数理解析研究所講究録
第 1547 巻 2007 年 94-99

Perron Type Theorems for Nonlinear Functional Difference Equations

大阪府立大学工学部 松井和幸 (Kazuyuki Matsui)
大阪府立大学工学部 松永秀章 (Hideaki Matsunaga)
Department of Mathematical Sciences
Osaka Prefecture University

1 Introduction

Denote by $\mathbf{Z}$, $\mathbf{Z}^+$ and $\mathbf{Z}^-$ the set of all integers, the set of all nonnegative integers and the set of all nonpositive integers, respectively. Let $\mathbf{C}^k$ be the k-dimensional complex Euclidean space with any convenient norm $|\cdot|$ and $\mathcal{B}^\gamma$ be the Banach space defined by

$$\mathcal{B}^\gamma := \{\phi : \mathbf{Z}^- \to \mathbf{C}^k \mid \sup_{s\in\mathbf{Z}^-} |\phi(s)|\gamma^s < \infty\}$$

equipped with the norm $\|\phi\|_{\mathcal{B}^\gamma} = \sup_{s\in\mathbf{Z}^-} |\phi(s)|\gamma^s$, where γ is a positive constant with $\gamma \geq 1$. For any function $x : (-\infty, m] \to \mathbf{C}^k$ and any $n \in \mathbf{Z}$ with $n \leq m$, we define a function $x_n : \mathbf{Z}^- \to \mathbf{C}^k$ by $x_n(s) = x(n+s)$ for $s \in \mathbf{Z}^-$.

In this paper we are concerned with the nonlinear functional difference equation

$$x(n+1) = L(x_n) + f(n, x_n) \tag{1.1}$$

as a perturbation of the linear autonomous equation

$$x(n+1) = L(x_n), \tag{1.2}$$

where $L : \mathcal{B}^\gamma \to \mathbf{C}^k$ is a bounded linear operator and $f : \mathbf{Z}^+ \times \mathcal{B}^\gamma \to \mathbf{C}^k$ is continuously Frechét differentiable with respect to the second variable. The purpose of this paper is to present some results on the asymptotic behavior of solutions of Eq. (1.1), which correspond to the following Perron type theorems (see, e.g., [2, Chapter 8]) for *ordinary* difference equations.

For the nonlinear difference equation

$$y(n+1) = Ay(n) + g(n, y(n)), \tag{1.3}$$

where A is a $k \times k$ complex matrix and $g : \mathbf{Z}^+ \times \mathbf{C}^k \to \mathbf{C}^k$ is a continuous function, Coffman [1] established the following result provided that the nonlinear term $g(n, y(n))$ is small as $n \to \infty$.

Theorem A [1, Theorem 5.1]. *Suppose that*

$$\frac{|g(n,y)|}{|y|} \to 0 \quad as \ (n,y) \to (\infty, 0). \tag{1.4}$$

If y is a solution of Eq. (1.3) *such that $y(n) \neq 0$ for all large n and $y(n) \to 0$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \sqrt[n]{|y(n)|} = |\lambda_j|,$$

where λ_j is an eigenvalue of the matrix A. Moreover, $|\lambda_j| \leq 1$.

In a recent paper [5], Pituk gave the following result for the linear difference equation

$$y(n+1) = [A + B(n)]y(n), \tag{1.5}$$

where A and $B(n)$ are $k \times k$ complex matrices.

Theorem B [5, Theorem 1]. *Suppose $\|B(n)\|$, the norm of the matrix $B(n)$, satisfies*

$$\lim_{n\to\infty} \|B(n)\| = 0. \tag{1.6}$$

If y is a solution of Eq. (1.5), *then either $y(n) = 0$ for all large n or*

$$\lim_{n\to\infty} \sqrt[n]{|y(n)|} = |\lambda_j|,$$

where λ_j is an eigenvalue of the matrix A.

By an argument similar to the direct proof of Theorem B in [5], one can easily obtain the following result for Eq. (1.3) under the condition

$$|g(n,y)| \leq \beta(n)|y| \quad \text{for} \ (n,y) \in \mathbf{Z}^+ \times \mathbf{C}^k, \tag{1.7}$$

where $\beta : \mathbf{Z}^+ \to [0, \infty)$ is a function satisfying

$$\lim_{n\to\infty} \beta(n) = 0, \tag{1.8}$$

instead of (1.4).

