is parametrized as follows (see [5]):

$$(s_1, s_2, s_4) = \left(4, s, \frac{(s-4)^2}{4}\right), \quad s \in \mathbb{C} \setminus \{6\}.$$

3 Real polynomials and flip relation

For the case of quadratic rational maps or cubic polynomial maps, the real section of the moduli space coincides with the real moduli space, the space of conjugacy classes of maps with real coefficients (see [7] and [8]). However, this phenomena do not succeed to the case $n = 4$. We have the following results.

Theorem 1 ([6])

$$\Psi_4(\mathrm{M}_4(\mathbb{R})) = \mathbb{R}^3 \cap \Sigma(4) \cap \{\mathrm{G}_4(s_1, s_2, s_4) = 0 \text{ has a real root}\}.$$

The assertion of Theorem 1 is directly obtained by the following three lemmas 1, 2 and 3.

Let $\mathrm{Poly}_4(\mathbb{R})$ be the space of all complex valued quartic polynomial maps with real coefficients.

Lemma 1 *If* $p \in \mathrm{Poly}_4(\mathbb{R})$ *then* $\Psi_4(\langle p \rangle)$ *is in* $\mathbb{R}^3$.

Proof. This lemma is clear from the multipliers of p satisfy one of the following three cases: 'four real values', 'two real and a pair of complex conjugates', or 'two pair of complex conjugates'. ∎

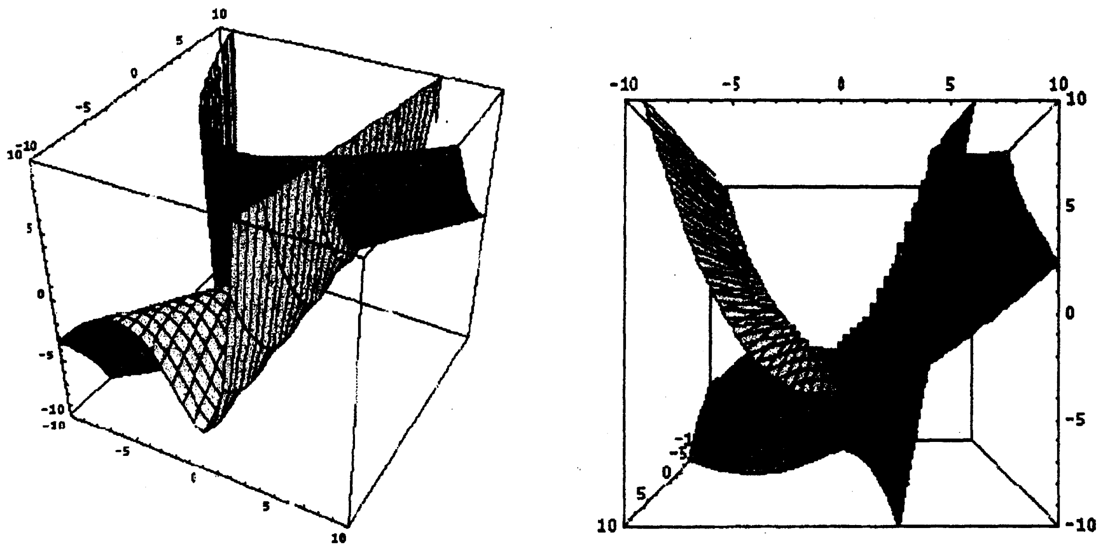

Figure 1: Both figures indicate the surface where discriminant of $G_4(s_1, s_2, s_4) = 0$ vanishes, on the space $\mathbb{R}^3 = \{(s_1, s_2, s_4)\}$.

Lemma 2 *For a point* $(s_1, s_2, s_4) \in \mathbb{R}^3 \cap \Sigma(4)$ *with* $s_1 \neq 4$, *let* $\langle p \rangle \in \Psi_4^{-1}(s_1, s_2, s_4)$. *Then, the equation* $G_4(s_1, s_2, s_4) = 0$ *has two real roots if and only if* $\langle p \rangle \in M_4(\mathbb{R})$.

Proof. At first, we remark that $G_4(s_1, s_2, s_4) = 0$ has real coefficients, since point (s_1, s_2, s_4) in $\mathbb{R}^3$.

