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Mathematical program with vector equilibrium problem constraints in Banach space

国立中山大学応用数学系 (台湾) 木村 健志 (KENJI KIMURA)

正修科技大学資訊管理系 (台湾) 劉 永誠 (YEONG-CHENG LIOU)

国立中山大学応用数学系 (台湾) 姚 任之 (JEN-CHIH YAO)

Abstract

In this paper, we study mathematical program with vector equilibrium constraint problems in reflexive Banach spaces. In 2006, Sufficient conditions to obtain closedness of the solutions mapping for a parametric vector equilibrium problem are established; see [5]. In 2005, an existence result of optimal solutions on non-compact set in reflexive Banach space has been established by Liou and Yao; see [7]. On the result, weakly closedness of the constraint set for upper-level problem are required. Therefore sufficient conditions to obtain weakly closedness of the graph of the solutions mapping are mainly investigated.

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1. INTRODUCTION

Throughout the paper, we assume that every topological space is Hausdorff and every field of vector space is real, and int A denotes the topological interior of a set A.

Let Ω_1 and Ω_2 be two nonempty subsets of a topological space X and a topological vector space, (in short, t.v.s.), Y, respectively. Let Z be a t.v.s., and $\operatorname{int} C(x) \subset Z$ be a domination structure generated by set-valued mapping $C : \Omega_1 \to 2^Z$ at $x \in \Omega_1$, such that C has solid pointed convex cone values. Suppose that the constraint map Ω is a set-valued mapping from Ω_1 to $2^{\Omega_2} \setminus \{\emptyset\}$. Let g be a vector-valued function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z. We consider the following parametric vector equilibrium problem (PVEP): for a given $x \in \Omega_1$,

(PVEP) finding $y^* \in \Omega(x)$ such that
$g(x, y^*, v) \notin -\operatorname{int} C(x)$ for all $v \in \Omega(x)$,

whose solution mapping S_E is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by.

$$S_E(x) = \{y \in \Omega(x) : g(x, y, v) \notin -\operatorname{int} C(x), \text{ for all } v \in \Omega(x)\}. \tag{1}$$

If Ω, C, and g have a constant value for $x \in \Omega_1$, respectively, then the problem (PVEP) is reduced to an ordinary vector equilibrium problem. Liou et al. [6] introduce a weak PVVI as follows: for a given $x \in \Omega_1$,

$$\text{(PVVI)} \qquad \begin{array}{l} \text{finding } y^* \in \Omega(x) \text{ such that} \\ \nabla_y \varphi(x, y^*)(y^* - v) \notin -\text{int}\, C \text{ for all } v \in \Omega(x), \end{array}$$

where $\varphi = (\varphi_1, \ldots, \varphi_p) : \Omega_1 \times \Omega_2 \to \mathbb{R}^p$, $\varphi(x, \cdot)$ is differentiable in $\Omega(x)$ for a given $x \in \Omega_1$ and $\text{int}\, C \subset Z$ is a domination structure generating a partial ordering on Z; see Yu [11]. It is clear that PVVI is a special case of PVEP.

The purpose of this paper is to establish some existence results for PVEP and give some applications of PVEP, particularly to the mathematical programs with vector equilibrium constraints. To this end, we will give some preliminaries which will be used for the rest of this paper in Section 2. We will establish some existence results and closedness of the graph of the solution map for PVEP In Section 3. Finally we will establish some existence results for the mathematical program with equilibrium constraints as applications of PVEP.

2. Preliminaries

We recall the cone-convexity of vector-valued functions by Tanaka [9]. Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. The mapping f is said to be C-convex on K if for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$, we have

$$\lambda f(x_1) + (1 - \lambda) f(x_2) \in f(\lambda x_1 + (1 - \lambda) x_2) + C.$$

As a special case, if $Z = \mathbb{R}$ and $C = \mathbb{R}_+$ then C-convexity is the same as ordinary convexity.