Theorem C. *Suppose* (1.7) *and* (1.8) *hold. If y is a solution of Eq.* (1.3), *then either $y(n) = 0$ for all large n or*

$$\lim_{n\to\infty} \sqrt[n]{|y(n)|} = |\lambda_j|,$$

where λ_j is an eigenvalue of the matrix A.

Only recently, the second author and Murakami [3] discussed the asymptotic behavior of solutions of the linear functional difference equation

$$x(n+1) = L(x_n) + G(n)x_n, \tag{1.9}$$

where L, $G(n) : \mathcal{B}^\gamma \to \mathbf{C}^k$ are bounded linear operators. In order to state their results, we need the characteristic matrix and the characteristic equation of Eq. (1.2) defined by

$$\Delta(z) := zI - L(\omega_z I), \qquad |z| > \frac{1}{\gamma},$$
$$\det \Delta(z) = \det\big(zI - L(\omega_z I)\big) = 0, \qquad |z| > \frac{1}{\gamma},$$

respectively, where I is the $k \times k$ identity matrix and ω_z is defined as $\omega_z(s) = z^s$, $s \in \mathbf{Z}^-$.

Theorem D [3, Theorem 2.1]. *Suppose* $\|G(n)\|$, *the operator norm of* $G(n)$, *satisfies*

$$\lim_{n\to\infty} \|G(n)\| = 0. \tag{1.10}$$

If x is a solution of Eq. (1.9), *then either*

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} \le \frac{1}{\gamma}$$

or

$$\lim_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} = |\lambda|,$$

where λ is a root of $\det \Delta(\lambda) = 0$ *with* $|\lambda| > 1/\gamma$.

Theorem E [3, Theorem 2.2]. *Suppose* (1.10) *holds. If x is a solution of Eq.* (1.9), *then either*

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} \le \frac{1}{\gamma}$$

or

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} = |\lambda|,$$

where λ is a root of $\det \Delta(\lambda) = 0$ *with* $|\lambda| > 1/\gamma$.

Note that Theorems D and E correspond to Theorem B in some sense. Our goal is to extend Theorem C to Eq. (1.1).

2 Preliminaries

We consider the nonhomogeneous functional difference equation

$$x(n+1) = L(x_n) + p(n), \tag{2.1}$$

where $L : \mathcal{B}^\gamma \to \mathbf{C}^k$ is a bounded linear operator and $p : \mathbf{Z} \to \mathbf{C}^k$. For any $(\tau, \phi) \in \mathbf{Z} \times \mathcal{B}^\gamma$, there exists a unique function $x : \mathbf{Z} \to \mathbf{C}^k$ such that $x(\tau + s) = \phi(s)$ for any $s \in \mathbf{Z}^-$ and x satisfies Eq. (2.1) for all $n \geq \tau$. The function x is called a solution of Eq. (2.1) through (τ, ϕ) and is denoted by $x(\cdot, \tau, \phi; p)$. For any $n \in \mathbf{Z}^+$, we define an operator $T(n)$ on $\mathcal{B}^\gamma$ by

$$[T(n)\phi](s) = x(n+s, 0, \phi; 0) \quad \text{for } \phi \in \mathcal{B}^\gamma, \ s \in \mathbf{Z}^-.$$

$T(n)$ is called the solution operator of the homogeneous difference equation (1.2). One can easily see that the operator $T(n)$ is bounded and linear, and it satisfies the following semigroup property:

$$T(n)T(m) = T(n+m) \quad \text{for } n, m \in \mathbf{Z}^+.$$

Therefore, we obtain the relation $T(n) = T^n$ for $n \in \mathbf{Z}^+$ where $T = T(1)$.

Let $\Gamma(s)$, $s \in \mathbf{Z}^-$, be a matrix function defined by

$$\Gamma(s) = \begin{cases} I, & s = 0, \\ O, & s = -1, -2, \dots, \end{cases}$$

where O is the $k \times k$ zero matrix. It can easily be verified that if $y \in \mathbf{C}^k$, then $\Gamma y \in \mathcal{B}^\gamma$ and $\|\Gamma y\|_{\mathcal{B}^\gamma} = |y|$.

