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On eventually uniformly asymptotical stability to finite coverings of periodic points for difference equations

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Abstract

In this paper we discuss the Moroshima's example, which implies a kind of eventually asymptotical stability of solutions for a difference equation $x(n+1) = f(x(n))$ for $n = 0, 1, 2, \cdots$. We define new definitions of eventual stability of periodic points in the meaning of the large in the same way as ones of Lakshmikantham et. al. and Yoshizawa. By applying the Lyapunov's second method we give eventual stability criteria in the large of the difference equation. In order to illustrate our main results on eventual stability an example of a set of 2-periodic points for eventual stability is given with an analytical estimation. Finally we show another criteria to the eventual stability for difference equations. The criteria is corresponding to Yoshizawa's result on the eventual stability of ordinary differential equations.

1 Introduction

In 1977 Morishima[3] gave results on the stability, oscillation and chaos of periodic points concerning the following difference equation.

$$x(n+1) = \frac{A(n)}{A(n)+B(n)} \quad \text{for } n = 0, 1, \cdots \tag{E}$$

and

$$\begin{aligned} A(n) &= \max[\frac{a}{b}x(n) + \{1-(1+a)x(n)\}, 0], \\ B(n) &= \max[(1-x(n))\{\frac{a}{b} - \frac{x(n)(1-(1+a)x(n))}{(1-x(n))^2}\}, 0] \end{aligned}$$

Here a, b are positive parameters. His results[3] with $a = 0.6, b = 1$ were studied concerning the chaos of Eq(E) independently with Li-Yorke[2] in 1975.

Morishima[4] studied the chaotic behavior and the stability of orbits of

$$x(n+1) = f(x(n)), \tag{1.1}$$

where $f : [0,1] \to [0,1]$ is continuous, $x : \mathbf{Z}_+ = \{0, 1, 2, \cdots\} \to [0,1]$ is the price of the commodity and also he discussed some type of stability of periodic points, where the stability is not globally uniformly asymptotically stable but every orbits of (1.1) has unstable properties in the beginning and the stable behavior from some iterations.

In this paper we show results on the globally asymptotical stability for periodic points of (1.1) as well as we discuss the globally eventually asymptotical stability. See Lakshikantham-Leela[1], Yoshizawa[5] concerning the eventual stability for the case of ordinary differential equations.

2 Notations

Consider difference equation (1.1) in $I^m \subset \mathbf{R}^m$ with $I = [0,1]$ and positive integer m. Denote $x(n) = (x_1(n), x_2(n), \cdots, x_m(n))^T$, where T means the transpose, is a relative price vector of $m-$commodities, where $0 \leq x_j(n) \leq 1$ for $j = 1, 2, \cdots, m$ and $\sum_{j=1}^{m} x_j(n) = 1$ for $n \in \mathbf{Z}_+$. See [3, 4] in detail. A function $f : I^m \to I^m$ is continuous.

Let k be a positive integer. Denote a set of $k-$peridic points by $P(k) = \{x^* \in I^m\}$. $x^* \in P(k)$ if and only if $f^i(x^*) \neq f^j(x^*)$ for $1 \leq i \neq j \leq k$ and $f^k(x^*) = x^*$. Denote by $x(n; n_0, x_0)$ a solution of (1.1) for $n \geq n_0$ with $x(n_0; n_0, x_0) = x_0$ satisfying the initial condition $(n_0, x_0) \in \mathbf{Z}_+ \times I^m$. Denote by $\| x \|$ a norm of $x \in \mathbf{R}^m$. For $r > 0$ we denote the following neighborhoods: when a point $x_0 \in I^m, B(x_0, r) = \{x \in I^m : \| x - x_0 \| < r\}$; when a subset $P \subset I^m, S(P, r) = \cup_{x \in P} B(x, r)$.

A set of $k-$periodic points $P(k)$ is called eventually uniformly stable [EV-US] if for each $\varepsilon > 0$ there exist $N_0 \in \mathbf{Z}_+$ and $\delta > 0$ such that for every $x_0 \in S(P(k), \delta)$ and every $n_0 \geq N_0$, it holds that each solution $x(n; n_0, x_0) \in S(P(k), \varepsilon)$ for $n \geq n_0$, *i.e.*,