Definition 1 (C-quasiconvex, [2, 8, 9]). Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-quasiconvex on K* if it satisfies one of the following two equivalent conditions:

(i) for each $x_1, x_2 \in K$ and $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda) x_2) \in z - C, \text{ for all } z \in C(f(x_1), f(x_2)),$$

where $C(f(x_1), f(x_2))$ is the set of upper bounds of $f(x_1)$ and $f(x_2)$, i.e.,

$$C(f(x_1), f(x_2)) := \{z \in Z : z \in f(x_1) + C \text{ and } z \in f(x_2) + C\}.$$

(ii) for each $z \in Z$,

$$A(z) := \{x \in K : f(x) \in z - C\}$$

is convex or empty.

First statement is defined by Luc [8] and the second is by Ferro [2].

Remark 1 (See Tanaka [9]). Some readers recall the following Helbig's definition which is stronger than Luc and Ferro's definition. When Z is locally convex space and C is closed, the definition is equivalent to C-naturally quasiconvex defined by Tanaka [9].

Definition 2 (Helbig's C-quasiconvexity, [4, 9]). Let X be a vector space, and Z also a locally convex space with a partial ordering defined by a closed pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be (Helbig's) *C-quasiconvex on K* if for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$, and each $\varphi \in C^*, \varphi(f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\varphi(f(x_1)), \varphi(f(x_2))\}$, where C^* stands for the topological dual cone of C.

Example 1. $f : \mathbb{R} \to \mathbb{R}^2$ is defined by $f(x) = (x, -|x|)$ for $x \in [-1,1]$ and $C = \{(x,y) \in \mathbb{R}^2 : y \geq |x|\}$. Then we can see that f is Luc and Ferro's C-quasiconvex, but not Helbig's.

Definition 3 (C-properly quasiconvex, [9]). Let X be a vector space, and Z also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to Z. Then, f is said to be *C-properly quasiconvex on K* if for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$ we have either

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - C,$$

or

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - C.$$

Definition 4 (C-continuity, [8, 10]). Let X be a topological space, and Z a topological vector space with a partial ordering defined by a solid pointed convex cone C. Suppose that f is a vector-valued function from X to Z. Then, f is said to be *C-continuous at $x \in X$* if it satisfies one of the following three equivalent conditions:

(i) $f^{-1}(x + \operatorname{int} C)$ is open.

(ii) For any neighbourhood $V_{f(x)} \subset Z$ of $f(x)$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in V_{f(x)} + C$ for all $u \in U_x$.

(iii) For any $k \in \operatorname{int} C$, there exists a neighbourhood $U_x \subset X$ of x such that $f(u) \in f(x) - k + \operatorname{int} C$ for all $u \in U_x$.

Moreover a vector-valued function f is said to be *C-continuous on X* if f is C-continuous at every x on X.

Remark 2. Whenever $Z = \mathbb{R}$ and $C = \mathbb{R}_+$, C-continuity and $(-C)$-continuity are the same as ordinary lower and upper semicontinuity, respectively. In [10, Definition 2.1 (pp.314–315)] corresponding to ordinary functions, C-continuous function is called C-lower semicontinuous function, and $(-C)$-continuous function is called C-upper semicontinuous function.

Definition 5 (see [1]). Let X and Y be two topological spaces, $T : X \to 2^Y$ a set-valued mapping.

(i) T is said to be *lower semicontinuous* (l.s.c. for short) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $z \in U$, $T(z) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(ii) The graph of T, denoted by $\mathrm{Gr}(T)$ is the following set:

$$\{(x, y) \in X \times Y : y \in T(x)\}.$$

Definition 6 (Parameterized cone continuity). Let P be a topological space. Let X and Z be two t.v.s.. Suppose that C is a set-valued mapping from P to 2^Z such that C has solid convex cone values, and suppose that K is a set-valued mapping from P to $2^X \setminus \{\emptyset\}$. Then vector-valued function $f : P \times X \times X \to Z$ is said to be *parametarized C-continuous on $P \times X$ with respect to K*, if for each $p \in P$ and $x \in K(p)$ such that

$$f(p, x, y) \in \operatorname{int} C(p) \text{ for some } y \in K(p),$$

there exists a neighborhood $\mathcal{U}$ of (p, x) such that for all $(\tilde{p}, \tilde{x}) \in \mathcal{U} \cap \mathrm{Gr}(K)$

$$f(\tilde{p}, \tilde{x}, \hat{y}) \in \operatorname{int} C(\tilde{p}) \text{ for some } \hat{y} \in K(\tilde{p}).$$

We denote f is parametarized w-C-continuous on $P \times X$ with respect to K if we consider the continuity in weak topology.