If $G_4(s_1, s_2, s_4) = 0$ has two real roots, we can choose real values as coefficients of p. Therefore $\langle p \rangle \in M_4(\mathbb{R})$.

Conversely, we suppose $G_4(s_1, s_2, s_4) = 0$ has a pair of complex conjugate roots, and let $p = z^4 + c_2 z^2 + c_1 z + c_0$ be a corresponding monic and centered representative. Then, from the argument of case c. in the proof of Theorem 2 described in below, the coefficient c_2 is never real and $\langle p \rangle \notin M_4(\mathbb{R})$. Therefore $G_4(s_1, s_2, s_4) = 0$ have two real roots, counted including multiplicity. ∎

Lemma 3 *For each point* $(4, s_2, s_4) \in (\{4\} \times \mathbb{R}^2) \cap \Sigma(4)$, *the linear equation* $G_4(s_1, s_2, s_4) = 0$ *has a real root. That is, there is a unique conjugacy class in* $M_4(\mathbb{R})$ *corresponding to this point.*

Proof. Let $p(z) = z^4 + c_2z^2 + c_1z + c_0$ be a representative such that $\Psi_4(\langle p\rangle) = (4, s_2, s_4)$.

From the transformation formula (2), $s_1 = 4 = -8c_1 + 12$, we have $c_1 = 1$. Substituting $s_1 = 4$ to $G_4(s_1, s_2, s_4) = 0$, we have

$$131072(s_2^2 - 8s_2 - 4s_4 + 16)c - 32768(s_2 - 6)^3 = 0.$$

Therefore, the unique root c is always real valued. Hence, we can choose real value c_2 with $c_2^3 = c$. Then, c_0 is also real valued from the transformation formula. Thus we have the assertion. ∎

We remark that, the intersection of the plane $\{(4, s_2, s_4)\}$ and the locus where the discriminant of G_4 is vanished coincides with the set $\{(4, s, \frac{(s-4)^2}{4})\}$. This set corresponds to

$$\mathcal{E}(4) \cup \{(4, 6, 1)\}.$$

Now, we introduce the following "flip relation" between two classes, to prove Theorem 2.

Definition 2 Two conjugacy classes $\langle p\rangle$ and $\langle q\rangle \in M_4$ are under *flip relation*, denoted by $\langle p\rangle \leftrightarrows \langle q\rangle$, if there exist two representatives $p \in \langle p\rangle$, $q \in \langle q\rangle$ such that $p(\overline{z}) = \overline{q(z)}$.

The flip relation is not an equivalence relation, because $\langle p\rangle \not\leftrightarrows \langle p\rangle$ in general.

The following theorem contrasts with Theorem 1.

Theorem 2 ([6]) *Suppose that two classes $\langle p\rangle$ and $\langle q\rangle$ with $\langle p\rangle \neq \langle q\rangle$ satisfy $\Psi_4(\langle p\rangle) = \Psi_4(\langle q\rangle) \in \mathbb{R}^3$. Then, the following two are equivalent.*

(1) $\langle p\rangle \leftrightarrows \langle q\rangle$.

(2) *The equation $G_4(s_1, s_2, s_4) = 0$ has no real root.*

Remark 1 In the case of $s_1 \neq 4$, $G_4(s_1, s_2, s_4) = 0$ is a quadratic equation. Therefore the condition (2) in Theorem 2 equivalents to the condition "The equation $G_4(s_1, s_2, s_4) = 0$ has negative discriminant". As we see later, for every point (s_1, s_2, s_4) in $\Psi_4(M_4(\mathbb{R}))$, the corresponding equation $G_4(s_1, s_2, s_4) = 0$ has real coefficients. Hence, if one of the roots is real, the other is also real.

The flip relation correlates between periodic cycles of p and q by symmetric relation for real axis. The property of a periodic cycle of q, attracting, repelling etc., implies the same one of p. (See Figure 2.) We will show this fact as the following proposition.

