数理解析研究所講究録
第 1544 巻 2007 年 212-215

On an equivalence relation between efficiency and ideal efficiency

島根大学大学院総合理工学研究科 [1] 塗矢哲也 (Tetsuya Nuriya)
島根大学総合理工学部 [2] 黒岩大史 (Daishi Kuroiwa)

[1] Interdisciplinary Graduate School of Science and Engineering, Shimane University
[2] Interdisciplinary Faculty of Science and Engineering, Shimane University

Abstract

In previous researches, the set of all weakly efficient points and the set of all properly efficient points in vector optimization were represented by using the sets of all efficient points with respect to convex cones. In this paper, we consider a representation of the set of all ideal efficient points by using the sets of all efficient points, which is a representation similar to the above.

1 Introduction

Let E be a topological vector space and K be a convex cone of E which is not the whole space. A binary relation "$\leq_K$" on E is defined by

$$x \leq_K y \overset{\text{def}}{\Longleftrightarrow} y \in x + K. \quad (x, y \in E)$$

First, we state well-known notions of efficiency in vector optimization, see [6].

Definition 1. *Let A be a nonempty subset of E and K a convex cone of E which is not the whole space. We say that*

(1) *$x \in A$ is an ideal efficient point of A with respect to K if $y \in x + K$ for all $y \in A$;*

(2) *$x \in A$ is an efficient point of A with respect to K if $y \in x + K$ for all $y \in A$ with $x \in y + K$;*

(3) *$x \in A$ is a properly efficient point of A with respect to K if there exists a convex cone L of E, which is not the whole space, such that $K \setminus l(K)$ is contained in $\mathrm{int}L$ and x is an efficient point of A with respect to L, where $l(K) = K \cap (-K)$;*

(4) *Supposing that $\mathrm{int}K$ is nonempty, $x \in A$ is a weakly efficient point of A with respect to K if x is an efficient point of A with respect to $\mathrm{int}K \cup \{\theta\}$;*

The set of all efficient points (resp. properly efficient points, weakly efficient points and ideal efficient points) of A *with respect to* K *is denoted by* $\mathrm{Min}(A \mid K)$ *(resp.* $\mathrm{PrMin}(A \mid K)$*,* $\mathrm{WMin}(A \mid K)$ *and* $\mathrm{IMin}(A \mid K)$*).*

$\mathrm{WMin}(A \mid K)$ and $\mathrm{PrMin}(A \mid K)$ can be represented by using the sets of all efficient points with respect to convex cones. Concretely, it is clear that $\mathrm{WMin}(A \mid K) = \mathrm{Min}(A \mid \mathrm{int}K \cup \{\theta\})$, and a representation of the set of all properly efficient points was given by Nieuwenhuis.

Theorem 1 ([1]). *Let* A *be a nonempty subset of* E *and* K *be a convex cone. Assume that a sequence* $\{K_n\}_{n\in\mathbf{N}}$ *of closed convex cones such that* $K\backslash\{\theta\} \subset \mathrm{int}K_n$ *and* $K_n \subset K_{n+1}$ *for each* $n \in \mathbb{N}$*, and* $K = \bigcap_{n\in\mathbf{N}} K_n$*. Then, we have*

$$\mathrm{PrMin}(A \mid K) = \bigcup_{n\in\mathbf{N}} \mathrm{Min}(A \mid K_n).$$

In this paper, we consider a representation of $\mathrm{IMin}(A \mid K)$, which is the same as $\mathrm{WMin}(A \mid K)$ and $\mathrm{PrMin}(A \mid K)$, by using the sets of all efficient points. First, we state a representation of $\mathrm{IMin}(A \mid K)$ when K is closed, and we give scalarization for ideal efficiency by using the representation of $\mathrm{IMin}(A \mid K)$.

2 Main results

Remember that a convex cone K is said to be pointed if and only if $l(K) = \{\theta\}$, where θ is the origin of E.

Theorem 2. *Let* A *be a nonempty subset of* E*, and* K *a closed pointed convex cone of* E*, which is not the whole space. Then the following equations hold:*

$$\mathrm{IMin}(A \mid K) = \mathrm{Min}(A + (-K)^c \cup \{\theta\} \mid K).$$

Example 1. *Let* $E = \mathbb{R}^2$*,* $K = \mathbb{R}^2_+$*,* $A_1 = \{(x, y) \mid x \geq 0, y \leq 0, \|(x, y)\| \leq 1\}$ *and* $A_2 = \{(x, y) \mid \|(x, y)\| \leq 1\}$*, where* $\mathbb{R}_+$ *is the set of all nonnegative real numbers. Then,*

$$\begin{aligned} A_1 + (-K)^c \cup \{\theta\} &= (0, -1) + \mathbb{R}^2 \setminus (-\mathbb{R}^2_+ \setminus \{(0, 0)\}), \\ A_2 + (-K)^c \cup \{\theta\} &= (-1, -1) + \mathbb{R}^2 \setminus (-\mathbb{R}^2_+), \end{aligned}$$

and we can check

$$\begin{aligned} \mathrm{IMin}(A_1 \mid K) &= \mathrm{Min}(A_1 + (-K)^c \cup \{\theta\} \mid K) = \{(0, -1)\}, \\ \mathrm{IMin}(A_2 \mid K) &= \mathrm{Min}(A_2 + (-K)^c \cup \{\theta\} \mid K) = \emptyset. \end{aligned}$$

