数理解析研究所講究録
第 1547 巻 2007 年 120-124

On the uniqueness of nodal radial solutions of superlinear elliptic equations in a ball

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1. Introduction

We consider the Dirichlet problem

$$(1.1)\qquad \begin{cases} \Delta u+K(|x|)|u|^{p-1}u=0 & \text{in } B, \\ u=0 & \text{on } \partial B, \end{cases}$$

where $B=\{x\in\mathbf{R}^N : |x|<1\}$, $N\geq 3$, $p>1$, $K\in C^2[0,1]$ and $K(r)>0$ for $0\leq r\leq 1$.

In this paper we investigate the numbers of radial solutions $u=u(|x|)$ of (1.1). Let $u(r)$ be a radial solution of (1.1), where $r=|x|$. Then $u(r)$ satisfies the second order ordinary differential equation

$$(1.2)\qquad u''+\frac{N-1}{r}u'+K(r)|u|^{p-1}u=0$$

for $0<r<1$, and the boundary condition

$$(1.3)\qquad u'(0)=u(1)=0.$$

We consider solutions u of problem (1.2)–(1.3) satisfying $u(0)>0$ only, since if u is a solution of (1.2)–(1.3), so is $-u$.

In this paper we study the uniqueness of solutions of the problem (1.2)–(1.3) having exactly $k-1$ zeros in $(0,1)$, where $k\in\mathbf{N}$. Hence we consider the following problem:

$$(1.4)\qquad \begin{cases} u''+\dfrac{N-1}{r}u'+K(r)|u|^{p-1}u=0, \quad 0<r<1, \\ u'(0)=u(1)=0, \quad u(0)>0, \\ u \text{ has exactly } k-1 \text{ zeros in } (0,1). \end{cases}$$

We define the constant λ and the function as follows:

$$\lambda=\frac{(N-2)p-(N+2)}{2};\quad V(r)=\frac{rK'(r)}{K(r)}.$$

It is known (Kusano and M. Naito [3], [4]) that if

$$(1.5)\qquad V(r)\leq\lambda,\quad 0\leq r\leq 1,$$

then problem (1.4) has no solution for every $k \in \mathbf{N}$. Letting $r \to +0$ in (1.5), we have $p \geq (N+2)/(N-2)$. On the other hand, the following existence result for (1.4) has been obtained by Y. Naito [6, Theorem 3]:

Theorem A. *If* $1 < p < (N+2)/(N-2)$, *then, for every* $k \in \mathbf{N}$, *there exists at least one solution of* (1.4).

In the case $K(r) \equiv 1$ for $r \in [0,1]$, we easily see that problem (1.4) has at most one solution, since if u is solution of (1.2), so is $v(r) = \alpha u(\alpha^{(p-1)/2} r)$ for $\alpha > 0$. (See also [1, Lemma 2.3].) Hence from Theorem A it follows that if $1 < p < (N+2)/(N-2)$, then (1.4) has a unique solution for every $k \in \mathbf{N}$. Moreover if $p \geq (N+2)/(N-2)$ and $K(r) \equiv 1$, then (1.5) holds, which implies that (1.4) has no solution for every $k \in \mathbf{N}$.

Ni [7] and Ni and Nussbaum [8] considered the problem

$$(1.6) \qquad \begin{cases} u'' + \dfrac{N-1}{r} u' + f(r,u) = 0, \quad 0 < r < 1, \\ u'(0) = u(1) = 0, \end{cases}$$

and derived the sufficient conditions for the uniqueness of positive solutions of (1.6). Applying the results in [7] and [8, Theorem 2.47], we conclude that problem (1.4) with $k = 1$ has at most one solution if either

$$\frac{(N+3) - p(N-1)}{2} \leq V(r) \leq \frac{(N-1)(p-1)}{2}, \quad 0 < r < 1$$

or

$$(1.7) \qquad (N-2)p - N \leq V(r) \leq (N-2)p + N - 4, \quad 0 < r < 1.$$

In the case $K(r) = r^l$, $l \geq 0$, Nagasaki [5] showed that if $1 < p < (N+2+2l)/(N-2)$, then (1.4) has a unique solution for every $k \in \mathbf{N}$, and that if $p \geq (N+2+2l)/(N-2)$, then (1.4) has no solution for every $k \in \mathbf{N}$. In the case $l > 0$, we have $K(0) = 0$. On the other hand, we assume that $K(0) > 0$ in this paper. Yanagida [10] proved that, for each $k \in \mathbf{N}$, problem (1.4) has at most one solution if $V(r)$ is nonincreasing. We can apply his result whether $K(0) = 0$ or $K(0) > 0$.