$$d(x(n; n_0, x_0), P(k)) < \varepsilon.$$

Here a distance between a point $x \in \mathbf{R}^m$ and a subset $S \subset \mathbf{R}^m$ is defined by $d(x, S) = \inf\{\| x - a \| : a \in S\}$. A set of $k-$periodic points $P(k)$ is called eventually uniformly attractive to finite coverings [EV-UA-FC] if each finite covering $\{C_q \subset I^m : \cup_{q=1}^{Q} C_q \supset I^m\}$ and each $\varepsilon > 0$, there exist $N_0 \in \mathbf{Z}_+$ and $T_0 \in \mathbf{Z}_+$ such that for every $1 \leq q \leq Q$, every $x_0 \in C_q$ and every $n_0 \geq N_0$, it holds that every solution $x(n; n_0, x_0) \in S(P(k), \varepsilon)$ for $n \geq n_0 + T_0$, *i.e.*,

$$d(x(n; n_0, x_0), P(k)) < \varepsilon.$$

The set of $k-$periodic points $P(k)$ is called eventually uniformly asymptotically stable to finite coverings [EV-UAS-FC] if $P(k)$ is [EV-US] and [EV-UA-FC].

3 Criterion of Eventual Stability

Assume that Eq(1.1) has a set of k-periodic points

$$P(k) = \{x_1, x_2, \cdots, x_k\}$$

for $k = 1, 2, \cdots$. We show two criterion for eventually uniformly asymptotically stable of $P(k)$ by applying Lyapunov's second method. In case $k = 1, P(1)$ is a set of fixed point.

Let a set of functions denote

$$CIP = \{a : I \to \mathbf{R}_+ \text{ is continuous, strictly increasing and positively definite}\}$$

and $R_+ = [0, \infty)$. Denote $A - B = \{x \in A : x \notin B\}$ for sets $A, B \subset I^m$.

In the following theorem we give eventually uniformly asymptotically stable to finite coverings of $P(k)$.

Theorem. k-periodic points $P(k)$ is eventually uniformly asymptotically stable to finite coverings under that there exists a function $V : \mathbf{Z}_+ \times I^m \to \mathbf{R}_+$ satisfying the following condition (a)-(b).

(a) For any $r > 0$ there exist a nonnegative integer $N_0 \geq 0$ and two functions $a_r, b_r \in CIP$ such that
$$a_r(d(x, P(k))) \leq V(n, x) \leq b_r(d(x, P(k))$$
for any $n \geq N_0$ and any $x \in I^m - S(P(k), r)$.

(b) Let $\Delta V(n, x) = V(n + k, f^k(x)) - V(n, x)$ for $(n, x) \in \mathbf{Z}_+ \times I^m$. For any $r > 0$ there exist a nonnegative integer $N_0 \geq 0$ and a function $c_r \in CIP$ such that
$$\Delta V(n, x) \leq -c_r(d(P(k), x))$$
for any $n \geq N_0$ and any $x \in I^m - S(P(k), r)$.

Outline of Proof At first, we get the following inequalities.

$$\tilde{a_r}(d(x, P(k))) \leq V(n, x) \leq b_r(d(x, P(k))); \tag{3.2}$$
$$\Delta V(n, x) \leq -\tilde{c_r}(d(x, P(k))). \tag{3.3}$$

where $\tilde{a_r}(d) = \min[a_r(d), c_r(d)]$ and $\tilde{c_r}(d) = \frac{1}{2}\tilde{a_r}(d)$ for $d > 0$. For a sufficiently large $\alpha_1 > 0$ and small $\alpha_2 > 0$ and any $p_\omega \in P(k)$ it can be seen that $I^m \subset S(P(k), \alpha_1)$ and that

$$\text{if } x \in B(p_\omega, \alpha_2), \text{ then } f^k(x) \in B(p_\omega, \alpha_1). \tag{3.4}$$

For any $\varepsilon > 0$ define

$$\phi_\omega(\varepsilon) = \inf\{V(n, x) : \varepsilon \leq \| x - p_\omega \| \leq \alpha_1, n \geq n_0\}. \tag{3.5}$$

We get

$$V(n, x) < \phi_\omega(\varepsilon) \text{ for } x \in B(p_\omega, \delta_\omega),\ n_0 \geq N_0. \tag{3.6}$$

Second, it cam be seen that there exist $1 \leq k(1), k(2) \leq k$ and $\delta > 0$ as follows:

$$\begin{aligned} &\exists p_{k(1)} \in P(k), 0 < \exists \delta < \delta_\omega : \forall y \in B(p_{k(1)}, \delta), \forall n_0 \geq N_0; \\ &\forall \ell = 1, 2, \cdots, \exists p_{k(2)} \in P(k) : x(n_0 + \ell k; n_0, y)) \in B(p_{k(2)}, \varepsilon). \end{aligned} \tag{3.7}$$

Hence, $P(k)$ is [EV-US], because for any $0 < \varepsilon < \alpha_2$ there exist a positive $\delta < \min\{\delta_\omega : 1 \leq \omega \leq k\}$ and an integer $N_0 \geq 0$ such that for any $n_0 \geq N_0$ and any $n \geq n_0$ if $x_0 \in S(P(k), \delta)$, then $x(n; n_0, x_0) \in S(P(k), \varepsilon)$.

