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GENERIC CONDITIONS FOR DUCK SOLUTIONS IN R^4

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ABSTRACT. A slow-fast system in R^4 includes a possibility having a constrained surface with a 2-dimensional differentiable manifold. The system in R^4 having such a constrained surface is analyzed in this paper. Although it is difficult to analyze these systems in general, we will give some sufficient conditions to make it possible. Reducing the system to the problem in R^3 with a transversality condition, it is possible to show the existence of the duck solutions by using Benoit's criterion. In the case reducing the system to the problem in R^2 directly with another transversality condition, it can be done up a direct analysis using a 1-dimensional differentiable manifold.

1.INTRODUCTION

A slow fast system in R^4 includes a possibility having a constrained surface with 1-dimensional or 2-dimensional or 3-dimensional differentiable manifold. In this paper, we take up the system in R^4 with a 2-dimensional constrained surface. There are two different approaches, which is an indirect method and the other is a direct one to find the duck solutions in R^4 ([5]). A typical example of this system is a 2-paralleled FitzHugh-Nagumo equations. S.A.Campbell, one of authors of [3], investigated first the coupled FitzHugh-Nagumo equations as a bifurcation problem. In the system, we, I and S.A.Campbell, have already proved the existence of the winding duck solutions in R^4 ([4]). As the associated slow-fast system (or singular perturbation problem) has a 2-dimensional slow manifold (constrained surface), we can reduce it to the slow-fast one in R^3. It turns to have two kinds of projected slow-fast systems in R^3: one has 2-dimensional constrained surface, the other has 1-dimensional constrained surface. Giving transversality conditions in each case, it will be shown that there exists the duck in the original system. Recently, we, I and Miki and Nishino, investigated a trading dynamical economics model using both methods. See ([6]).

2.SLOW-FAST SYSTEM IN R^3

Let us consider the following slow-fast system:

$$\begin{aligned} \epsilon dx/dt &= h(x,y,\epsilon), \\ dy_1/dt &= f_1(x,y,\epsilon), \qquad (2.1) \\ dy_2/dt &= f_2(x,y,\epsilon), \end{aligned}$$

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where $x \in R^1$, $y = (y_1, y_2) \in R^2$, are variables, and ϵ is a parameter, which is infinitesimally small in the sense of non-standard analysis of Nelson. We give the following assumptions in the system(2.1).

$(A1)$ $h \in C^2$, $f = (f_1, f_2) \in C^1$ are defined on $R^3 \times R^1$,

$(A2)$ The set $S_1 = \{(x, y) \in R^3 | h(x, y, 0) = 0\}$ is a 2-dimensional differentiable manifold and the set S_1 intersects the set $T_1 = \{(x, y) \in R^3 | \partial h(x, y, 0)/\partial x = 0\}$ transversely so that the pli set $PL = \{(x, y) \in S_1 \cap T_1\}$ is a 1-dimensional differentiable manifold.

$(A3)$ $f_1(x, y, 0) \neq 0$, or $f_2(x, y, 0) \neq 0$ at any point $(x, y) \in PL$.

Let $(x(t, \epsilon), y(t, \epsilon))$ be a solution of (2.1). When $\epsilon = 0$, differentiating $h(x, y, 0)$ with respect to the time t, the following equation holds:

$$h_{y1}(x, y, 0) f_1(x, y, 0) + h_{y2}(x, y, 0) f_2(x, y, 0) + h_x(x, y, 0) dx/dt = 0, \tag{2.2}$$

where $h_i(x, y_1, y_2, 0) = \partial h(x, y_1, y_2, 0)/\partial i$, $i = x, y_1, y_2$. The above system(2.1) restricted in S_1 becomes the following system:

$$\begin{aligned} dy_1/dt &= f_1(x, y, 0), \\ dy_2/dt &= f_2(x, y, 0), \\ dx/dt &= -\{h_{y1}(x, y, 0) f_1(x, y, 0) + h_{y2}(x, y, 0) f_2(x, y, 0)\}/h_x(x, y, 0), \end{aligned} \tag{2.3}$$

where $(x, y) \in S_1 \setminus PL$. The system (2.1) coincides with the system (2.3) at any point $p \in S_1 \setminus PL$. In order to avoid the degeneracy of the system (2.3), let us consider the following system:

$$\begin{aligned} dy_1/dt &= -h_x(x, y, 0) f_1(x, y, 0), \\ dy_2/dt &= -h_x(x, y, 0) f_2(x, y, 0), \\ dx/dt &= h_{y1}(x, y, 0) f_1(x, y, 0) + h_{y2}(x, y, 0) f_2(x, y, 0). \end{aligned} \tag{2.4}$$

As the system(2.4) is well defined at any point of R^3, it is well defined indeed at any point of PL. The solutions of the system(2.4) coincide with those of the system(2.3) on $S_1 \setminus PL$ except the velocity when they start from the same initial points.