Definition 7 (Joint $-C(p)$-continuity). For each $(\hat{p}, \hat{x}, \hat{y}) \in \Omega_1 \times \Omega_2 \times \Omega_2$, a neighborhood $\mathcal{V}_{\hat{p}}$ of $\hat{p}$, and a neighborhood $\mathcal{V}_{\hat{g}}$ of $g(\hat{p}, \hat{x}, \hat{y})$, there exist $\mathcal{U}_{\hat{p}} (\subset \mathcal{V}_{\hat{p}})$, $\mathcal{U}_{\hat{x}}$, and $\mathcal{U}_{\hat{y}}$ such that

$$g(p, x, y) \in (\mathcal{V}_{\hat{g}} - \operatorname{int} C(\hat{p})) \text{ for all } (p, x, y) \in \mathcal{U}_{\hat{p}} \times \mathcal{U}_{\hat{x}} \times \mathcal{U}_{\hat{y}},$$

where $\mathcal{U}_{\hat{p}}, \mathcal{U}_{\hat{x}}$, and $\mathcal{U}_{\hat{y}}$ stand for neighborhoods of $\hat{p}, \hat{x}$ and $\hat{y}$, respectively.

Proposition 1. *Let Ω_1 and Ω_2 be two nonempty subsets of two normal spaces, respectively. Let Z be a normal t.v.s., and C a set-valued mapping from Ω_1 to 2^Z, such that C has solid pointed convex cone values. Suppose that Ω is a set-valued mapping from Ω_1 to $2^{\Omega_2} \setminus \{\emptyset\}$, and that g is a vector-valued function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z. Also assume the following conditions:*

(i) *g is $-C(p)$-continuous on $\Omega_1 \times \Omega_2 \times \Omega_2$, jointly;*
(ii) *Ω is l.s.c. on Ω_1;*
(iii) *the set-valued map $W(p) = Z \setminus -\operatorname{int} C(p)$ has closed graph.*

Then g is parametarized $-C$-continuous on $\Omega_1 \times \Omega_2$ with respect to Ω.

Proof. Suppose for each $\hat{p} \in \Omega_1$ and $\hat{x} \in \Omega(\hat{p})$ such that $g(\hat{p}, \hat{x}, \hat{y}) \in -\operatorname{int} C(\hat{p})$ for some $\hat{y} \in \Omega(\hat{p})$.

Then there is a $\hat{z} \in -\operatorname{int} C(\hat{p})$ such that $(\hat{z} - \operatorname{cl} C(\hat{p}))$ is a closed neighborhood of $g(\hat{p}, \hat{x}, \hat{y})$.

On the other hand $\{\hat{p}\} \times (\hat{z} - \operatorname{cl} C(\hat{p}))$ is a closed subset of $\Omega_1 \times Z$ such that

$$\mathrm{Gr}(W) \cap (\{\hat{p}\} \times (\hat{z} - \operatorname{cl} C(\hat{p}))) = \emptyset.$$

Since $\Omega_1 \times Z$ is normal space and, by condition (iii), $\mathrm{Gr}(W)$ is a closed subset of $\Omega_1 \times Z$, there exist a neighborhood $\mathcal{V}_{\hat{p}}$ of $\hat{p}$ and a neighborhood $\mathcal{V}_D$ of $(\hat{z} - \operatorname{cl} C(\hat{p}))$ such that

$$\mathrm{Gr}(W) \cap (\mathcal{V}_{\hat{p}} \times \mathcal{V}_D) = \emptyset,$$

