数理解析研究所講究録
第 1548 巻 2007 年 53-60

Russian Options with Finite Time Horizon: A Laplace Transform Approach*

北海道大学・経済学研究科 木村 俊一 (Toshikazu Kimura)
Graduate School of Economics and Business Administration
Hokkaido University

1 Introduction

Russian options are *path-dependent* contingent claims that give the holder the right to receive the realized supremum value of the underlying asset prior to his exercise time. The holder can exercise the option at any time, *i.e.*, the option is of American-style. Shepp and Shiryaev [12, 13] introduced the Russian option, assuming no maturity date for the exercise; see Duffie and Harrison [3] for a financial justification of their results. Shepp and Shiryaev showed that there exists an optimal threshold level of the asset price below which it is advantageous to exercise the option, provided that the asset pays dividends. This type of Russian options can be regarded as a special case of American lookback options. More specifically, it is the perpetual fixed-strike lookback call option with null strike price (Pedersen [10]). In common with lookback options, Russian options are *not* genuine option contracts, because they pay the holder the supremum asset price, always finishing in-the-money. This means that high premiums are charged for Russian options in compensation for "reduced regret" [12].

This paper deals with Russian options with *finite time horizon*, *i.e.*, there is a finite expiry or maturity date for the exercise. The holder may exercise the option at any time but during the option's lifetime. Recently, some researchers have contributed theoretical results to the valuation of finite-lived Russian options. Ekström [5] showed the existence and continuity of an *optimal stopping* or *early exercise* boundary for the Russian option. Also, Ekström proved that the option value is given by the solution of a certain boundary value problem, from which he analyzed asymptotic behavior of the optimal stopping boundary near expiration. This free boundary problem was further studied by Duistermaat *et al.* [4] who suggested a numerical algorithm for valuing the Russian option; see Kyprianou and Pistorius [9] for related theoretical work. Peskir [11] proved that the optimal stopping boundary can be characterized by the solution of a nonlinear integral equation arising from the early exercise premium representation.

Except for Duistermaat *et al.* [4], there is no quantitative research of the finite-lived Russian option, which is principally due to the lack of efficient tools for solving the free boundary problem. Duistermaat *et al.* [4] have used the method of randomization of Carr [2] who proposed that the value of an American vanilla option can be approximated by a randomization of the maturity date using an n-stage Erlangian distribution. As $n \to \infty$, it is possible to show convergence to the value of the American option. This idea has its origin in the classic theory of integral transforms, and it goes by the name of the Post-Widder inversion formula [15]; see Abate and Whitt [1, Section 8] for an algorithm based on the Fourier series. Duistermaat *et al.* developed a recursive algorithm for computing the n-th approximations of the value and the early exercise boundary of the finite-lived Russian option. The complexity of their algorithm comes from the expression of the n-stage Erlangian distribution, and it is directly concerned with the implementation and speed of the algorithm. The purpose of this paper is to provide another quantitative method for computing both the option value and the early exercise boundary.

*This paper is an abbreviated version of Kimura [8]. All proofs, remarks and some computational results are omitted due to the page restriction. This research was supported in part by the Grant-in-Aid for Scientific Research (No. 16310104) of the Japan Society for the Promotion of Science in 2004–2008.

2 Basic Framework

2.1 Optimal stopping problem

The setup is the standard Black-Scholes-Merton framework where the price of the underlying asset evolves according to a geometric Brownian motion: Let $(S_t)_{t\geq 0}$ be the price process of the underlying asset, which is defined by

$$S_t = s\exp\left\{(r-\delta-\tfrac{1}{2}\sigma^2)t+\sigma W_t\right\}, \qquad t \geq 0, \tag{2.1}$$

where $S_0 = s > 0$, $r > 0$ is the risk-free rate of interest, $\delta \geq 0$ is the continuous dividend rate, $\sigma > 0$ is the volatility coefficient of the asset price, and $W \equiv (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The filtration $\mathbb{F} \equiv (\mathcal{F}_t)_{t\geq 0}$ is a natural one generated by W and the probability measure $\mathbb{P}$ is chosen so that the stock has mean rate of return r. For the price process $(S_t)_{t\geq 0}$ and a constant $m \geq s$, define the supremum process as

$$M_t = m \vee \sup_{0\leq u\leq t} S_u, \qquad t \geq 0, \tag{2.2}$$

where $a \vee b = \max\{a, b\}$.