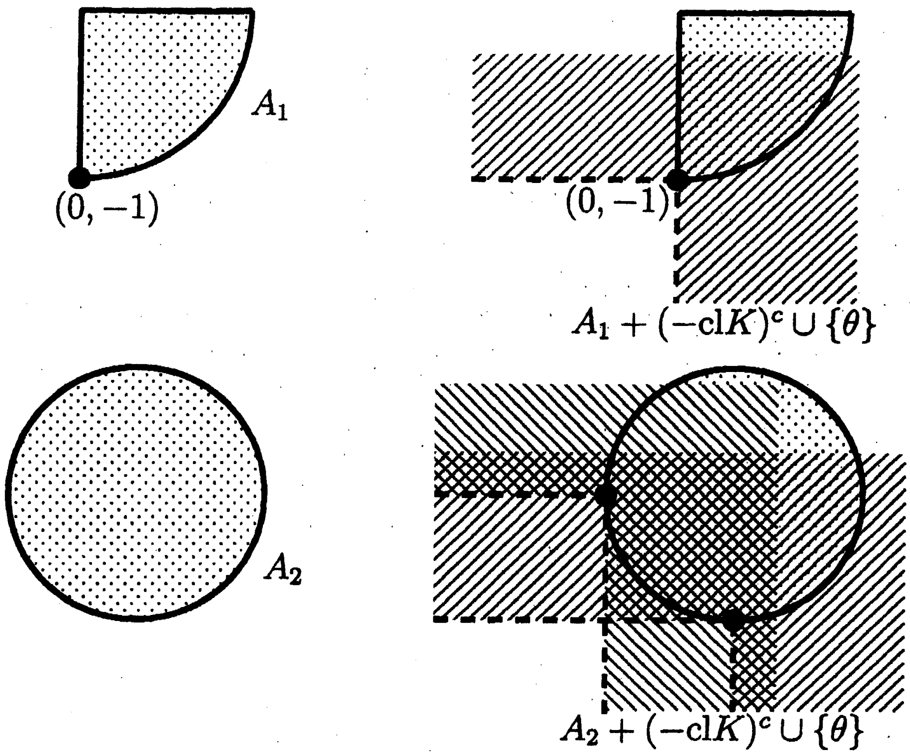

Figure of Example 1.

Next, we give a scalarization for ideal efficiency by using the representation of the set of all ideal efficient points. In the previous researches, we obtained the following scalarization for weak efficiency.

Definition 2 ([5]). *A function $f : E \to \mathbb{R}$ is said to be monotonic with respect to K if*

$$x \leq_K y, x \neq y \Rightarrow f(x) < f(y),$$

and f is said to be weakly monotonic with respect to K if

$$x \leq_K y \Rightarrow f(x) \leq f(y), \quad x \leq_{\mathrm{int}K \cup \{\theta\}} y, x \neq y \Rightarrow f(x) < f(y).$$

Theorem 3 ([5]). *Let A be a nonempty subset of $\mathbb{R}^n$ and K be a closed pointed convex cone. Then,*

(1) *x is a weakly efficient point of A with respect to K if and only if there exists a continuous weakly monotonic function f from E to $\mathbb{R}$ such that $f(x)$ is the minimum of f on A.*

(2) *x is a properly efficient point of A with respect to K if and only if there exist a closed pointed convex cone L of E with $K \setminus \{\theta\} \subset \mathrm{int}L$ and a continuous function f from E to $\mathbb{R}$ such that f is monotonic with respect to L and $f(x)$ is the minimum of f on A.*

By using Theorem 2 and 3, we have the following theorem.

Theorem 4. *Let A be a nonempty subset of E and K a closed pointed convex cone. The following assertions are true:*

(1) *if x is an ideal efficient point of A with respect to K, then there exists a continuous weakly monotonic function f from E to $\mathbb{R}$ such that $f(x)$ is the minimum of f on $A + (-K)^c \cup \{\theta\}$;*

(2) *if there exist a closed pointed convex cone L of E with $K \setminus \{\theta\} \subset \mathrm{int}L$ and a continuous function f from E to $\mathbb{R}$ such that f is monotonic with respect to L and $f(x)$ is the minimum of f on $A + (-\mathrm{cl}L)^c \cup \{\theta\}$, then x is an ideal efficient point of A with respect to K.*

References

[1] J. W. Nieuwenhuis, Properly efficient and efficient solutions for vector maximization problems in Euclidean space, *Journal of Mathematical Analysis and Applications* **84** (1981), 311–317.

[2] M. I. Henig, A cone separation theorem, *Journal of Optimization Theory and Applications* **36** (1982) 3, 451–455.

[3] M. I. Henig, Proper efficiency with respect to cones, *Journal of Optimization Theory and Applications* **36** (1982) 3, 387–407.

[4] S. Helbig, Approximation of the efficient point set by perturbation of the ordering cone, *Methods and Models of Operations Research* **36** (1982) 3, 387–407.

[5] D. T. Luc, On scalarizing method in vector optimization, *Lecture Notes in Economic and Math.Systems* **273** (1986), 46–51.

[6] D. T. Luc, Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems **319**. *Springer-Verlag, Berlin,* 1989.