and so $\mathrm{Gr}(W) \cap (\mathcal{V}_{\hat{p}} \times (\hat{z} - \operatorname{int} C(\hat{p}))) = \emptyset$. Since $(\hat{z} - \operatorname{int} C(\hat{p}))$ is a neighborhood of $g(\hat{p}, \hat{x}, \hat{y})$, by condition (i), we can choose $\mathcal{U}_{\hat{p}}(\subset \mathcal{V}_{\hat{p}})$, $\mathcal{U}_{\hat{x}}$, and $\mathcal{U}_{\hat{y}}$ such that for all $(p, x, y) \in \mathcal{U}_{\hat{p}} \times \mathcal{U}_{\hat{x}} \times \mathcal{U}_{\hat{y}}$,

$$g(p, x, y) \in ((\hat{z} - \operatorname{int} C(\hat{p})) - \operatorname{int} C(\hat{p})) = (\hat{z} - \operatorname{int} C(\hat{p})),$$

where $\mathcal{U}_{\hat{p}}, \mathcal{U}_{\hat{x}}$, and $\mathcal{U}_{\hat{y}}$ stand for neighborhoods of $\hat{p}, \hat{x}$ and $\hat{y}$, respectively.

Next by condition (ii) noting $\Omega(\hat{p}) \cap \mathcal{U}_{\hat{y}} \neq \emptyset$, we can choose a neighborhood $\mathcal{U}'_{\hat{p}}$ of $\hat{p}$ such that

$$\Omega(p) \cap \mathcal{U}_{\hat{y}} \neq \emptyset \text{ for all } p \in \mathcal{U}'_{\hat{p}}.$$

Let $\mathcal{U} = (\mathcal{U}_{\hat{p}}) \cap \mathcal{U}'_{\hat{p}} \times \mathcal{U}_{\hat{x}}$ which is a neighborhood of $(\hat{p}, \hat{x})$. Then for each $(p', x') \in \mathcal{U} \cap \mathrm{Gr}(\Omega)$, since $p' \in U'_{\hat{p}}$, $\Omega(p') \cap \mathcal{U}_{\hat{y}} \neq \emptyset$, there exists $y' \in \Omega(p') \cap \mathcal{U}_{\hat{y}}$. Therefore for the (p', x', y')

$$g(p', x', y') \in (\hat{z} - \operatorname{int} C(\hat{p})),$$

and hence

$$(p', g(p', x', y')) \in \mathcal{V}_{\hat{p}} \times \mathcal{V}_D.$$

Consequently, $(p', g(p', x', y')) \notin \mathrm{Gr}(W)$ and hence

$$g(p', x', y') \in -\operatorname{int} C(p').$$

□

Definition 8 (KKM-map). Let X be a topological vector space, and K a nonempty subset of X. Suppose that F is a multifunction from K to 2^X. Then, F is said to be a *KKM-map*, if

$$\operatorname{co}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} F(x_i)$$

for each finite subset $\{x_1, \ldots, x_n\}$ of X.

Remark 3. Obviously, if F is a KKM-map, then $x \in F(x)$ for each $x \in K$.

Lemma 1 (Fan-KKM; see [3]). *Let X be a topological vector space, and K a nonempty subset of X; and let G be a multifunction from K to 2^X. Suppose that G is a KKM-map and that $G(x)$ is a closed subset of X for each $x \in K$. If $G(\hat{x})$ is compact for at least one $\hat{x} \in K$, then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

3. Existence results for PVEP and weakly closedness of solutions graph

Throughout the rest of the paper, let Y and Y be two real reflexive Banach spaces, and Z a real Hausdorff topological vector space.