It can be seen that (1.1) is uniformly bounded as follows:

$$\forall\alpha > 0, \exists\beta(\alpha) > 0 : \forall n_0 \geq 0, \| x(n; n_0, x) \| < \beta(\alpha) \text{ for } \| x \| < \alpha, n \geq n_0. \quad (3.8)$$

Finally, if Eq(1.1) is not [EV-UA-FC], then we lead to a contradiction Therefore $P(k)$ is [EV-UA-FC].

In case where $k = 1$ the above theorem leads to an eventual stability theorem of fixed point for (1.1).

Corollary. Eq(1.1) has a fixed point x^*. The point x^* is eventually uniformly asymptotically stable to finite coverings under that there exists a function $V : \mathbf{Z}_+ \times I^m \to \mathbf{R}_+$ satisfying Condition (a)-(b).

(a) For any $r > 0$ there exist an integer $N_0 \geq 0$ and two functions $a_r, b_r \in CIP$ such that

$$a_r(\| x - x^* \|) \leq V(n, x) \leq b_r(\| x - x^* \|)$$

for any integers $n \geq N_0$ and any initial points $x \in I^m - \{x^*\}$.

(b) Let $\Delta V(n, x) = V(n+1, f(x)) - V(n, x)$ for $(n, x) \in \mathbf{Z}_+ \times I^m$. For any $r > 0$ there exist an integer $N_0 \geq 0$ and a function $c_r \in CIP$ such that

$$\Delta V(n, x) \leq -c_r(\| x - x^* \|)$$

for any $n \geq N_0$ and any $x \in I^m - \{x^*\}$.

4 Illustration of Theorem

We illustrate Theorem in the case $k = 2$ and $P(2) = \{0.5, 0.7\}$ in the space $\mathbf{R}$ with a numerical result. Consider Morishima's example as follows.

$$x(n+1) = f(x(n)) = \frac{A(n)}{A(n) + B(n)}.$$

Here $A(n) = \max[x + bE_1(x(n)), 0], B(n) = \max[1 - x + bE_2(x(n)), 0]$ and $a = 0.6$ and $E_1(x) = -x + \frac{1-x}{a}, E_2(x) = -\frac{xE_1(x)}{1-x}$. See [3] in detail. Then, in $b = 0.6$, we get

$$f(x) = \frac{1.8x^2 - 4.8x + 3}{9.6x^2 - 13.8x + 6}, \quad f'(x) = \frac{21.24x^2 - 36x + 12.6}{(9.6x^2 - 13.8x + 6)^2}.$$

Let

$$V(x) = d(x, P(2)) = \min[|x - 0.5|, |x - 0.7|]$$

for $x \in I$. Let $a_r(d) = b_r(d) = d$ $(d > 0)$ to any $r > 0$. Then $a_r, b_r \in CIP$ and it holds that Condition(a) of Theorem is satisfied. It can be seen that

$$\begin{aligned}
\Delta V(x) &= \min(|f^2(x) - 0.5|, |f^2(x) - 0.7|) - d(x, P(2)) \\
&= \min(|f^2(x) - f^2(0.5)|, |f^2(x) - f^2(0.7)|) - d(x, P(2)) \\
&= \min_{x^*=0.5,0.7} \left| \int_0^1 \frac{df^2}{dx}(x^* + \theta(x - x^*))(x - x^*) d\theta \right| - d(x, P(2))
\end{aligned}$$

$$
\begin{aligned}
&\leq \min_{x^*=0.5,0.7} \max_{x\in I} |\frac{df^2}{dx}(x)||x-x^*| - d(x,P(2)) \\
&= \max_{x\in I} |\frac{df^2}{dx}(x)| d(x,P(2)) - d(x,P(2)) \\
&= (\max_{x\in I} |f'(f(x))f'(x)| - 1) d(x,P(2)).
\end{aligned}
$$

It holds that $\Delta V(x) \leq \max_{x\in I}(f'(f(x))f'(x) - 1)V(x)$.

We shall show that $\Delta V(x) \leq -cV(x)$ for $x \in I$ with a real number $c > 0$. Putting $y(x) = f'(f(x))f'(x) - 1$, when $y(x) < 0$, then there exists a positive number c such that

$$\Delta V(x) \leq -cV(x). \tag{4.9}$$

Putting $C(d) = cd$, we have

$$C \in CIP : \Delta V(x) \leq -C(V(x)).$$

Therefore it holds that Condition (b) of Theorem is satisfied.

