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# Positive solutions for semilinear elliptic equations involving Dirac measures

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We are concerned with the problem of finding positive solutions with prescribed isolated singularities to semilinear elliptic equations. Choosing a finite set of points $\{a_i\}_{i=1}^m$ in $\mathbf{R}^N$ and a set of positive numbers $\{c_i\}_{i=1}^m$, we consider the existence of positive solutions of the problem

$$-\Delta u+u=u^p+\kappa\sum_{i=1}^m c_i\delta_{a_i}\quad \text{in } \mathcal{D}'(\mathbf{R}^N), \tag{1.1$_\kappa$}$$

with the condition at infinity

$$u(x)\to 0\quad \text{as } |x|\to\infty, \tag{1.2}$$

where $N\geq 3$, $1<p<N/(N-2)$, $\kappa\geq 0$ is a parameter, and $\delta_a$ is the Dirac delta function supported at $a\in\mathbf{R}^N$. We denote the Laplacian on $\mathbf{R}^N$ by $\Delta$ and the class of distributions on $\mathbf{R}^N$ by $\mathcal{D}'(\mathbf{R}^N)$.

We recall some known results concerning the singularities of possible solutions of the equation. Let $\Omega$ be a bounded domain in $\mathbf{R}^N$ containing 0. By the works due to Lions [14] and Brezis and Lions [6], we obtain the following result.

**Theorem A** [14, 6]. *Assume that $u\in C^2(\Omega\setminus\{0\})$ satisfies*

$$-\Delta u+u=u^q\quad \text{in } \Omega\setminus\{0\} \tag{1.3}$$

*with $q>1$ and $u\geq 0$ a.e. in $\Omega$. Then $u\in L^q_{\mathrm{loc}}(\Omega)$ and*

$$-\Delta u+u=u^q+\kappa\delta_0\quad \text{in } \mathcal{D}'(\Omega) \tag{1.4}$$

*for some $\kappa\geq 0$. Furthermore, the following* (i) *and* (ii) *hold.*

(i) *In the case $1<q<N/(N-2)$, if $\kappa=0$ in* (1.4) *then $u\in C^2(\Omega)$, and if $\kappa>0$ then $u$ behaves like a multiple of the fundamental solution $E_0$ for $-\Delta$ in $\mathbf{R}^N$, i.e., $-\Delta E_0=\delta_0$ in $\mathcal{D}'(\mathbf{R}^N)$.*

(ii) *In the case $q \geq N/(N-2)$, there holds $\kappa = 0$ in* (1.4).

For the proof, see Theorem 1 in [6] and Corollary 1, Theorem 2, and Remark 2 in [14].

It should be mentioned that Johnson, Pan, and Yi [13] showed the existence and asymptotic behaviour of singular positive radial solution $u$ of (1.3) with $1 < q < (N+2)/(N-2)$. In particular, they showed that, if $N/(N-2) < q < (N+2)/(N-2)$, there exists a positive solution $u$ of (1.3) satisfying $u(x) \sim c|x|^{-2/(p-1)}$ as $|x| \to 0$ for some constant $c > 0$. Then, in this case, the singularity of $u$ at $x = 0$ exists, but is not visible in the sense of distribution.

In this paper, we investigate the existence of positive solutions with prescribed isolated singularities to the equation in $\mathbf{R}^N$. By (ii) of Theorem A, if $p \geq N/(N-2)$ then $(1.1)_\kappa$ with $\kappa > 0$ has no positive solution $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. Hence, the condition $1 < p < N/(N-2)$ is necessary for the existence of positive solutions $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ of $(1.1)_\kappa$ with $\kappa > 0$.