Given a finite time horizon $T > 0$, the arbitrage-free value of the Russian option at time $t \in [0, T]$ is given by

$$V(s,m,t) = \operatorname*{ess\,sup}_{0\leq\theta_t\leq T-t} \mathbb{E}_{s,m}[e^{-r\theta_t}M_{\theta_t}], \tag{2.3}$$

where θ_t is a stopping time of the filtration $\mathbb{F}$ and the conditional expectation $\mathbb{E}_{s,m}[\,\cdot\,] \equiv \mathbb{E}[\,\cdot\,|\,\mathcal{F}_0] = \mathbb{E}[\,\cdot\,|\,S_0 = s, M_0 = m]$ is calculated under the risk-neutral probability measure $\mathbb{P}$. The random variable $\theta_t^* \in [0, T-t]$ is called an optimal stopping time if $V(s,m,t) = \mathbb{E}_{s,m}[e^{-r\theta_t^*}M_{\theta_t^*}]$. It is clear from (2.1)–(2.3) that $V(s,m,t) \geq m$, V is nondecreasing in s and m, and V is nonincreasing in t. Ekström [5] proved that the value function $V \equiv V(s,m,t)$ is continuous, *i.e.*, V is uniformly continuous in s, m and t separately. Solving the optimal stopping problem (2.3) is equivalent to finding the points (S_t, M_t, t) for which early exercise before maturity is optimal. Let

$$\mathcal{D} = \{(s,m,t) \in \mathbb{R}_+ \times [s, +\infty) \times [0,T]\}$$

be the whole domain, and $\mathcal{E}$ and $\mathcal{C}$ denote the *exercise region* and *continuation region*, respectively. In terms of the value function $V(s,m,t)$, the continuation region $\mathcal{C}$ is defined by

$$\mathcal{C} = \{(s,m,t);\ V(s,m,t) > m\},$$

which is an open set since V is continuous. The exercise region $\mathcal{E}$ is the complement of $\mathcal{C}$ in $\mathcal{D}$ and the optimal stopping time θ_t^* satisfies

$$\theta_t^* = \inf\{u \in [0, T-t];\ (S_u, M_u, t+u) \in \mathcal{E}\}.$$

Since V is nondecreasing in s, $(s,m,t) \in \mathcal{C}$ implies $(x,m,t) \in \mathcal{C}$ for all x satisfying $s \leq x \leq m$. Hence, there exists a function $\underline{S}(m,t)$ with $0 \leq \underline{S}(m,t) \leq m$ such that

$$\underline{S}(m,t) = \inf\{s \in [0,m];\ (s,m,t) \in \mathcal{C}\}, \tag{2.4}$$

and $(\underline{S}(m,t))_{t\in[0,T]}$ is called the *early exercise boundary*. The boundary function $\underline{S}(m,t)$ is nondecreasing in t since V is nondecreasing in t, and it is continuous in t if $\delta > 0$; see Theorem 2 in Ekström [5]. In terms of the function $\underline{S}(m,t)$, the continuation region $\mathcal{C}$ can be represented as

$$\mathcal{C} = \{(s,m,t);\ \underline{S}(m,t) < s \leq m\}.$$

2.2 Free boundary problem

It has been known that the optimal stopping problem (2.3) of finding the option value V can be deduced to a parabolic free boundary problem (see Theorem 1 in Ekström [5]): The value V of the Russian option with finite time horizon is given by a solution of the PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} + (r-\delta)s\frac{\partial V}{\partial s} - rV = 0, \qquad \underline{S}(m,t) < s \le m, \tag{2.5}$$

together with the boundary conditions

$$\left|\begin{array}{l} \lim\limits_{s \downarrow \underline{S}} V(s,m,t) = m, \\ \lim\limits_{s \downarrow \underline{S}} \dfrac{\partial V}{\partial s} = 0, \\ \lim\limits_{m \downarrow s} \dfrac{\partial V}{\partial m} = 0, \end{array}\right. \tag{2.6}$$

and the terminal condition

$$V(s,m,T) = m. \tag{2.7}$$

The boundary conditions in (2.6) are respectively called the *value matching*, *smooth pasting* and *Neumann* conditions in order.