$(A4)$ For any point $(x, y) \in S_1$, either of the following holds;

$$h_{y1}(x, y, 0) \neq 0, h_{y2}(x, y, 0) \neq 0, \tag{2.5}$$

that is, the surface S_1 can be expressed as $y_1 = \varphi_1(x, y_2)$ or $y_2 = \varphi_2(x, y_1)$ in the neighborhood of PL. Let $y_2 = \varphi_2(x, y_1)$ exist, then the projected system(2.6) is obtained:

$$\begin{aligned} dy_1/dt &= -h_x(x, y_1, \varphi_2(x, y_1), 0) f_1(x, y_1, \varphi_2(x, y_1), 0), \\ dx/dt &= h_{y1}(x, y_1, \varphi_2(x, y_1), 0) f_1(x, y_1, \varphi_2(x, y_1), 0) + \\ & h_{y2}(x, y_1, \varphi_2(x, y_1), 0) f_2(x, y_1, \varphi_2(x, y_1), 0). \end{aligned} \tag{2.6}$$

If we take $y_1 = \varphi_1(x, y_2)$, it can be analyzed as the same way.

$(A5)$ All the singular points of the system(2.6) are nondegenerate, that is, the matrix induced from the linearized system of (2.6) at a singular point has two nonzero eigenvalues.

Remark. All these points are contained in the set $PS = \{(x,y) \in PL | dx/dt = 0\}$, which is called *pseudo singular points.* Note that these points are the singular points in the system(2.4).

Definition2.1. Let $p \in PS$ and μ_1, μ_2 be two eigenvalues of the matrix associated with the linearized system of (2.6) at p. The point p is called *pseudo singular saddle* if $\mu_1 < 0 < \mu_2$ and called *pseudo singular node* if $\mu_1 < \mu_2 < 0$ or $\mu_1 > \mu_2 > 0$.

Definition2.2. A solution $(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon))$ of the systems(2.1) are called *ducks*, if there exist standard $t_1 < t_0 < t_2$ such that

(1) ${}^*(x(t_0,\epsilon), y(t_0,\epsilon), z(t_0,\epsilon)) \in S_1$, where the set${}^*(X)$ denotes the standard part of the set$X$,

(2) for $t \in (t_1, t_0)$ the segment of the trajectory $(x(t,\epsilon), y(t,\epsilon), z(t,\epsilon))$ is infinitesimally close to the attracting part of the slow curves (the constrained surface),

(3) for $t \in (t_0, t_2)$, it is infinitesimally close to the repelling part of the slow curves, and

(4) the attracting and repelling parts of the trajectory are not infinitesimally small.

Theorem2.1(Benoit). If the system has a pseudo singular saddle or node point with no resonance, then it has duck solutions.

3.Slow-fast system in R^4

Now, let us consider a slow-fast system(3.1):

$$\begin{aligned} \epsilon dx_1/dt &= h_1(x_1,x_2,y_1,y_2,\epsilon), \\ \epsilon dx_2/dt &= h_2(x_1,x_2,y_1,y_2,\epsilon), \\ dy_1/dt &= f_1(x_1,x_2,y_1,y_2,\epsilon), \\ dy_2/dt &= f_2(x_1,x_2,y_1,y_2,\epsilon), \end{aligned} \tag{3.1}$$

where $f = (f_1, f_2)$ and $h = (h_1, h_2)$ are defined on $R^4 \times R^1$ and ϵ is infinitesimally small.

We assume that the system(3.1) satisfies the following generic conditions $(B1)-(B5)$:

$(B1)$ f is of class C^1 and h is of class C^2.

$(B2)$ The set $S_2 = \{(x,y) \in R^4 | h(x,y,0) = 0\}$ is a 2-dimensional differentiable manifold and the set S_2 intersects the set $T_2 = \{(x,y) \in R^4 | det[\partial h(x,y,0)/\partial x] = 0\}$ transversely so that the generalized pli set $GPL = \{(x,y) \in S_2 \cap T_2\}$ is a 1-dimensional differentiable manifold.