We review some known results concerning related problems. Lions [14] studied the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = u^p + \kappa\delta_0 & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\Omega$ is a bounded domain in $\mathbf{R}^N$ containing 0 with smooth boundary $\partial\Omega$. It was shown in [14] that there exists $\kappa^* > 0$ such that (1.5) has at least two positive solutions for each $\kappa \in (0, \kappa^*)$ and no such solution for $\kappa > \kappa^*$. Later, Baras and Pierre [4] studied the existence of positive solutions for the problem

$$\begin{cases} -\Delta u = u^p + \kappa\mu & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $\mu$ is a positive bounded Radon measure in $\Omega$. In [4] they showed that (1.6) has at least one positive solution for each sufficiently small $\kappa > 0$ by investigating the corresponding integral equations. See also Roppongi [16]. Amann and Quittner [3] exhibited the existence of $\kappa^* > 0$ such that (1.6) has at least two positive solutions for $0 < \kappa < \kappa^*$ and no solution for $\kappa > \kappa^*$. Bidaut-Veron and Yarur [5] gave the existence results and a priori estimates for (1.6) including the case where $\mu$ is unbounded. In [3], [5], they also consider the problems involving measures as boundary data. We also refer a survey by Veron [19], [20], and the references therein. In [17] the second author studied the existence of positive solutions for the problem

$$-\Delta u + f(u) = \sum_{i=1}^m c_i\delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N)$$

in the cases where $f$ is nonnegative. In [17] he also showed the nonexistence of positive solutions for some $f$ with sign changing.

Concerning nonhomogeneous semilinear elliptic problems of the form

$$-\Delta u + u = u^q + \kappa f(x) \quad \text{in } \mathbf{R}^N$$

with $q > 1$ and $f \in H^{-1}(\mathbf{R}^N)$, we refer to Zhu [21], Deng and Li [10], [11], Cao and Zhou [7], and Hirano [12]. They successfully showed the existence of at least two positive solutions of the problems under suitable conditions. See also [18, 8] for closely related problems.

In order to state our results, we introduce some notations. Let $E_1$ denote the fundamental solution for $-\Delta + I$ in $\mathbf{R}^N$, that is,

$$E_1(x) = E_1(|x|) = \frac{1}{(2\pi)^{N/2}|x|^{(N-2)/2}} K_{(N-2)/2}(|x|) \quad \text{for } x \in \mathbf{R}^N \setminus \{0\},$$

where $K_\nu$ is the modified Bessel function of order $\nu$. We see that $E_1$ has the following properties:

$$E_1(x) \sim \frac{1}{(N-2)N\omega_N|x|^{N-2}} \quad \text{as } |x| \to 0, \quad \text{and}$$
$$E_1(x) \sim c_1|x|^{-(N-1)/2}e^{-|x|} \quad \text{as } |x| \to \infty,$$

where $\omega_N$ denotes the volume of the unit ball in $\mathbf{R}^N$ and $c_1 > 0$ is a constant depends on $N$. In particular, $E_1 \in C^\infty(\mathbf{R}^N \setminus \{0\})$ and $E_1 \in L^r(\mathbf{R}^N)$ for all $1 \le r < N/(N-2)$. Define $f_0$ by

$$f_0(x) = \sum_{i=1}^m c_i E_1(x - a_i).$$

Then $f_0 \in C^\infty(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $f_0 \in L^r(\mathbf{R}^N)$ for all $1 \le r < N/(N-2)$, and $f_0$ satisfies

$$-\Delta f_0 + f_0 = \sum_{i=1}^m c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

In this paper we refer to $u$ as a positive solution of $(1.1)_\kappa$ if $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ satisfies $(1.1)_\kappa$ in the sense of distribution and $u > 0$ a.e. in $\mathbf{R}^N$.

**Proposition 1.1.** *Let $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ be a positive solution of $(1.1)_\kappa$ with $\kappa > 0$. Then $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $u(x) > 0$ for $x \in \mathbf{R}^N \setminus \{a_i\}_{i=1}^m$. Assume, in addition, that (1.2) holds. Then $u \in L^q(\mathbf{R}^N)$ for all $q \in [1, N/(N-2))$ and $u$ satisfies*

$$u = E_1 * [u^p] + \kappa f_0 \quad \text{a.e. in } \mathbf{R}^N \tag{1.7$_\kappa$}$$

*and $u(x) = O(E_1(x))$ as $|x| \to \infty$, where the symbol $*$ denotes the convolution.*

For each $\kappa > 0$, we define $U_j^\kappa$ for $j = 0, 1, 2, \ldots$, inductively, by

$$U_0^\kappa = \kappa f_0 \quad \text{and} \quad U_j^\kappa = E_1 * \left[(U_{j-1}^\kappa)^p\right] + \kappa f_0 \quad \text{for } j = 1, 2, \ldots. \tag{1.8}$$