From (2.3), we see that the value V depends on time only through the time $T-t$ remaining to maturity. For notational convenience, we introduce the time-reversed quantities

$$\widetilde{V}(s,m,\tau) = V(s,m,T-\tau) = V(s,m,t),$$

and

$$\widetilde{\underline{S}}(m,\tau) = \underline{S}(m,T-\tau) = \underline{S}(m,t),$$

with the change of variables $\tau := T-t$. It follows from the definition (2.3) of the value function V that

$$\widetilde{V}(ks,km,\tau) = k\widetilde{V}(s,m,\tau), \tag{2.8}$$

for arbitrary $k \in \mathbb{R}_+$. In particular, if we set $k = m^{-1}$, then

$$\widetilde{V}(s,m,\tau) = m\widetilde{V}(\frac{s}{m},1,\tau),$$

which permits a reduction in the dimensionality of the problem by a similarity variable. That is, we may find a solution of the form

$$\widetilde{V}(s,m,\tau) = mW(\xi,\tau), \tag{2.9}$$

with the change of variables $\xi := s/m$.

Using the relations

$$\begin{aligned} \frac{\partial V}{\partial s} &= \frac{\partial W}{\partial \xi}, & \frac{\partial^2 V}{\partial s^2} &= \frac{1}{m}\frac{\partial^2 W}{\partial \xi^2}, \\ \frac{\partial V}{\partial m} &= W - \xi\frac{\partial W}{\partial \xi}, & \frac{\partial V}{\partial t} &= -m\frac{\partial W}{\partial \tau}, \end{aligned}$$

we can rewrite the PDE (2.5) as

$$-\frac{\partial W}{\partial \tau} + \tfrac{1}{2}\sigma^2\xi^2\frac{\partial^2 W}{\partial \xi^2} + (r-\delta)\xi\frac{\partial W}{\partial \xi} - rW = 0, \qquad \underline{\xi}(\tau) < \xi \le 1, \tag{2.10}$$

where $\underline{\xi}(\tau)$ $(\in [0,1])$ is defined by

$$\underline{\xi}(\tau) = \frac{1}{m}\underline{\widetilde{S}}(m,\tau),$$

being nonincreasing in τ. The boundary conditions for W are given by

$$\left|\begin{array}{l} \lim_{\xi \downarrow \underline{\xi}(\tau)} W(\xi,\tau) = 1, \\ \lim_{\xi \downarrow \underline{\xi}(\tau)} \dfrac{\partial W}{\partial \xi} = 0, \\ \lim_{\xi \uparrow 1} \left(W - \dfrac{\partial W}{\partial \xi}\right) = 0, \end{array}\right. \tag{2.11}$$

and the initial condition is

$$W(\xi,0) = 1. \tag{2.12}$$

3 Valuation with Laplace-Carlson Transforms

For $\lambda > 0$, define the Laplace-Carlson transform (LCT) of the time-reversed quantity $W(\xi,\tau)$ as

$$W^*(\xi,\lambda) = \mathcal{LC}[W(\xi,\tau)](\lambda) \equiv \int_0^\infty \lambda e^{-\lambda\tau} W(\xi,\tau)\mathrm{d}\tau.$$

Similarly, we denote the LCT of $\underline{\xi}(\tau)$ by attaching the asterisk, *i.e.*, $\underline{\xi}^*(\lambda) = \mathcal{LC}[\underline{\xi}(\tau)](\lambda)$. No doubt, there is no essential difference between the LCT and the Laplace transform (LT) defined by

$$\widehat{W}(\xi,\lambda) = \mathcal{L}[W(\xi,\tau)](\lambda) \equiv \int_0^\infty e^{-\lambda\tau} W(\xi,\tau)\mathrm{d}\tau.$$

Obviously, we have $W^*(\xi,\lambda) = \lambda\widehat{W}(\xi,\lambda)$ for $\lambda > 0$. The principal reason why we prefer LCTs to LTs is that LCTs generate relatively simpler formulas than LTs for option pricing problems because constant values are invariant after taking transformation. In the context of option pricing, LCTs have been adopted in the randomization of Carr [2] as an initial approximation.