Denote

$$p = 21.24x^2 - 36x + 12.6, \quad q = (9.6x^2 - 13.8x + 6)^2,$$

then we have $f' = p/q$ and $\max_{x\in I} |p/q| < 1$. In fact

$$
\begin{aligned}
&\frac{p^2 - q^2}{q^2} \\
&= \frac{(p-q)(p+q)}{q^2} \\
&= [(21.24x^2 - 36x + 12.6) - (9.6x^2 - 13.8x + 6)^2] \\
&\times [21.24x^2 - 36x + 12.6 + (9.6x^2 - 13.8x + 6)^2]/q^2 \\
&= [-92.16x^4 + 264.96x^3 - 284.4x^2 - 118.8x - 23.4] \\
&\times [21.24(x - (1.18)^{-1})^2 + 12.6 - (9/5.09) + (9.6x^2 - 13.8x + 6)^2]/q^2
\end{aligned}
$$

and $12.6 - (9/5.09) > 0$, $264.96x^3 - 284.4x^2 = 264.96x^2(x - 284.4/264.96) < 0$ for $0 \leq x \leq 1$, then we have $|f'(x)| \leq \max_{x\in I} \frac{|p|}{|q|} < 1$. Hence it holds that on $x \in [0,1]$

$$y(x) = f'(f(x))f'(x) - 1 \leq (\max_{x\in I} \frac{|p|}{|q|})^2 - 1 < 0.$$

Since y is continuous and $[0,1]$ is compact, then there exists a positive number c such that $y(x) \leq -c < 0$ on $[0,1]$.

5 Future Study

In this paper we considered a definition of [EV-UAS-FC] (eventually uniformly asymptotic stability to finite coverings) in the same way as theory of ordinary differential equations.

We showed a theorem for [EV-UAS-FC] of difference equation $x(n+1) = f(x(n))$ by Lyapunov's second method but including a computational result and also analytical estimation of ΔV.

Moreover we illustrated the eventual stability theorem by applying it to the Morishima's example.

In [5] Yoshizawa gave an eventual stability theorem where the following ordinary differential equation

$$x' = F(t,x) \text{ for } t \in \mathbf{R}_+ = [0,\infty), x \in \mathbf{R}^m. \qquad (ODE)$$

Here F is continuous on $\mathbf{R}_+ \times \mathbf{R}^m$. Let $N \subset \mathbf{R}^m$ be closed. Moreover it is assumed that the solutions for (ODE) are uniform-bounded and there exists a Lyapunov function $V(t,x) : \mathbf{R}_+ \times \mathbf{R}^m \to \mathbf{R}_+$ which satisfies the following conditions(i) and (ii).

(i) $a(d(x,N)) \leq V(t,x) \leq b(d(x,N), \| x \|)$ for any (t,x), where a is continuous and $a(r) > 0$ for $r > 0$. $b(r, s(r))$ is continuous and increasing in r and $b(r, s(r)) \to 0$ as $r \to 0$ for $s = s(r) \geq 0$, which is dependent on r.

(ii) $V'(t,x) + V^*(t,x) \to 0$, uniformly on $0 < \lambda \leq d(x,N) \leq \mu$, $x \in S_\alpha = \{\| x \| \leq \alpha\}$ for any λ, μ, α as $t \to \infty$. Here

$$V'(t,x) = \limsup_{h \to +0} \frac{V(t+h, x+hF(t,x)) - V(t,x)}{h}$$

and V^* is a continuous function such that $V^*(t,x) \geq W(x)$ for any (t,x), where W is positively definite with respect to N.

Then the set N is an eventually uniform-asymptotically stable set of (ODE) in the large. Moreover if N satisfies $F(t,N) \subset N$ for any t, then N is a uniform-asymptotically stable set in the large.

In the case of the $k-$periodic points $P(k)$ to (1.1) w consider $N = P(k)$. It is expected that $P(k)$ is eventually uniform-asymptotically stable in the large if $\Delta V(n,x) + V^*(n,x) \to 0$ as $n \to \infty$ uniformly on $0 < \lambda \leq d(x, P(k)) \leq \mu$, $x \in S_\alpha = \{\| x \| \leq \alpha\}$ for any λ, μ, α. Moreover $V^*(n,x) \geq W(x)$ for any $n = 0, 1, 2, \cdots$, and $x \in I^m$ and W is continuous and positively definite to N provided with the above condition (i) in [5].

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