Theorem 1. *Let Ω_1 and Ω_2 be two nonempty subsets of X and Y, respectively. Let C be a set-valued mapping from Ω_1 to 2^Z, such that C has solid pointed convex cone values. Suppose that Ω is a set-valued mapping from Ω_1 to $2^{\Omega_2} \setminus \{\emptyset\}$ and that g is a vector-valued function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z. Also we assume the following conditions:*

(i) *Ω has closed convex values for each $x \in \Omega_1$;*

(ii) *for each* $(x, y, v) \in \Omega_1 \times \Omega(x) \times \Omega(x)$ *satisfying* $g(x, y, v) \in -\operatorname{int} C(x)$, *there exists an weak neighborhood* $\mathcal{U}_y$ *of* y *such that for all* $y' \in (\mathcal{U}_y \cap \Omega(x))$

$$g(x, y', v') \in -intC(x) \text{ for some } v' \in \Omega(x).$$

(iii) $g(x, y, \cdot)$ *is* $C(x)$*-quasiconvex on* $\Omega(x)$ *for each* $x \in \Omega_1, y \in \Omega(x)$;

(iv) $g(x, y, y) \notin -\operatorname{int} C(x)$ *for each* $x \in \Omega_1, y \in \Omega(x)$.

(v) *for each* $x \in \Omega_1$ *there exist* $\hat{v} \in \Omega(x)$ *and a weakly compact set* $\mathcal{B} \subset Y$ *such that* $\hat{v} \in \mathcal{B}$ *and*

$$g(x, y, v) \in -\operatorname{int} C(x) \text{ for all } y \in (\Omega(x) \setminus \mathcal{B}).$$

Then the problem (PVEP) has at least one solution for each $x \in \Omega_1$.

Proof. Let

$$G(v) := \{y \in \Omega(x) : g(x, y, v) \notin -\operatorname{int} C(x)\} \; v \in G(v),$$

for each $x \in \Omega_1$. First, we show that $G(v)$ is a KKM-map, for each $x \in \Omega_1$. Suppose to the contrary that there exists $\alpha_i \in [0, 1]$, $y_i \in \Omega(x)$ $(i = 1, \ldots, n)$ such that

$$\sum_{i=1}^{n} \alpha_i y_i = y \notin \bigcup_{i=1}^{n} G(y_i).$$

Then we have $y \in \Omega(x)$ because, by condition (i), $\Omega(x)$ is convex. Hence

$$f(x, y, y_i) \in -\operatorname{int} C(x), \; i = 1, \ldots, n.$$

This means that

$$f(x, y, \sum_{i=1}^{n} \alpha_i y_i) = f(x, y, y) \in -\operatorname{int} C(x),$$

because of condition (iii), and contradicts condition (iv).

Next, from conditions (i) and (ii), for each $v \in \Omega(x)$, $G(v)$ is a weakly closed set, and by condition (iv), $G(v) \neq \emptyset$, and also from condition (v), $G(\hat{v})$ is a weakly compact set. Thus we can apply Lemma 1, to get

$$S_E(x) = \bigcap_{v \in \Omega(x)} G(v) \neq \emptyset,$$

for each $x \in \Omega_1$, where S_E denotes the solutions map defined by (1). □

Condition (ii) can be replaced as follows: $g(x, \cdot, v)$ is weakly $-C(x)$-continuous on $\Omega(x)$ for each $x \in \Omega_1, v \in \Omega(x)$; and if we assume Ω has weakly compact values, then condition (v) can be removed. Hence we also obtain the following corollary.

Corollary 1. *Let* Ω_1 *and* Ω_2 *be two nonempty subsets of* X *and* Y, *respectively. Let* C *be a set-valued mapping from* Ω_1 *to* 2^Z, *such that* C *has solid pointed convex cone values. Suppose that* Ω *is a set-valued mapping from* Ω_1 *to* $2^{\Omega_2} \setminus \{\emptyset\}$ *and that* g *is a vector-valued function from* $\Omega_1 \times \Omega_2 \times \Omega_2$ *to* Z. *Also we assume the following conditions:*

(i) Ω *has weakly compact convex values for each* $x \in \Omega_1$;

(ii) $g(x, \cdot, v)$ *is weakly* $-C(x)$*-continuous on* $\Omega(x)$ *for each* $x \in \Omega_1, v \in \Omega(x)$;

(iii) $g(x, y, y) \notin -\operatorname{int} C(x)$ *for each* $x \in \Omega_1, y \in \Omega(x)$.

Then the problem (PVEP) has at least one solution for each $x \in \Omega_1$.

