Convergence rates of asymptotic solutions to Hamilton-Jacobi equations in Euclidean $n$ space

Yasuhiro FUJITA (University of Toyama)*

This is a survey of the paper [F]. Let us consider the Cauchy problem for the Hamilton-Jacobi equation

\begin{align}
(0.1) & \quad u_t(x, t) + H(x, Du(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
(0.2) & \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n.
\end{align}

By [FIL, I], under suitable assumptions on $H, u_0$, it is shown that the Cauchy problem (0.1)-(0.2) admits a unique solution $u \in C(\mathbb{R}^n \times [0, \infty))$ and there is a pair $(c, v) \in \mathbb{R} \times C(\mathbb{R}^n)$ such that

\begin{equation}
(0.3) \quad \lim_{t \to \infty} (u(x, t) + ct) = v(x) \quad \text{locally uniformly in } \mathbb{R}^n.
\end{equation}

Furthermore, $v$ is a solution of

\begin{equation}
(0.4) \quad H(x, Dv(x)) = c \quad \text{in } \mathbb{R}^n.
\end{equation}

In this talk, we are interested in rates of convergence of (0.3). We assume the following:

(A1) \quad H \in C(\mathbb{R}^n \times \mathbb{R}^n).

(A2) \quad For each $x \in \mathbb{R}^n$, $H(x, \cdot)$ is convex in $\mathbb{R}^n$.

(A3) \quad \lim_{r \to \infty} \inf \{H(x, p) \mid x \in B(0, R), \ p \in \mathbb{R}^n \setminus B(0, r)\} = \infty \quad \text{for } R > 0.

(A4) \quad There exists a pair $(\theta_0, c, w^+, w^-) \in (0, \infty) \times \mathbb{R} \times C(\mathbb{R}^n) \times C^1(\mathbb{R}^n)$ such that $w^+$ and $w^-$ are, respectively, a subsolution and a supersolution of (0.4) and

\begin{equation}
(0.5) \quad 0 \leq w^+(x) - w^-(x) \leq \frac{1}{\theta_0}(c - H(x, Dw^-(x))) \quad \text{in } \mathbb{R}^n.
\end{equation}

If (A1)-(A4) is fulfilled, then we define the function $\hat{w}$ by

\begin{equation}
(0.6) \quad \hat{w}(x) = \sup \{w(x) \mid w \in W\} \quad \text{in } \mathbb{R}^n,
\end{equation}

where $W$ is the set of all $w \in C(\mathbb{R}^n)$ such that $w^- + w$ is a viscosity subsolution of (0.4)

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for the constant \( c \) of (A4), and \( w \) satisfies the inequality
\[
0 \leq w(x) \leq w^+(x) - w^-(x) \quad \text{in } \mathbb{R}^n.
\]

**Remark 1.** \( 0 \leq \hat{w}(\cdot) \in \text{Lip}_{loc}(\mathbb{R}^n) \), and \( w^- + \hat{w} \) is a solution of (0.4). □

Besides (A1)-(A4), we assume:

(A5) There exists a function \( \varphi \in C(\mathbb{R}^n) \) such that \( \inf \{w^-(x) + \hat{w}(x) + \varphi(x) | x \in \mathbb{R}^n \} > -\infty \) and the following comparison principle holds: Let
\[
\Phi := \left\{ u \in C(\mathbb{R}^n \times [0, \infty)) \left| \inf \{u(x,t) + \varphi(x) | x \in \mathbb{R}^n, t \in [0,T)\} > -\infty \text{ for } T > 0 \right\} \right. ,
\]
If \( u_1 \in C(\mathbb{R}^n \times [0, \infty)) \) and \( u_2 \in \Phi \) are, respectively, a subsolution and a supersolution of (0.1) and satisfy \( u_1(\cdot, 0) \leq u_2(\cdot, 0) \) in \( \mathbb{R}^n \), then \( u_1 \leq u_2 \) in \( \mathbb{R}^n \times [0, \infty) \).