Let $\widetilde{V}^* \equiv \widetilde{V}^*(s,m,\lambda) = \mathcal{LC}[\widetilde{V}(s,m,\tau)](\lambda)$ be the LCT of the time-reversed value $\widetilde{V}$. From the PDE (2.10) with the conditions (2.11) and (2.12), we obtain a closed-form solution as follows:

Theorem 1 The LCT of the time-reversed value $\widetilde{V}(s,m,\tau)$ of the Russian option with finite time horizon $T < \infty$ is given by

$$\widetilde{V}^*(s,m,\lambda) = \begin{cases} \dfrac{m}{\alpha_2-\alpha_1}\dfrac{r}{\lambda+r}\left\{\alpha_2\left(\dfrac{s}{m\underline{\xi}^*}\right)^{\alpha_1} - \alpha_1\left(\dfrac{s}{m\underline{\xi}^*}\right)^{\alpha_2}\right\} + \dfrac{\lambda m}{\lambda+r}, & m\underline{\xi}^* < s \le m \\ m, & 0 < s \le m\underline{\xi}^*, \end{cases} \tag{3.1}$$

where the parameters $\alpha_1 > 1$ and $\alpha_2 < 0$ are two real roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\alpha^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\alpha - (\lambda + r) = 0, \tag{3.2}$$

and the LCT $\underline{\xi}^* \equiv \underline{\xi}^*(\lambda) \le 1$ is a unique positive solution of the functional equation

$$\frac{\alpha_1(1-\alpha_2)}{\alpha_2(1-\alpha_1)}\left(\underline{\xi}^*\right)^{\alpha_1-\alpha_2} + \frac{\lambda}{r}\frac{\alpha_1-\alpha_2}{\alpha_2(1-\alpha_1)}\left(\underline{\xi}^*\right)^{\alpha_1} = 1. \tag{3.3}$$

Proposition 2 The LCTs of the time-reversed Greeks

$$\Delta^* = \mathcal{LC}\left[\frac{\partial \widetilde{V}}{\partial s}\right], \quad \Gamma^* = \mathcal{LC}\left[\frac{\partial^2 \widetilde{V}}{\partial s^2}\right] \quad \text{and} \quad \Theta^* = \mathcal{LC}\left[\frac{\partial \widetilde{V}}{\partial \tau}\right]$$

for $s \in (m\underline{\xi}^*, m]$ are respectively given by

$$\Delta^* = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\frac{r}{\lambda+r}\frac{m}{s}\left\{\left(\frac{s}{m\underline{\xi}^*}\right)^{\alpha_1} - \left(\frac{s}{m\underline{\xi}^*}\right)^{\alpha_2}\right\},$$

$$\Gamma^* = \frac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}\frac{r}{\lambda+r}\frac{m}{s^2}\left\{(\alpha_1-1)\left(\frac{s}{m\underline{\xi}^*}\right)^{\alpha_1} - (\alpha_2-1)\left(\frac{s}{m\underline{\xi}^*}\right)^{\alpha_2}\right\},$$

$$\Theta^* = \frac{\lambda r m}{\lambda+r}\left[\frac{1}{\alpha_2-\alpha_1}\left\{\alpha_2\left(\frac{s}{m\underline{\xi}^*}\right)^{\alpha_1} - \alpha_1\left(\frac{s}{m\underline{\xi}^*}\right)^{\alpha_2}\right\} - 1\right].$$

Proposition 3 For the early exercise boundary $(\underline{S}(m,t))_{t\in[0,T]}$ of the Russian option with finite time horizon $T < \infty$, we have

$$\lim_{t\to T}\underline{S}(m,t) = m. \tag{3.4}$$

Applying Abelian theorem on the terminal value of LTs to the LCT $\widetilde{V}^*(s,m,\lambda)$, we can obtain the well-known result for the *perpetual* case; see Duffie and Harrison [3] and Shepp and Shiryaev [12]. There exist several different proofs for valuing the perpetual Russian option [3, 7, 9, 10, 12, 13]. To make this paper self-contained, however, we provide the result and a brief proof from the view point of the Laplace transform approach.