$(B3)$ The value of f is nonzero at any point $p \in GPL$.

$(B4)$ For any $(x,y) \in S_2$, $rank[\partial h(x,y,0)/\partial x] = 2$ and $rank[\partial h(x,y,0)/\partial y] = 2$. Then the surface S_2 can be expressed as $y = \varphi(x)$ or $x = \psi(y)$ in the neighborhood of GPL. Note that we use the notations $x = (x_1,x_2)$, $y = (y_1,y_2)$ and $\varphi(x) = (\varphi_1(x), \varphi_2(x))$, and $\psi(y) = (\psi_1(y), \psi_2(y))$.

Let the latter of $(B4)$ be satisfied, then the following two projected systems (3.2), (3.3) in R^3 can be reduced under the condition $|dx_1/dt - dx_2/dt|$ is limited, that is, $\epsilon|dx_1/dt - dx_2/dt|$ tends to zero as ϵ tends to zero:

$$\begin{aligned} \epsilon dx_1/dt &= h_1(x_1, \psi_2(y), y_1, y_2, \epsilon), \\ dy_1/dt &= f_1(x_1, \psi_2(y), y_1, y_2, \epsilon), \\ dy_2/dt &= f_2(x_1, \psi_2(y), y_1, y_2, \epsilon), \end{aligned} \tag{3.2}$$

since the relation $x_2 = \psi_2(y)$ is established from the above assumption. To analyze the vector field of the system(3.2) on the constrained surface, we use $h_2(x_1, x_2, y_1, y_2, 0)$ instead of $h_1(x_1, \psi_2(y), y_1, y_2, 0)$, if x_2 is a functional of y in h_1. Because, it is complicated to analyze the system as is using h_1. Actually, we need the above condition in such a case. Therefore, this approach is called *an indirect method.*

Using the other relation $x_1 = \psi_1(y)$, we can get the following:

$$\begin{aligned} \epsilon dx_2/dt &= h_2(\psi_1(y), x_2, y, \epsilon). \\ dy_1/dt &= f_1(\psi_1(y), x_2, y, \epsilon), \\ dy_2/dt &= f_2(\psi_1(y), x_2, y, \epsilon). \end{aligned} \tag{3.3}$$

On the set S_2, differentiating both sides of $h(x, \varphi(x), 0) = 0$ by x,

$$[h_x] + [h_y]D\varphi = 0, \tag{3.4}$$

where $D\varphi$ is a derivative with respect to x, thus the following (3.5) is established:

$$D\varphi(x) = -[h_y]^{-1}[h_x]. \tag{3.5}$$

On the other hand,

$$dy/dt = D\varphi(x)dx/dt, \tag{3.6}$$

because of $y = \varphi(x)$. We can reduce the slow system to the following:

$$D\varphi(x)dx/dt = f(x, \varphi(x)). \tag{3.7}$$

Using (3.5), the system (3.7) is described by

$$[h_x]dx/dt = -[h_y]f(x, \varphi(x)). \tag{3.8}$$

Put $[h_x] = A$ simply, then

$$dx/dt = -B[h_y]f(x, \varphi(x)), \tag{3.9}$$

where $AB = BA = (detA)I$.

The system(3.9) is the time scaled reduced system projected into R^2. Again, we assume the set $T_2 = \{(x, y) \in R^4 | detA = 0\} \neq \phi$.

($B5$) All the singular points of the system(3.9) are nondegenerate, that is, the matrix induced from the linearized system of (3.9) at a singular point has two nonzero eigenvalues.

Remark. All these points are contained in the set $GPS = \{(x,y) \in GPL | detA = 0\}$, which is called the set of *generalized pseudo singular points.*

As this approach transforms the original system to the time scaled reduced system directly, it is called *a direct method.*

Definition3.1. Let $p \in GPS$ and μ_1, μ_2 be two eigenvalues of the matrix associated with the linearized system of (3.9) at $p \in R^4$. The point p is called *generalized pseudo singular saddle* if $\mu_1 < 0 < \mu_2$ and called *generalized pseudo singular node* if $\mu_1 < \mu_2 < 0$ or $\mu_1 > \mu_2 > 0$.

Definition3.2. If there exists a duck in the both systems (3.2) and (3.3) at the common pseudo singular point in R^4, it is called *a duck in* R^4. If there exists a duck in only one of the above systems, it is called *a partial duck* in R^4.