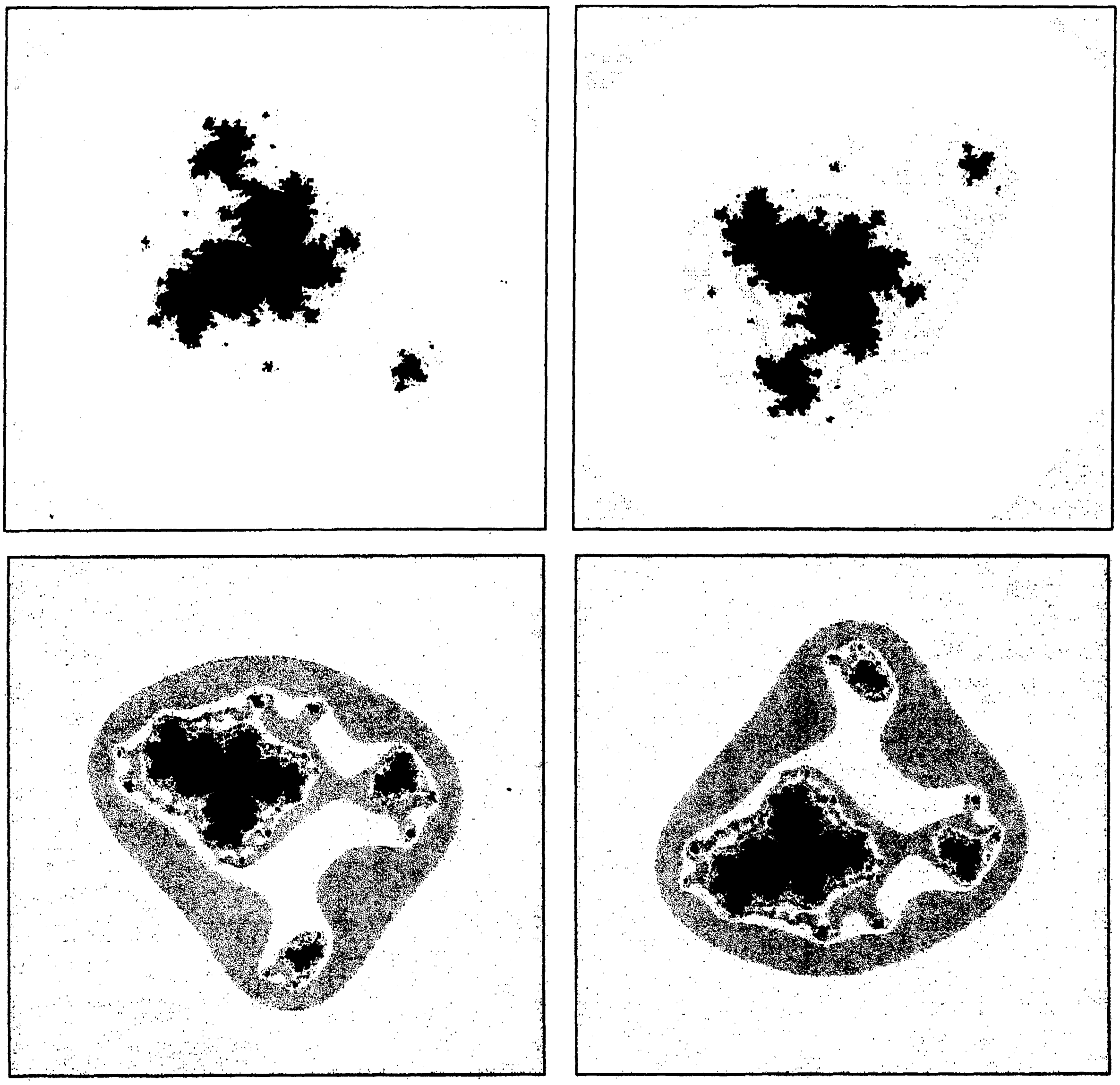

Figure 2: The left figure indicates the dynamics of $p_1(z) = z^4 + iz^2 + iz$ (upper), and $p_2(z) = z^4 - 0.39685iz^2 + iz - 0.66933$ (lower). $\Psi_4(\langle p_1 \rangle) = \Psi_4(\langle p_2 \rangle) = (12 - 8i, 30 - 64i, 125 - 48i)$. The map p_1 has a parabolic fixed point at the origin with multiplier i. The right figure indicates the dynamics of $q_1(z) = z^4 - iz^2 - iz$ (upper), and $q_2(z) = z^4 + 0.39685iz^2 - iz - 0.66933$ (lower). $\Psi_4(\langle q_1 \rangle) = \Psi_4(\langle q_2 \rangle) = (12 + 8i, 30 + 64i, 125 + 48i)$. The map q_1 has a parabolic fixed point at the origin with multiplier $-i$. These maps satisfy $\langle p_1 \rangle \leftarrow\!\!\!\!\!\!\rightarrow \langle q_1 \rangle$ and $\langle p_2 \rangle \leftarrow\!\!\!\!\!\!\rightarrow \langle q_2 \rangle$, and dynamics of these maps are symmetry for the real axis.