Proposition 4 Let $V_\infty(s,m)$ be the value of the perpetual Russian option. For $\delta > 0$, we have

$$V_\infty(s,m) = \begin{cases} \dfrac{m}{\alpha_2^\circ - \alpha_1^\circ}\left\{\alpha_2^\circ\left(\dfrac{s}{m\underline{\xi}_\infty}\right)^{\alpha_1^\circ} - \alpha_1^\circ\left(\dfrac{s}{m\underline{\xi}_\infty}\right)^{\alpha_2^\circ}\right\}, & m\underline{\xi}_\infty < s \le m \\ m, & 0 < s \le m\underline{\xi}_\infty, \end{cases} \tag{3.5}$$

where $\alpha_i^\circ = \lim_{\lambda\to 0}\alpha_i(\lambda)$ $(i = 1,2)$ are two real roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\alpha^2 + (r - \delta - \tfrac{1}{2}\sigma^2)\alpha - r = 0, \tag{3.6}$$

and

$$\underline{\xi}_\infty = \left(\frac{\alpha_2^\circ(1-\alpha_1^\circ)}{\alpha_1^\circ(1-\alpha_2^\circ)}\right)^{\frac{1}{\alpha_1^\circ-\alpha_2^\circ}}. \tag{3.7}$$

It is worthwhile noting here that the expressions for $V_\infty(s,m)$ in (3.5) and $\underline{\xi}_\infty$ in (3.7) of the perpetual Russian option are *symmetric* with respect to the roots α_1° and α_2°. Furthermore, motivated by some observations in numerical experiments, we obtain an interesting symmetric property of the optimal threshold level $\underline{\xi}_\infty$. Symbolic computation with a mathematical software yields

Proposition 5 Denote $\underline{\xi}_\infty \equiv \underline{\xi}_\infty(r,\delta)$ for $r,\delta > 0$. Then, $\underline{\xi}_\infty(r,\delta)$ is the symmetric function of r and δ, *i.e.*,

$$\underline{\xi}_\infty(r,\delta) = \underline{\xi}_\infty(\delta,r). \tag{3.8}$$

4 Computational Results

As shown in the previous section, Laplace transforms are useful to do asymptotic analysis via Abelian theorems. However, the primary value of the transforms is in time-dependent analysis of the original functions via analytical or numerical transform inversion. In particular, numerical inversion is most important when a transform cannot be analytically inverted by manipulating tabled formulas, which is the normal case in option pricing problems. Numerical inversion is also important when a Laplace transform is implicitly defined, *e.g.*, as the solution of a certain functional equation. Actually, this is the case of our problem: To invert the LCTs $\widetilde{V}^*(s,m,\lambda)$ and $\underline{\xi}^*(\lambda)$, we first have to solve the functional equation (3.3) for $\underline{\xi}^*(\lambda)$. Among many numerical methods for Laplace transform inversion, the Gaver-Stehfest method [6, 14] is especially convenient for such implicitly defined Laplace transforms, since it works with the transform evaluated only at real arguments.

Consider the LCT $G^*(\lambda) = \mathcal{LC}[G(\tau)](\lambda)$ for a given function $G(\tau) \in L^1(\mathbb{R}_+)$. Gaver [6] developed an inversion algorithm based on the asymptotic result

$$G(\tau) = \lim_{n\to\infty} G_n(\tau), \qquad \tau \geq 0$$

where $G_n(\tau) \equiv G_n^{(n)}(\tau)$ $(n \geq 1)$ is defined by using a sequence $\{G_n^{(m)}(\tau);\ n, m \geq 1\}$ generated by the recursion

$$\begin{cases} G_0^{(m)}(\tau) = G^*\left(m\dfrac{\log 2}{\tau}\right), & n = 0 \\ G_n^{(m)}(\tau) = \left(1+\dfrac{m}{n}\right)G_{n-1}^{(m)}(\tau) - \dfrac{m}{n}G_{n-1}^{(m+1)}(\tau), & n \geq 1. \end{cases} \tag{4.1}$$

To accelerate the convergence of $(G_n(\tau))_{n\geq 1}$ to $G(\tau)$, Stehfest [14] proposed an extrapolation formula

$$\overline{G}_n(\tau) = \sum_{k=1}^{n} \frac{(-1)^{(n-k)}k^n}{k!(n-k)!} G_k(\tau), \tag{4.2}$$

which has been known under an alias of the n-point Richardson extrapolation scheme in the context of option pricing. The procedure for generating the n-th approximation $\overline{G}_n(\tau)$ is called the Gaver-Stehfest method; see Abate and Whitt [1] for details. To compute the root $\underline{\xi}^*(\lambda) \in [0,1]$ of the functional equation (3.3) for a given $\lambda > 0$, we simply use the Newton method. This is due to the existence and uniqueness of the root in the interval $[0,1]$.