(A6) \( u_0 \in C(\mathbb{R}^n) \), and there exists a pair \( (K,F) \in \mathbb{R} \times C([0, \infty)) \) such that
\[
\begin{align*}
(0.8) & \quad F \geq 0 \text{ in } [0, \infty), \quad \limsup_{s \to 0} \frac{F(s)}{s} < \infty, \\
(0.9) & \quad K + w^-(x) \leq u_0(x) \leq K + w^-(x) + F(\hat{w}(x)) \text{ in } \mathbb{R}^n.
\end{align*}
\]

**Remark 2.** We give a sufficient condition for (A5). Assume that (A1), (A3) and (A4) hold and that \( H(x,\cdot) \) is strictly convex in \( \mathbb{R}^n \) for each \( x \in \mathbb{R}^n \) instead of (A2). Furthermore, assume that there exist functions \( \psi_i \in \text{Lip}_{loc}(\mathbb{R}^n) \) and \( \sigma_i \in C(\mathbb{R}^n) \), with \( i = 0,1 \), such that for \( i = 0,1 \),
\[
\begin{align*}
(0.10) & \quad \lim_{|x| \to \infty} \sigma_i(x) = \infty, \quad \inf \{w^-(x) + \hat{w}(x) + \psi_0(x) | x \in \mathbb{R}^n \} > -\infty, \\
& \quad H(x, -D\psi_i(x)) \leq -\sigma_i(x) \text{ almost every } x \in \mathbb{R}^n . \\
& \quad \lim_{|x| \to \infty} (\psi_1(x) - \psi_0(x)) = \infty.
\end{align*}
\]
Then (A5) holds for \( \varphi = \psi_0 \) by [I, Theorem 4.1]. As for examples satisfying these conditions, we give in our talk. □

**Lemma 1.** Let \( F \) be the function of (A6). Then, there exists a function \( G \in C((0, \infty)) \cap C^1((0, \infty)) \) such that
\[
(0.11) \quad G(0) = 0, \quad s + F(s) \leq G(s) \leq sG'(s) \text{ in } (0, \infty). \quad \square
\]

In the following, we assume (A1)-(A6). We define the constant \( \theta \in (0, \infty] \) by
\[
(0.12) \quad \theta = \sup \{ \theta_0 | \theta_0 \text{ fulfills (0.5)} \}.
\]
Theorem 1. Assume (A1)-(A6).

(i) \( \theta = \infty \) if and only if \( w^- \) is a solution of (0.4).

(ii) Let \( u \in \Phi \) be a solution of the Cauchy problem (0.1)-(0.2).

(a) If \( \theta = \infty \), then \( \dot{w} = 0 \) in \( \mathbb{R}^n \) and \( u(x, t) + ct = K + w^{-}(x) \) in \( \mathbb{R}^n \times [0, \infty) \).

(b) If \( \theta < \infty \), then

\[
-\hat{w}(x)e^{-\theta t} \leq u(x, t) + ct - (K + w^{-}(x)) \leq [G(\hat{w}(x)) - \hat{w}(x)]e^{-\theta t}
\]

in \( \mathbb{R}^n \times [0, \infty) \),

where \( G \) is the function of Lemma 1. \( \square \)

Next, we give an example such that even if (A1)-(A5) hold, the rate of convergence in (0.3) is just equal to \( t^{-1} \) as \( t \to \infty \) provided (A6) is violated. For \( a, b \in \mathbb{R} \), let

\[
a^+ = \max\{a, 0\}, \quad a \lor b = \max\{a, b\} \quad \text{and} \quad a \land b = \min\{a, b\}.
\]

Example 1. For \( \alpha > 0 \), let

\[
H(x, p) = \alpha x \cdot p + \frac{1}{2}|p|^2 - \frac{\alpha^2}{2}(1 - |x|^2)^+ \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^n,
\]

\[
u_0(x) = \frac{\alpha}{2} \quad \text{in} \quad \mathbb{R}^n.
\]

Then, we have:

(i) The assumptions (A1)-(A5) hold for the constants \( c = -\alpha^2/2, \ \theta_0 = \alpha \) and the functions

\[
w^+(x) = \zeta_1(x), \quad w^-(x) = \zeta_k(x) \quad (k \in (0, 1)), \quad \varphi(x) = \frac{\alpha}{2}|x|^2 \quad \text{in} \quad \mathbb{R}^n,
\]

where

\[
\zeta_\ell(x) = \frac{\alpha}{2}(1 - |x|^2) + \alpha\ell \int_1^{|x|\lor 1} \sqrt{r^2 - 1} \, dr \quad \text{for} \quad x \in \mathbb{R}^n, \ \ell \in (0, 1).
\]

However, there is no pair \( (K, F) \in \mathbb{R} \times C([0, \infty)) \) for which (A6) holds.