However very little is known about the uniqueness of solutions of (1.4) for the case where $V(r)$ is not nonincreasing.

The main result of this paper is as follows.

Theorem 1.1. *Assume that*

$$(1.8) \quad \Big[V(r) - p(N-2) - N + 4\Big]\Big[V(r) - p(N-2) + N\Big] - 2rV'(r) < 0, \quad 0 < r < 1.$$

Then, for each $k \in \mathbf{N}$, *the solution of* (1.4) *exists and is unique.*

Remark 1.1. Letting $r \to +0$ in (1.8), we have $p \leq N/(N-2)$. Hence by Theorem A, we see that if (1.8) holds, then (1.4) has at least one solution for each $k \in \mathbf{N}$.

We have the following corollary of Theorem 1.1.

Corollary 1.1. *Assume that* (1.7) *holds and* $V'(r) > 0$ *for* $0 < r < 1$. *Then, for each* $k \in \mathbf{N}$, *the solution of* (1.4) *exists and is unique.*

Example 1.1. We consider the case where $N = 3$, $p = 2$ and $K(r) = e^{2r}$. Then $V(r) = 2r$ and we can not apply the results in [7], [8] and [10]. On the other hand, since

$$\Big[V(r) - p(N-2) - N + 4\Big]\Big[V(r) - p(N-2) + N\Big] - 2rV'(r) = -4r(1-r) - 1$$

for $0 < r < 1$, Theorem 1.1 implies that the solution of (1.4) exists and is unique.

In Theorem 1.1 we can not remove condition (1.8). Indeed we have the following result.

Theorem 1.2. *Let* $1 < p < (N+2)/(N-2)$. *For each* $k \in \mathbf{N}$, *there exists* $K \in C^\infty[0,1]$ *such that* $K(r) > 0$ *for* $0 \le r \le 1$ *and that* (1.4) *has at least three solutions.*

2. Outline of the proof of Theorem 1.1

In this section we give the outline of the proof of Theorem 1.1. The proof of Theorem 1.1 is based on the shooting method. Namely we consider the solution $u(r,\alpha)$ of (1.2) satisfying the initial condition

$$u(0) = \alpha > 0, \quad u'(0) = 0, \tag{2.1}$$

where $\alpha > 0$ is a parameter. Since $K \in C^2[0,1]$, we see that $u(r,\alpha)$ exists on $[0,1]$ is unique, u, $u' \in C^1([0,1] \times (0,\infty))$ and $u_\alpha(r,\alpha) = \frac{\partial}{\partial \alpha} u(r,\alpha)$ is a solution of the linearized problem

$$\begin{cases} w'' + \dfrac{N-1}{r} w' + pK(r)|u(r,\alpha)|^{p-1} w = 0, \quad r \in (0,1], \\ w(0) = 1, \quad w'(0) = 0. \end{cases} \tag{2.2}$$

(See, for example, [9, §6 and 13].)

We note that $u(r,\alpha)$ and $u'(r,\alpha)$ cannot vanish simultaneously. In fact, if $u(r_0,\alpha) = u'(r_0,\alpha) = 0$ for some $r_0 \in (0,1]$, then, by the uniqueness of the initial value problem, $u(r,\alpha) \equiv 0$ for $r \in (0,1]$, which contradicts (2.1).

We define z_i to be the i-th zero of $u(r,\alpha)$, if such a z_i exist is. Then we easily find that

$$(-1)^i u'(z_i,\alpha) = (-1)^i \frac{d}{dr} u(z_i,\alpha) > 0 \quad \text{for } i = 1, 2, \ldots. \tag{2.3}$$

To prove Theorem 1.1, we need the following lemma.

Lemma 2.1. *Assume that there exists the* k*-th zero* z_k *of* $u(r,\alpha)$ *in* $(0,1]$. *Let* w *be the solution of* (2.2). *If* (1.8) *holds, then* $(-1)^i w(z_i) > 0$ *for* $i = 1, 2, \ldots, k$.