The following result plays an important role in this paper, which yields a representation formula for solutions of Eq. (2.1) in the phase space $\mathcal{B}^\gamma$. Hereafter, we use the usual convention

$$\sum_{\tau}^{m} = 0 \quad \text{for } m < \tau.$$

Proposition 2.1 [4, Theorem 2.1]. *Let $(\tau, \phi) \in \mathbf{Z} \times \mathcal{B}^\gamma$ be given. Then the segment $x_n(\tau, \phi; p)$ of solution $x(\cdot, \tau, \phi; p)$ of Eq. (2.1) satisfies the following relation in $\mathcal{B}^\gamma$:*

$$x_n(\tau, \phi; p) = T(n-\tau)\phi + \sum_{s=\tau}^{n-1} T(n-s-1)(\Gamma p(s)) \quad \textit{for } n \geq \tau. \tag{2.2}$$

By repeating almost the same argument as in [4, Lemma 4.2], one can see that any λ belonging to the spectrum $\sigma(T)$ of $T := T(1)$ with $|\lambda| \geq 1/\gamma$ is characterized as a root of $\det \Delta(z) = 0$. Let ρ be any constant satisfying $\rho > 1/\gamma$ and $\det \Delta(z) \neq 0$ for all z with $|z| = \rho$, and consider the set

$$\Sigma_\rho := \{\lambda \in \mathbf{C} \mid \det \Delta(\lambda) = 0, \ \ |\lambda| > \rho\}.$$

Then Σ_ρ is a finite set because Σ_ρ does not intersect with the essential spectrum of T, and therefore, the space $\mathcal{B}^\gamma$ is decomposed as a direct sum

$$\mathcal{B}^\gamma = U \oplus S, \tag{2.3}$$

where $U := U_\rho$ and $S := S_\rho$ are some invariant closed subspaces of $\mathcal{B}^\gamma$ which correspond to Σ_ρ. In the following, we use the notations $T^S \equiv T|_S : S \to S$ and $T^U \equiv T|_U : U \to U$. Note that $\sigma(T^U) = \Sigma_\rho$ and $\sigma(T^S) = \sigma(T) \setminus \Sigma_\rho$.

3 Main Results

The following theorems are our main results.

Theorem 3.1. *Suppose that*

$$|f(n,\phi)| \leq \beta(n)\|\phi\|_{\mathcal{B}^\gamma} \quad for \ \ (n,\phi) \in \mathbf{Z}^+ \times \mathcal{B}^\gamma, \tag{3.1}$$

where $\beta : \mathbf{Z}^+ \to [0,\infty)$ *is a function satisfying*

$$\lim_{n\to\infty} \beta(n) = 0. \tag{3.2}$$

If x is a solution of Eq. (1.1), *then either*

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} \leq \frac{1}{\gamma}$$

or

$$\lim_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} = |\lambda|,$$

where λ is a root of $\det \Delta(\lambda) = 0$ *with* $|\lambda| > 1/\gamma$.

Theorem 3.2. *Suppose* (3.1) *and* (3.2) *hold. If x is a solution of Eq.* (1.1), *then either*

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} \leq \frac{1}{\gamma}$$

or

$$\limsup_{n\to\infty} \sqrt[n]{|x(n)|} = |\lambda|,$$

where λ is a root of $\det \Delta(\lambda) = 0$ *with* $|\lambda| > 1/\gamma$.

Note that Theorem 3.1 is a extension of Theorem C. We also notice that Theorems 3.1 and 3.2 correspond to Theorems D and E, respectively.

The following result plays an essential role in the proof of Theorem 3.1, which coincides with [3, Proposition 4.1].

Proposition 3.1. *Suppose that the conditions* (3.1) *and* (3.2) *hold. Let x be a solution of Eq.* (1.1) *such that*

$$\limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} > \frac{1}{\gamma},$$

and let ρ be any constant satisfying

$$\frac{1}{\gamma} < \rho < \limsup_{n\to\infty} \sqrt[n]{\|x_n\|_{\mathcal{B}^\gamma}} \quad \text{and} \quad \det \Delta(z) \neq 0$$

for all z with $|z| = \rho$. Then

$$\lim_{n\to\infty} \frac{\|\Pi^S x_n\|_{\mathcal{B}^\gamma}}{\|\Pi^U x_n\|_{\mathcal{B}^\gamma}} = 0,$$

where Π^S and Π^U are the projection operators corresponding to the decomposition of $\mathcal{B}^\gamma$.

One can prove Theorems 3.1 and 3.2 by slightly modifying the proof of Theorems D and E, respectively. So we will omit the proof of the above theorems and the proposition.

References

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