Proposition 1 *Suppose $p \not\sim q$, i.e. $\langle p\rangle \neq \langle q\rangle$, and $\langle p\rangle \leftarrow\!\!\!\!\rightarrow \langle q\rangle$. And suppose two representatives p and q are chosen as (3). If z_0 is a periodic point of p of period n with multiplier μ, then $\overline{z_0}$ is a periodic point of q of period n with multiplier $\overline{\mu}$. Therefore, the dynamics of p and q are symmetry for the real axis.*

Proof. The equation $\overline{p^n(z)} = q^n(\overline{z})$ holds by $\overline{p(z)} = q(\overline{z})$.

Since z_0 is a periodic point of p of period n, we obtain $\overline{z_0} = q^n(\overline{z_0})$. The fact of a relation between two multipliers is given by

$$\overline{\mu} = \overline{\{p^n(z_0)\}'} = \left\{\overline{p^n(z_0)}\right\}' = \{q^n(\overline{z_0})\}'.$$

∎

Remark that the coefficients of the z, i.e. c_1 or $\overline{c_1}$, are invariant under the action of cube roots of unity.

From now on, we assume that p and q are monic and centered representative quartic polynomial maps. If necessary, we consider p or q are the map that is taken conjugate by $z \mapsto \omega z$.

If $\langle p\rangle \leftarrow\!\!\!\!\rightarrow \langle q\rangle$, we can choose two representatives with following forms:

$$p = z^4 + c_2 z^2 + c_1 z + c_0, \qquad q = z^4 + \overline{c_2} z^2 + \overline{c_1} z + \overline{c_0}. \tag{3}$$

Lemma 4 *Under the condition that $\langle p\rangle \leftarrow\!\!\!\!\rightarrow \langle q\rangle$ with $\langle p\rangle \neq \langle q\rangle$, if $\Psi_4(\langle p\rangle) = \Psi_4(\langle q\rangle) = (s_1, s_2, s_4)$, then $(s_1, s_2, s_4) \in \mathbb{R}^3$.*

Proof. By the condition $\langle p\rangle \leftarrow\!\!\!\!\rightarrow \langle q\rangle$, the following is directly given by the transformation formula (2) (cf. Figure 2),

$$\langle p\rangle \in \Psi_4^{-1}(s_1, s_2, s_4) \quad \text{and} \quad \langle q\rangle \in \Psi_4^{-1}(\overline{s_1}, \overline{s_2}, \overline{s_4}).$$

Therefore, the assertion is clearly obtained. (See Figure 3.) ∎

Proof of Theorem 2. First, we suppose that $\langle p\rangle \leftarrow\!\!\!\!\rightarrow \langle q\rangle$. From Lemma 4, we have $(s_1, s_2, s_4) \in \mathbb{R}^3$. Hence, $\mathrm{G}_4(s_1, s_2, s_4) = 0$ has real coefficients.

As $s_1 \in \mathbb{R}$, each representative corresponding to the point (s_1, s_2, s_4) has a real coefficient of z. Therefore, two representatives p and q are given as follows:

$$p = z^4 + c_2 z^2 + c_1 z + c_0, \qquad q = z^4 + \overline{c_2} z^2 + c_1 z + \overline{c_0} \qquad (c_1 \in \mathbb{R}).$$

Now, we need to consider the following three cases.

a. The equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$ has multiple roots:
In this case, there is a unique conjugacy class corresponding to the point (s_1, s_2, s_4). While, from the assumption $\langle p\rangle \neq \langle q\rangle$ and $\Psi_4(\langle p\rangle) = \Psi_4(\langle q\rangle) = (s_1, s_2, s_4)$, we have $\#\Psi_4^{-1}(s_1, s_2, s_4) = 2$. This result contradicts with $\#\Psi_4^{-1}(s_1, s_2, s_4) = 1$.