Take $q_0 \in (p, N/(N-2))$ arbitrarily, and define $\{q_j\}$ by

$$\frac{1}{q_j} = \frac{1}{q_0} - \left(\frac{2}{N} - \frac{p-1}{q_0}\right) j = \frac{1}{q_{j-1}} - \left(\frac{2}{N} - \frac{p-1}{q_0}\right) \quad \text{for } j = 1, 2, \ldots. \tag{1.9}$$

From $p < N/(N-2)$ and $q_0 > p$, it follows that $2/N - (p-1)/q_0 > 0$. Then, by choosing suitable $q_0$ if necessary, there exists an positive integer denoted by $j_0$ satisfying

$$\frac{1}{q_{j_0-1}} > 0 > \frac{1}{q_{j_0}}. \tag{1.10}$$

We use the notation $C_0(\mathbf{R}^N) = \{u \in C(\mathbf{R}^N) : u(x) \to 0 \quad \text{as } |x| \to \infty\}$.

**Proposition 1.2.** *For each $\kappa \in (0, \infty)$, the following* (i) -(iii) *are equivalent to each other :*

(i) $u = w + U^{\kappa}_{j_0} \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ *is a positive solution of* $(1.1)_\kappa$–$(1.2)$;

(ii) $w \in C_0(\mathbf{R}^N)$ *is positive in* $\mathbf{R}^N$ *and satisfies*

$$w = E_1 * \left[(w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p\right] \quad \text{in } \mathbf{R}^N; \tag{1.11$_\kappa$}$$

(iii) $w \in H^1(\mathbf{R}^N)$ *is a weak positive solution of*

$$-\Delta w + w = (w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p \quad \text{in } \mathbf{R}^N, \tag{1.12$_\kappa$}$$

*that is, $w > 0$ a.e. in $\mathbf{R}^N$ and satisfies*

$$\int_{\mathbf{R}^N} (\nabla w \cdot \nabla \psi + w\psi)\, dx = \int_{\mathbf{R}^N} \left((w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p\right) \psi dx \tag{1.13$_\kappa$}$$

*for any $\psi \in H^1(\mathbf{R}^N)$.*

By Proposition 1.2, the problem $(1.1)_\kappa$–$(1.2)$ can be reduced to the problems $(1.11)_\kappa$ in $C_0(\mathbf{R}^N)$ and $(1.12)_\kappa$ in $H^1(\mathbf{R}^N)$. We will investigate the problems $(1.11)_\kappa$ and $(1.12)_\kappa$ by an approach based on adaptation of the methods by [1, 2, 9, 14].

Our main results are stated in the following theorems.

**Theorem 1.** *There exists $\kappa^* \in (0, \infty)$ such that*

(i) *if $0 < \kappa < \kappa^*$ then the problem* $(1.1)_\kappa$–$(1.2)$ *has a positive minimal solution $\underline{u}_\kappa$, that is, $\underline{u}_\kappa \le u$ a.e. in $\mathbf{R}^N$ for any positive solution $u$ of* $(1.1)_\kappa$–$(1.2)$. *Furthermore, if $0 < \kappa < \hat{\kappa} < \kappa^*$ then $\underline{u}_\kappa < \underline{u}_{\hat{\kappa}}$ a.e. in $\mathbf{R}^N$;*

(ii) *if $\kappa > \kappa^*$ then the problem* $(1.1)_\kappa$–$(1.2)$ *has no positive solution.*

**Theorem 2.** *If $\kappa = \kappa^*$ then the problem* $(1.1)_\kappa$–$(1.2)$ *has a unique positive solution.*

**Theorem 3.** *If* $0 < \kappa < \kappa^*$ *then the problem* $(1.1)_\kappa$–$(1.2)$ *has a positive solution* $\overline{u}_\kappa$ *satisfying* $\overline{u}_\kappa > \underline{u}_\kappa$.

Proofs of Theorems 1-3 can be found in [15]. In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of $(1.12)_\kappa$ and $(1.11)_\kappa$, respectively, to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of $(1.12)_\kappa$. We will prove Theorem 3 by employing the variational method with the Mountain Pass Lemma. In the proofs of Theorems 2 and 3, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions play a crucial role.

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