Theorem 2. *Let Ω_1 and Ω_2 be two nonempty subsets of two topological spaces, respectively. Let C be a set-valued mapping from Ω_1 to 2^Z, such that C has solid pointed convex cone values. Suppose that Ω is a set-valued mapping from Ω_1 to $(2^{\Omega_2} \setminus \{\emptyset\})$, g is a vector-valud function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, and S_E is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by (1). Also we assume that the following conditions:* *Let Ω_1 and Ω_2 be two nonempty subsets of two topological spaces, respectively. Let C be a set-valued mapping from Ω_1 to 2^Z, such that C has solid pointed convex cone values. Suppose that Ω is a set-valued mapping from Ω_1 to $(2^{\Omega_2} \setminus \{\emptyset\})$, g is a vector-valud function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, and S_E is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by (1). Also we assume the following conditions:*

(i) *Ω_1 is a weakly closed set;*
(ii) *Ω has weakly closed graph;*
(iii) *g is parametarized w-$(-C)$-continuous on $\Omega_1 \times \Omega_2$ with respect to Ω;*
(iv) *$S_E(x) \neq \emptyset$ for each $x \in \Omega_1$.*

Then the solution set $S_E(x)$ of problem (PVEP) has weakly closed graph.

Proof. Let $(x_\alpha, y_\alpha) \in \mathrm{Gr}(S_E)$ with $(x_\alpha, y_\alpha) \rightharpoonup (x, y)$. Then by conditions (i) and (ii), $x \in \Omega_1$ and $y \in \Omega(x)$. Therefore suppose to the contrary that $y \notin S_E(x)$, there exists $v \in \Omega(x)$ such that

$$g(x, y, v) \in -\mathrm{int}\, C(x).$$

Because of condition (iii), there is a weak neighborhood $\mathcal{U}$ of (x, y) such that for all $(\tilde{x}, \tilde{y}) \in \mathcal{U}$, there is $\tilde{v} \in \Omega(\tilde{x})$ such that $g(\tilde{x}, \tilde{y}, \tilde{v}) \in -\mathrm{int}\, C(\tilde{x})$. Then there exists $\bar{\alpha}$ such that for all $\alpha \geq \bar{\alpha}$, $y_\alpha \notin S_E(x_\alpha)$. This is a contradiction. □

Theorem 3. *Let Ω_1 and Ω_2 be two nonempty subsets of X and Y, respectively. Let C be a set-valued mapping from Ω_1 to 2^Z, such that C has solid pointed convex cone values. Suppose that Ω is a set-valued mapping from Ω_1 to $(2^{\Omega_2} \setminus \{\emptyset\})$, g is a vector-valud function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, and S_E is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by (1). Also we assume that the following conditions:*

(i) *Ω_1 is a weakly closed set;*
(ii) *Ω has weakly closed graph;*
(iii) *g is parametarized w-$(-C)$-continuous on $\Omega_1 \times \Omega_2$ with respect to Ω;*
(iv) *$g(x, y, \cdot)$ is $C(x)$-quasiconvex on $\Omega(x)$ for each $x \in \Omega_1$ and $y \in \Omega(x)$, and $g(x, y, y) \notin -\mathrm{int}\, C(x)$ for each $x \in \Omega_1$ and $y \in \Omega_2$;*
(v) *for each $x \in \Omega_1$ there exist $\hat{v} \in \Omega(x)$ and a weakly compact set $\mathcal{B} \subset Y$ such that $\hat{v} \in \mathcal{B}$ and*

$$g(x, y, v) \in -\mathrm{int}\, C(x) \text{ for all } y \in (\Omega(x) \setminus \mathcal{B}).$$

Then the problem (PVEP) has at least one solution, and S_E has weakly closed graph.

