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UNIQUENESS AND EXISTENCE FOR SPIRAL CRYSTAL GROWTH

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In [2] Ohtsuka studied on a crystal growth of spirals and proposed us to use a level set method. Since the conventional level set method (see [1]) could not express spiral curves having the orientation, he modifyed the conventional method by using a sheet structure function.

Let Ω be a bounded domain of $\mathbb{R}^2$ with the smooth boundary and let $B_{\rho_j}(a_j)$ be N screw dislocations in Ω, which are disks small enough with $a_j \in \Omega$ and $\rho_j > 0$ so that $\overline{B_{\rho_j}(a_j)} \subset \Omega$ and $B_{\rho_j}(a_j) \bigcap B_{\rho_k}(a_k) = \emptyset$ if $j \neq k$. We denote

$$W = \Omega \setminus \left(\bigcup_{j=1}^{N} \overline{B_{\rho_j}(a_j)} \right).$$

The level set equation for his spiral crystal growth on W is

$$u_t - |\nabla(u-\theta)| \left\{ \operatorname{div} \frac{\nabla(u-\theta)}{|\nabla(u-\theta)|} + C \right\} = 0 \quad \text{in } W, \tag{1}$$

with the boundary condition of Neumann type

$$\langle \nu, \nabla(u-\theta) \rangle = 0 \quad \text{on } \partial W. \tag{2}$$

Here C is a constant, ν is the unit normal vector of ∂W and $\theta(x) = \sum_{j=1}^{N} m_j \arg(x - a_j)$ for $m_j \in \mathbb{Z} \setminus \{0\}$. We note that $\theta(x)$ is a multi-valued function, but $\nabla\theta$ is single-valued. When Γ_t is the spiral curve, it must be defined

$$\Gamma_t = \left\{ x \in \overline{W} : u(t,x) - \theta(x) \equiv 0 \bmod 2\pi m\mathbb{Z} \right\}.$$

Here m is the greatest common divisor of $|m_j|$. Ohtsuka proved the following results.

COMPARISON THEOREM. *Let u and v be a viscosity subsolution and a supersolution of (1) and (2), respectively. If $u^*(0,\cdot) \leqslant v_*(0,\cdot)$, then we have $u^*(t,x) \leqslant v_*(t,x)$ for any $t > 0$.*

EXISTENCE THEOREM. *For any given $u_0 \in C(\overline{W})$ there exists a unique global-in-time viscosity solution $u \in C([0,\infty) \times \overline{W})$ of (1) and (2) with initial data $u(0,\cdot) = u_0$.*

This note is a short remark for the Ohtsuka's theory, that is, we would like to consider the uniqueness of Γ_t. It means that, for a given initial spiral Γ_0, we choose u_0 an initial function satisfying

$$\Gamma_0 = \left\{x \in \overline{W} : u_0(x) - \theta(x) \equiv 0 \bmod 2\pi m\mathbb{Z}\right\},$$

the Existence Theorem says that there exists a unique solution u, but we can choose v_0 an another initial function satisfying

$$\Gamma_0 = \left\{x \in \overline{W} : v_0(x) - \theta(x) \equiv 0 \bmod 2\pi m\mathbb{Z}\right\}$$

and we get a unique solution v. Our question is

$$\left\{x \in \overline{W} : u(t,x) - \theta(x) \equiv 0 \bmod 2\pi m\mathbb{Z}\right\}$$

and

$$\left\{x \in \overline{W} : v(t,x) - \theta(x) \equiv 0 \bmod 2\pi m\mathbb{Z}\right\}$$

are tracing the same spiral curve?

The paper [1] solved this uniqueness problem for the case of closed curves. The key step is to construct the order changing function satisfying $u_0(x) \leqslant G(v_0(x))$, when, generally, u_0 and v_0 are not maked order each other. Since G is nondecreasing, if $v(t,x)$ is a viscosity supersolution, then $G(v(t,x))$ is also a viscosity supersolution. By using the Comparison Theorem we see that $u(t,x) \leqslant G(v(t,x))$, which leads us to compair the level sets of u and v.

We try to extend this key idea to the spiral case. Applying the Ohtsuka's method in [2] we first introduce the covering space of $\overline{W}$ like

$$\mathfrak{X} = \left\{(x,\xi) \in \overline{W} \times \mathbf{R}^N : \xi = (\xi_1, \cdots, \xi_N),\ (\cos\xi_j, \sin\xi_j) = \frac{x - a_j}{|x - a_j|}\right\}$$

and assume that

$$\left\{(x,\xi)\in\mathfrak{X}:\ u_0(x)-\sum_{j=1}^{N}m_j\xi_j>0\right\}=\left\{(x,\xi)\in\mathfrak{X}:\ v_0(x)-\sum_{j=1}^{N}m_j\xi_j>0\right\}.$$

We construct an order changing function G with

$$u_0(x)-\sum_{j=1}^{N}m_j\xi_j\leqslant G\left(v_0(x)-\sum_{j=1}^{N}m_j\xi_j\right)\quad\text{for } (x,\xi)\in\mathfrak{X}.$$

The important properties for G are nondecreasing and satisfying the periodical condition ($\sharp$) $G(s)=G(s+2\pi m_j)-2\pi m_j$. Basically, G is modified from

$$G_1(s)=\sup\left\{(\tilde{u}_0(y,\eta))_+ :\ (y,\eta)\in\mathfrak{X},\ \tilde{v}_0(y,\eta)\leqslant s\right\}.$$

Here $\tilde{u}_0(y,\eta)=u_0(y)-\sum_{j=1}^{N}m_j\eta_j$, $\tilde{v}_0(y,\eta)=v_0(y)-\sum_{j=1}^{N}m_j\eta_j$ and $(a)_+=\max\{a,0\}$. Finally, we obtain

Invariance Lemma. *Let v be a viscosity supersolution with initial data $v(0,\cdot)=v_0$ and define*

$$w(t,x)=G(v(t,x)-\theta(x))+\theta(x)\tag{3}$$

in the sence of some meaning in the covering space (becase $\theta(x)$ is multi-valued). Then we have w is a viscosity supersolution with $w(0,\cdot)=w_0$.

The meaning of the definition (3) is the following: We denote that

$$\mathfrak{L}=\bigcup_{j=1}^{N}\mathfrak{L}_j,\quad \mathfrak{L}_j=\left\{x\in\overline{W}:\ \frac{x-a_j}{|x-a_j|}=(-1,0)\right\}$$

and $\Theta_j(x)=\operatorname{Arg}(x-a_j)$ is the principal value of the argument which is a function from $\overline{W}\setminus\mathfrak{L}_j$ to $(-\pi,\pi)$. Then, $\Theta(x)=\sum_{j=1}^{N}m_j\Theta_j(x)$ is a single-valued function with a jump discontinuity on $\mathfrak{L}$. However, since G is periodic like ($\sharp$), we see that

$$g(x)=\begin{cases}G(f(x)-\Theta(x))+\Theta(x) & \text{if } x\in\overline{W}\setminus\mathfrak{L},\\ \lim\limits_{\mathfrak{L}\not\ni y\to x}\{G(f(y)-\Theta(y))+\Theta(y)\} & \text{if } x\in\mathfrak{L}\end{cases}$$

is continuous on $\mathfrak{L}$.

We must discuss here about the constraction of an initial function u_0 for a given Γ_0, which gives us the existence result on the growth of Γ_t. The auther hopes it will be stated in a forthcoming paper.

This research was started by Maki Nakagawa as the master's thesis [3] in a simple case, which is supervised by the author. After that, Takeshi Ohtsuka and the author have revised and completed it.

References

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