From a financial point of view, the no-dividend case $\delta = 0$ is the most interesting one for the Russian option with finite time horizon, because we have to require the condition $\delta > 0$ when we deal with the perpetual Russian option. Tables 1 and 2 show the normalized option value $\widetilde{V}(s,m,\tau)/m$ for some cases with and without dividends, respectively. The initial value of the Newton method is fixed to 1 and the 4-point extrapolation is adopted in our inversion algorithm. We see from these tables that the premiums of Russian options with short maturity are not so expensive especially for $s/m < 1$, which implies that the (normalized) guaranteed discounted value $e^{-r\tau}$ is dominant in the option value for those cases. For cases with $\delta = 0$ and long maturity, the premiums are extremely high such that the commercial value of Russian options is doubtful. From these observations, we may say that the Russian option is intrinsically valuable when the maturity T is relatively short.

Figures 1(a) and 1(b) illustrate some curves of the normalized early exercise boundary $\underline{\xi}(t) = \underline{S}(m, T-t)/m$ of the Russian option with finite horizon $T = 10$ as functions of $t \in [0,10]$, where dashed lines represent the optimal threshold levels $\underline{\xi}_\infty$ for the associated perpetual cases. The effect of the interest rate r can be shown in Figure 1(a) and the dividend yield δ in Figure 1(b).

Table 1: Option values $\widetilde{V}(s,m,\tau)/m$ with dividends ($r = 0.05, \delta = 0.03$)

σ	s/m	$\tau = 1$	$\tau = .5$	$\tau = 10$	$\tau = \infty$
0.2	1.0	1.1308	1.2244	1.2600	1.2910
	0.9	1.0403	1.1150	1.1453	1.1723
	0.8	1.0008	1.0378	1.0571	1.0781
0.3	1.0	1.2188	1.4228	1.5273	1.6904
	0.9	1.1125	1.2890	1.3816	1.5273
	0.8	1.0426	1.1741	1.2517	1.3775
0.4	1.0	1.3130	1.6535	1.8572	2.3065
	0.9	1.1940	1.4950	1.6771	2.0803
	0.8	1.1014	1.3514	1.5049	1.8644

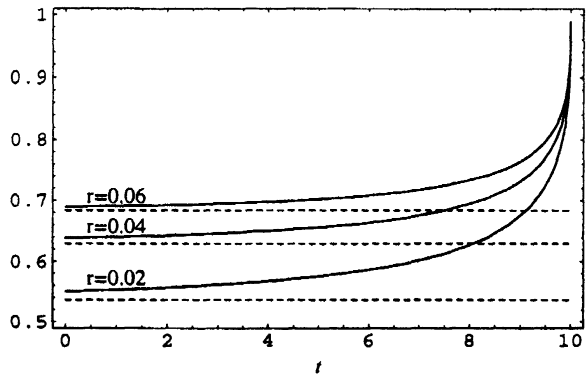
Table 2: Option values $\widetilde{V}(s,m,\tau)/m$ with no dividends ($r = 0.05, \delta = 0$)

σ	s/m	$\tau = 1$	$\tau = 5$	$\tau = 10$	$\tau = 100$
0.2	1.0	1.1475	1.3066	1.4144	2.0519
	0.9	1.0518	1.1835	1.2780	1.8478
	0.8	1.0061	1.0800	1.1545	1.6468
0.3	1.0	1.2372	1.5256	1.7391	3.2714
	0.9	1.1274	1.3793	1.5696	2.9456
	0.8	1.0508	1.2472	1.4101	2.6229
0.4	1.0	1.3329	1.7766	2.1287	5.0986
	0.9	1.2110	1.6045	1.9199	4.5903
	0.8	1.1138	1.4446	1.7202	4.0855