Theorem3.1. The transversality condition($B2$) is established if and only if the transversality condition($A2$) in Section2 is satisfied in the systems (3.2) and (3.3) at the common pseudo singular point.

Theorem3.2. The system(3.2) or (3.3) have a pseudo singular saddle (or pseudo singular node) point, if the system(3.1) has a generalized pseudo singular saddle (or pseudo singular node) point.

Theorem3.3. If the system(3.1) has a generalized pseudo singular saddle, or singular node point without resonance, the system(3.1) has a partial duck.

(Proof)
Theorem3.2 ensures that there exists the pseudo singular saddle or pseudo singular node in the system(3.2) or (3.3). Then, Theorem2.1 ensures the existence of a duck in these systems.

4. Proofs of Theorem3.1, and Theorem3.2

4.1 Proof of Theorem3.1
Let $\nabla h_i(x,y,0)$ denote a gradient vector of $h_i(x,y,0)$. The transversality between S_2 and T_2 at the generalized pseudo singular point $p = (x1_0, x2_0, y1_0, y2_0) \in R^4$ is checked as follows:

$$\text{(4.1)} \qquad rank \begin{pmatrix} \nabla h_1(p,0) \\ \nabla h_2(p,0) \\ \nabla det[\partial h(p,0)/\partial x] \end{pmatrix} = 3.$$

The transversality between S_1 and T_1 in the system (3.2) and (3.3) are checked as follows. Put

$$\text{(4.2)} \qquad \begin{aligned} g_1(x_1,y_1,y_2) &= h_1(x_1,\psi_2(y),y_1,y_2,0), \\ g_2(x_2,y_1,y_2) &= h_2(\psi_1(y),x_2,y_1,y_2,0), \end{aligned}$$

and then put

$$\text{(4.3)} \qquad \begin{pmatrix} \nabla g_1(p1) \\ \nabla \partial g_1(p1)/\partial x1 \end{pmatrix} = M_{p1},$$

where $p1 = (x1_0, y1_0, y2_0)$,

$$\left(\begin{array}{c}\nabla g_2(p2)\\ \nabla\partial g_2(p2)/\partial x2\end{array}\right) = N_{p2}, \tag{4.4}$$

where $p2 = (x2_0, y1_0, y2_0)$.

As the relation(4.1) is satisfied, $rankM_{p1}{=}rankN_{p2} = 2$ holds. In fact, the gradient vectors in (4.3) and (4.4) are independent, since only the coordinates are changed. Conversely, pulling back the equations(4.3), (4.4) to R^4, that is, embedding the corresponding 2-dimensional manifold into the original R^4, we can confirm that the relation(4.1) holds. In fact, the the second equation in (4.3), (4.4) is equivalent to the third one in (4.1). The proof is complete.

4.2 Proof of Theorem3.2

Let the system(3.1) have a generalized pseudo singular saddle point $p = (x1_0, x2_0, y1_0, y2_0) \in R^4$, that is, the point p is a singular point of the system(3.9). Note that this system is described on the constrained surface. The following slow-fast system describes the current state.

$$\begin{aligned}\epsilon dx_1/dt &= h_1(x_1, x_2, y_1, \varphi_2(x), \epsilon),\\ \epsilon dx_2/dt &= h_2(x_1, x_2, y_1, \varphi_2(x), \epsilon),\\ dy_1/dt &= f_1(x_1, x_2, y_1, \varphi_2(x), \epsilon),\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\epsilon dx_1/dt &= h_1(x_1, x_2, \varphi_1(x), y_2, \epsilon),\\ \epsilon dx_2/dt &= h_2(x_1, x_2, \varphi_1(x), y_2, \epsilon),\\ dy_2/dt &= f_2(x_1, x_2, \varphi_1(x), y_2, \epsilon).\end{aligned} \tag{4.6}$$

The above systems have a 1-dimensional slow manifold in R^3. We can reduce the systems(4.5), (4.6) to the system(2.6) in Section2: the latter(4.5) has the coordinates (x_1, y_1) and the former(4.6) has the coordinates (x_1, y_2). The orbits of the linearized systems(4.5), (4.6) are equivalent to the eigenvectors of the time scaled reduced system in the system(3.2). As the coordinate transformation is always done by using diffeomorphism, the corresponding eigenvalues are invariant in the sense of topological conjugacy. Therefore, the system(3.2) has a pseudo singular saddle point. In the case of the node point, it is useful as the same way. The proof is complete.

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