To show Theorem 1.1 we employ the Prüfer transformation for the solution $u(r,\alpha)$ of problem (1.2) and (2.1). For the solution $u(r,\alpha)$ with $\alpha > 0$, we define the functions $\rho(r,\alpha)$ and $\theta(r,\alpha)$ by

$$\begin{cases} u(r,\alpha) = \rho(r,\alpha)\sin\theta(r,\alpha), \\ r^{N-1}u'(r,\alpha) = \rho(r,\alpha)\cos\theta(r,\alpha), \end{cases} \tag{2.4}$$

where $' = d/dr$. Since $u(r,\alpha)$ and $u'(r,\alpha)$ cannot vanish simultaneously, we see that $\rho(r,\alpha)$ and $\theta(r,\alpha)$ are written in the forms

$$\rho(r,\alpha) = \left([u(r,\alpha)]^2 + r^{2(N-1)}[u'(r,\alpha)]^2\right)^{\frac{1}{2}} > 0$$

and

$$\theta(r,\alpha) = \arctan\frac{u(r,\alpha)}{r^{N-1}u'(r,\alpha)},$$

respectively. From u, $u' \in C^1([0,1]\times(0,\infty))$, it follows that ρ, $\theta \in C^1((0,1]\times(0,\infty))$. By a simple calculation we find that

$$\theta'(r,\alpha) = r^{-N+1}\cos^2\theta(r,\alpha) + r^{N-1}K(r)|\rho(r,\alpha)|^{p-1}|\sin\theta(r,\alpha)|^{p+1} > 0 \tag{2.5}$$

for $r \in (0,1]$, which shows that $\theta(r,\alpha)$ is strictly increasing in $r \in (0,1]$ for each fixed $\alpha > 0$. From (2.1) it follows that $\rho(0,\alpha) = \alpha$ and $\theta(0,\alpha) \equiv \pi/2 \pmod{2\pi}$. For simplicity we take $\theta(0,\alpha) = \pi/2$. It is easy to see that $u(r,\alpha)$ is a solution of (1.4) if and only if

$$\theta(1,\alpha) = k\pi. \tag{2.6}$$

Hence the number of solutions of (1.4) is equal to the number of roots $\alpha > 0$ of (2.6).

Lemma 2.2. *Let $k \in \mathrm{N}$ and let $u(r,\alpha_0)$ be a solution of* (1.4) *for some $\alpha_0 > 0$. Suppose that* (1.8) *holds. Then $\theta_\alpha(1,\alpha_0) > 0$.*

Proof. Observe that

$$\theta_\alpha(r,\alpha) = \frac{u_\alpha(r,\alpha)r^{N-1}u'(r,\alpha) - u(r,\alpha)r^{N-1}u'_\alpha(r,\alpha)}{[u(r,\alpha)]^2 + [r^{N-1}u'(r,\alpha)]^2}. \tag{2.7}$$

Since $z_k = 1$ and $u(1,\alpha_0) = 0$, we obtain

$$\theta_\alpha(1,\alpha_0) = \frac{u_\alpha(z_k,\alpha_0)}{u'(z_k,\alpha_0)}.$$

Note that $(-1)^k u'(z_k,\alpha_0) > 0$, by (2.3). From Lemma 2.1 it follows that

$$(-1)^k u_\alpha(z_k,\alpha_0) > 0,$$

which implies that $\theta_\alpha(1,\alpha_0) > 0$. The proof is complete.

Proof of Theorem 1.1. Recalling Remark 1.1, we see that (1.4) has at least one solution. We show that the solution of (1.4) is unique. Assume to the contrary that there exist numbers α_1 and α_2 such that $u(r,\alpha_1)$ and $u(r,\alpha_2)$ are solutions of (1.4) and $0 < \alpha_1 < \alpha_2$. Then $\theta(1,\alpha_1) = \theta(1,\alpha_2) = k\pi$. Lemma 2.2 implies that $\theta_\alpha(1,\alpha_1) > 0$ and $\theta_\alpha(1,\alpha_2) > 0$. Hence we see that $\theta(1,\alpha_0) = k\pi$ and $\theta_\alpha(1,\alpha_0) \le 0$ for some $\alpha_0 \in (\alpha_1,\alpha_2)$. This contradicts Lemma 2.2. Consequently, (1.4) has at most one solution. The proof of Theorem 1.1 is complete.

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