(ii) The Cauchy problem (0.1)-(0.2) admits a unique solution \( u \in \Phi \) given by

\[
\frac{\alpha}{2(\alpha t + 1)}(|x|^2 \land 1) \leq u(x, t) - \frac{\alpha^2}{2}t - \zeta_1(x) \leq \frac{\alpha}{2(\alpha t + 1)}|x|^2 \quad \text{in} \quad \mathbb{R}^n \times [0, \infty),
\]

where \( \Phi \) is the set of (A5) for \( \varphi \) of (i). \( \square \)

Finally, we give an example such that the precise rate of convergence in (0.3) is obtained by our sufficient condition.
Example 2. For $\alpha, \beta > 0$, let

$$H(x, p) = \alpha x \cdot p + \frac{1}{2}|p|^2 - \frac{\beta}{2}|x|^2 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n.$$  

Assume that $u_0 \in C(\mathbb{R}^n)$ and that there is a constant $\ell \in (1, \infty)$ such that

$$\frac{A}{2}|x|^2 \leq u_0(x) \leq \frac{A\ell}{2}|x|^2 \quad \text{in } \mathbb{R}^n,$$

where $A = \sqrt{\alpha^2 + \beta} - \alpha$. Then, we have:

(i) Let $k \in (0, 1)$. The assumptions (A1)-(A6) hold for

$$c = 0, \quad \theta_0 \in (0, A(1 + k) + 2\alpha]$$

$$w^+(x) = \frac{A}{2}|x|^2, \quad w^-(x) = \frac{Ak}{2}|x|^2, \quad \varphi(x) = \frac{\alpha}{2}|x|^2 \quad \text{in } \mathbb{R}^n,$$

$$K = 0, \quad F(s) = \frac{\ell - k}{1 - k}s \quad \text{in } [0, \infty).$$

In this case, $\theta$ is equal to $A(1 + k) + 2\alpha(=: \theta_k)$, and $\hat{w}(x) = A(1 - k)|x|^2/2$ in $\mathbb{R}^n$.

(ii) Let $\Phi$ be the set of (A5) which is defined for $\varphi$ of (i). Then, the Cauchy problem (0.1)-(0.2) admits a unique solution $u$ in $\Phi$. By letting $G(s) = F(s) + s$ in $[0, \infty)$, Theorem 1 leads to

$$-\frac{A(1 - k)}{2}|x|^2 e^{-\theta_k t} \leq u(x, t) - \frac{A}{2}|x|^2 \leq \frac{A(\ell - k)}{2}|x|^2 e^{-\theta_k t} \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

In particular, letting $k \nearrow 1$, we obtain

$$0 \leq u(x, t) - \frac{A}{2}|x|^2 \leq \frac{A(\ell - 1)}{2}|x|^2 e^{-\lambda t} \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

where $\lambda = 2\sqrt{\alpha^2 + \beta}$.

(iii) When $u_0(x) = A\ell|x|^2/2$ in $\mathbb{R}^n$, a unique solution $u \in \Phi$ is given by

$$u(x, t) = \frac{A|x|^2}{2} \left(1 + \frac{\lambda(\ell - 1)}{A(\ell - 1)(e^{\lambda t} - 1) + \lambda e^{\lambda t}}\right) \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

Hence, the rate $e^{-\lambda t}$ which is obtained in (ii) is optimal in this case.  \square

References


[I] H. Ishii, Asymptotic solutions for large time of Hamilton-Jacobi equations in Euclidean $n$ space, preprint.