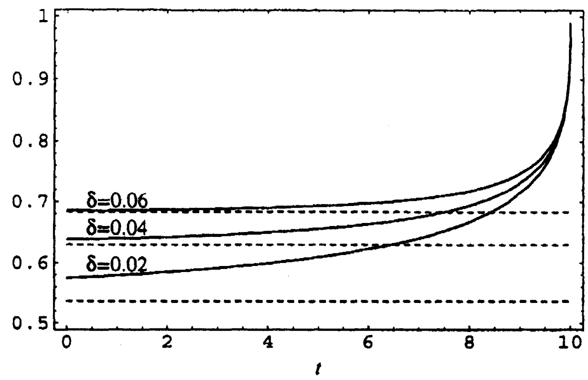
In these figures, we can see that each curve of the boundaries reaches the value 1 at maturity, which is consistent with Proposition 3. The algorithm works well even near expiration, depicting rapidly increasing curves as $t \to T$. Note that Figures 1(a) and 1(b) provide a numerical check for the symmetry relation proved in Proposition 5. All of the figures indicate a general property that the lower the threshold level $\underline{\xi}_\infty$, the slower convergence of $\underline{\xi}(\tau)$ as $\tau \to \infty$.

5 Conclusion

In this paper, we analyzed the Russian option with finite time horizon via the Laplace transform approach to obtain the LCTs of the option value, the early exercise boundary and some hedging parameters, all of which can be expressed in terms of the unique real root of a functional equation. Our numerical analysis showed that the accuracy of this root plays an important role in numerical inversion of Laplace transforms with the Gaver-Stehfest method that requires more than 20-digits precision. Although the Gaver-Stehfest method generates sufficiently accurate solutions for almost all cases as shown in Section 4, the solutions sometimes behave unstably for the situations where $V(s,m,t) \approx m$, typically occurred when $t \to T$ or $s \to \underline{S}(m,t)$. Removing this instability especially around the smooth-pasting point is an important problem to be solved as future work.



(a) $r = 0.02, 0.04, 0.06$, $\delta = 0.04$



(b) $r = 0.04$, $\delta = 0.02, 0.04, 0.06$

Figure 1: Early exercise boundaries $\underline{S}(m, T-t)/m$ $(T = 10, \sigma = 0.2)$

The Laplace transform approach is so general that it could be applied to other American-style path-dependent options whose payoff functions are sufficiently smooth with respect to state variables, *e.g.*, lookback, barrier, exchange and so on. Also, the approach could be extended to the cases that the underlying asset price has jumps and that it is discretely monitored. These extensions still remain as future work.

References

[1] Abate, J. and W. Whitt, "The Fourier-series method for inverting transforms of probability distributions," *Queueing Systems*, **10** (1992) 5–88.

[2] Carr, P., "Randomization and the American put," *Review of Financial Studies*, **11** (1998) 597–626.

[3] Duffie, J.D. and J.M. Harrison, "Arbitrage pricing of Russian options and perpetual lookback options," *Annals of Applied Probability*, **3** (1993) 641–651.

[4] Duistermaat, J.J., A.E. Kyprianou and K. van Schaik, "Finite expiry Russian options," Working Paper, Utrecht University, 2003.

[5] Ekström, E., "Russian options with a finite time horizon," *Journal of Applied Probability*, **41** (2004) 313–326.

[6] Gaver, D.P., "Observing stochastic processes and approximate transform inversion," *Operations Research*, **14** (1966) 444–459.

[7] Graversen, S.E. and G. Peškir, "On the Russian option: The expected waiting time," *Theory of Probability and their Applications*, **42** (1998) 416–425.

[8] Kimura, T., "Valuing Russian options with finite time horizon via Laplace transforms," Discussion Paper Series *A*, No. 2006-171, Graduate School of Economics and Business Administration, Hokkaido University, 2006.

[9] Kyprianou, A.E. and M.R. Pistorius, "Perpetual options and Canadization through fluctuation theory," *Annals of Applied Probability*, **13** (2003) 1077–1098.

[10] Pedersen, J.L., "Discounted optimal stopping problems for the maximum process," *Journal of Applied Probability*, **37** (2000) 972–983.

[11] Peskir, G., "The Russian option: Finite horizon," *Finance and Stochastics*, **9** (2005) 251–267.

[12] Shepp, L. and A.N. Shiryaev, "The Russian options: Reduced regret," *Annals of Applied Probability*, **3** (1993) 631–640.

[13] Shepp, L. and A.N. Shiryaev, "A new look at the 'Russian option'," *Theory of Probability and their Applications*, **39** (1994) 103–119.

[14] Stehfest, H., "Algorithm 368: Numerical inversion of Laplace transforms," *Communications of the ACM*, **13** (1970) 47–49.

[15] Widder, D.V., *The Laplace Transform*, Princeton University Press, Princeton, N.J., 1946.