Figure 3: The left figure indicates the dynamics of $p(z) = z^4 + (1.04113 - 0.28573i)z^2 + 0.82627z + (0.41265 - 0.10426i)$. The map p has two attracting fixed points at $0.33745 + 0.71902i$ and $0.21049 - 0.61365i$ with multiplier $0.8i$ and $-0.8i$ respectively. The right figure indicates the dynamics of $q(z) = z^4 + (1.04113 + 0.28573i)z^2 + 0.82627z + (0.41265 + 0.10426i)$. We remark that, $\Psi_4(\langle p\rangle) = \Psi_4(\langle q\rangle) = (5.38983, 7.80949, 4.58847) \in \mathbb{R}^3$, and $\langle p\rangle \leftrightarrow \langle q\rangle$.

b. The equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$ has two real roots:
   Let c_2^3 and $\tilde{c}_2^3$ be two real roots of $\mathrm{G}_4(s_1, s_2, s_4) = 0$. In this case we can choose two real values c_2, $\tilde{c}_2$. Then, two values c_0, $\tilde{c}_0$ are also real values, since $s_2 = 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48$. Hence the relation $\langle p\rangle \not\leftrightarrow \langle q\rangle$ is obtained, and this result contradicts with the assumption. (Moreover, $p \not\sim q$ is clear from $c_2^3 \neq \tilde{c}_2^3$.)

c. The equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$ has a pair of complex conjugate roots:
   Let c_2^3, $\tilde{c}_2^3$ be a pair of complex conjugate roots. We can choose two values c_2, $\tilde{c}_2$ as $c_2 = \overline{\tilde{c}_2}$ by $c_2^3 = \overline{\tilde{c}_2^3}$. Then, $c_0 = \overline{\tilde{c}_0}$ is given by $s_2 = 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48$.

Therefore the equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$ has negative discriminant.

Conversely, suppose the following quadratic equation,

$$\begin{aligned}262144(s_1-4)^2c^2 &+ 1024(27s_1^4 + (-144s_2 - 576)s_1^2 + (384s_2\\ &+1280)s_1 + 128s_2^2 - 256s_2 - 512s_4 - 768)c + (9s_1^2\\ &+24s_1 - 32s_2 - 48)^3 = 0,\end{aligned}$$

has negative discriminant, i.e.

$$\begin{aligned}1073741824(54s_1^5 &+ (-81s_2 - 27s_4 - 135)s_1^4 + (36s_2^2 - 144s_2 - 1008)s_1^3\\ &+(-4s_2^3 + 360s_2^2 + (144s_4 + 2976)s_2 + 576s_4 + 4192)s_1^2 + (-160s_2^3\\ &-2176s_2^2 + (-384s_4 - 6400)s_2 - 1280s_4 - 5376)s_1 + 16s_2^4 + 448s_2^3\\ &+(-128s_4 + 2176)s_2^2 + (256s_4 + 3840)s_2 + 256s_4^2 + 768s_4 + 2304) < 0.\end{aligned}$$

Then, for a monic and centered representative $p \in \langle p \rangle = \Psi_4^{-1}(s_1, s_2, s_4)$, the term of degree one, i.e. c_1, is given by $c_1 = -\frac{s_1 - 12}{8}$. Because the equation $\mathrm{G}_4(s_1, s_2, s_4) = 0$ has negative discriminant with real coefficients, there exists two roots $c = c_2^3,\ \tilde{c}_2^3$ satisfying $c_2^3 = \overline{\tilde{c}_2^3}$. We can choose $c_2 = \overline{\tilde{c}_2}$. Then, $c_0 = \overline{\tilde{c}_0}$ holds by $s_2 = 4c_2^3 - 16c_0c_2 + 18c_1^2 - 60c_1 + 48$. Now we obtain the following two polynomial maps belonging to (s_1, s_2, s_4):

$$p = z^4 + c_2 z^2 + c_1 z + c_0, \quad q = z^4 + \tilde{c}_2 z^2 + c_1 z + \tilde{c}_0 = z^4 + \overline{c_2} + c_1 z + \overline{c_0}.$$

Thus, $\langle p \rangle \leftharpoondown \langle q \rangle$ holds. ∎

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