Proof. The result follows from Theorems 1 and 2. □

Theorem 4. *Let Ω_1 and Ω_2 be two nonempty subsets of X and Y, respectively. Let C be a set-valued mapping from Ω_1 to 2^Z, such that C has solid pointed convex cone values*

values. Suppose that Ω is a set-valued mapping from Ω_1 to $(2^{\Omega_2} \setminus \{\emptyset\})$, g is a vector-valud function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, and S_E is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by (1). Also we assume that the following conditions:

(i) *Ω_1 is a closed set;*
(ii) *Ω has closed convex graph;*
(iii) *$\operatorname{int}(\cap_{x\in\Omega_1} C(x))$ is nonempty;*
(iv) *g is parametarized $(-C)$-continuous on $\Omega_1 \times \Omega_2$ with respect to Ω;*
(v) *$g(x, y, \cdot)$ is $C(x)$-quasiconvex on $\Omega(x)$ for each $x \in \Omega_1$ and $y \in \Omega(x)$, and $g(x, y, y) \notin -\operatorname{int} C(x)$ for each $x \in \Omega_1$ and $y \in \Omega_2$;*
(vi) *for each $x \in \Omega_1$ there exist $\hat{v} \in \Omega(x)$ and a compact set $\mathcal{B} \subset Y$ such that $\hat{v} \in \mathcal{B}$ and*

$$g(x, y, v) \in -\operatorname{int} C(x) \text{ for all } y \in (\Omega(x) \setminus \mathcal{B}).$$

(vii) *$g(\cdot, \cdot, v)$ is C-properly quasiconcave on $\Omega_1 \times \Omega_2$ for each $v \in \Omega_2$, where $C := \cap_{x\in\Omega_1} C(x)$.*

Then the problem (PVEP) has at least one solution for each $x \in \Omega_1$, the graph of S_E is weakly closed in $\Omega_1 \times \Omega_2$.

Proof. Using the same way with Theorem 1 and Theorem 2, we obtain nonemptyness of $S_E(x)$ for each $x \in \Omega_1$ and closedness of $\operatorname{Gr}(S_E)$. Moreover by conditions (iv) and (ix), $\operatorname{Gr}(S_E)$ is a convex set. Hence $\operatorname{Gr}(S_E)$ is weakly closed. □

4. Mathematical program with vector equilibrium constraints

As an application of weakly closedness result of solutions map for (PVEP), we investigate the existence of solution for a MPEC. Consider the following MPEC:

$$\text{(MPEC)} \qquad \min\{f(x, y) : y \in S_E(x)\},$$

where $f : \Omega_1 \times \Omega_2 \to (-\infty, \infty)$ and $S_E : \Omega_1 \to 2^{\Omega_2}$ is a set-valued mapping such that for each $x \in \Omega_1$, $S_E(x)$ is teh solution set of the following PVEP, consisting in finding $y \in \Omega$ such that

$$g(x, y, v) \notin -\operatorname{int} C(x) \text{ for all } v \in \Omega(x),$$

where g is a vector-valued function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, $C(x) \subset Z$ is a domination structure generated bu set-valued mapping $C : \Omega_1 \to 2^Z$ at $x \in \Omega_1$, and $\Omega : \Omega_1 \to 2^{\Omega_2} \setminus \{\emptyset\}$ stands for a constraint map.

Definition 9 (see [7]). Let K be a nonempty subset of a real Banach space E and h a function from K to $[-\infty, \infty]$.

(i) h is called proper if h is not identically equal to $+\infty$, or h is not identically equal to $-\infty$.
(ii) h is called weakly sequentially lower semicontinuous on K if for each $x \in K$ and each sequence $\{x_n\}$ in K, in weak topology,

$$x_n \rightharpoonup x \Rightarrow h(x) \leq \liminf_{n\to\infty} h(x_n).$$

(iii) h is called weakly coercive if in weak topology,

$$h(x) \to +\infty, \text{ as } \|x\|_E \to \infty \text{ on } K,$$

where $\|\cdot\|_E$ denotes the norm of E.

Lemma 2 (Theorem 2,[7]). *Let f be a function from $X \times Y$ to $(-\infty, infty]$. Suppose that*

(i) *f is proper and weakly sequentially lower semicontinuous on* $\mathrm{Gr}(S)$*;*
(ii) *f is weakly coercive;*
(iii) $\mathrm{Gr}(S_E)$ *is weakly closed.*

Then MPEC has at least one solution.

We have the following existence of MPEC.

Theorem 5. *Let Ω_1 and Ω_2 be two nonempty subsets of X and Y, respectively. Let C be a set-valued mapping from Ω_1 to 2^Z, such that C has solid poited convex cone values. Suppose that Ω is a set-valued mapping from Ω_1 to $2^{\Omega_2} \setminus \{\emptyset\}$, that g is a vector-valued function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, and that S is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by 1, and suppose that f is a vector-valued function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to* $\mathrm{Gr}(S_E)$*, where* $\mathrm{Gr}(S_E)$ *stands for the graph of S_E. Also we assume that the following conditions:*

(i) *f is proper and weakly sequentially lower semicontinuous on* $\mathrm{Gr}(S)$*;*
(ii) *f is weakly coercive;*
(iii) *Ω_1 is a weakly closed set;*
(iv) *Ω has weakly closed graph;*
(v) *g is parametarized w-$(-C)$-continuous on $\Omega_1 \times \Omega_2$ with respect to Ω;*
(vi) *$g(x, y, \cdot)$ is $C(x)$-quasiconvex on $\Omega(x)$ for each $x \in \Omega_1$ and $y \in \Omega(x)$, and $g(x, y, y) \notin -\mathrm{int}\, C(x)$ for each $x \in \Omega_1$ and $y \in \Omega_2$;*
(vii) *for each $x \in \Omega_1$ there exist $\hat{v} \in \Omega(x)$ and a weakly compact set $\mathcal{B} \subset Y$ such that $\hat{v} \in \mathcal{B}$ and*

$$g(x, y, v) \in -\mathrm{int}\, C(x) \textit{ for all } y \in (\Omega(x) \setminus \mathcal{B}).$$

Then the (MPEC) has at least one solution.

Proof. By Theorems 3, we have $S_E(x) \neq \emptyset$ and $\mathrm{Gr}(S_E)$ is weakly closed. Then we can apply, by condition (i) and (ii), Lemma 2, and so the MPEC has at least one solution. □

Theorem 6. *Let Ω_1 and Ω_2 be two nonempty subsets of X and Y, respectively. Let C be a solid pointed convex cone in Z. Suppose that Ω is a set-valued mapping from Ω_1 to $(2^{\Omega_2} \setminus \{\emptyset\})$, g is a vector-valud function from $\Omega_1 \times \Omega_2 \times \Omega_2$ to Z, and S_E is a set-valued mapping from Ω_1 to 2^{Ω_2} defined by (1). Also we assume that the following conditions:*

(i) *f is proper and weakly sequentially lower semicontinuous on* $\mathrm{Gr}(S)$*;*
(ii) *f is weakly coercive;*
(iii) *Ω_1 is a closed set;*
(iv) *Ω has closed convex graph;*

(v) $\operatorname{int}(\cap_{x\in\Omega_1} C(x))$ *is nonempty;*
(vi) g *is parametarized* $(-C)$*-continuous on* $\Omega_1 \times \Omega_2$ *with respect to* Ω*;*
(vii) $g(x, y, \cdot)$ *is* $C(x)$*-quasiconvex on* $\Omega(x)$ *for each* $x \in \Omega_1$ *and* $y \in \Omega(x)$*, and* $g(x, y, y) \notin -\operatorname{int} C(x)$ *for each* $x \in \Omega_1$ *and* $y \in \Omega_2$*;*
(viii) *for each* $x \in \Omega_1$ *there exist* $\hat{v} \in \Omega(x)$ *and a compact set* $\mathcal{B} \subset Y$ *such that* $\hat{v} \in \mathcal{B}$ *and*

$$g(x, y, v) \in -\operatorname{int} C(x) \text{ for all } y \in (\Omega(x) \setminus \mathcal{B}).$$

(ix) $g(\cdot, \cdot, v)$ *is* C*-properly quasiconcave on* $\Omega_1 \times \Omega_2$ *for each* $v \in \Omega_2$*, where* $C := \cap_{x\in\Omega_1} C(x)$.

Then the (MPEC) has at least one solution.

Proof. By Theorem 4 and Lemma 2, we have the result. □

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