Topics from Competitive Game Theory

Some of the recent works by the present author are given in the two parts A and B. A lot of interesting open problems are mentioned.

[A] 2- AND 3- PLAYER GAMES OF SCORE SHOWDOWN

1. 3-Player Games of Score Showdown Let $X_{ij}(i=1,2,3; j=1,2)$ be the $j$-th r.v. observed by Player $i$. Assume that $\{X_{ij}\}$ are i.i.d. with $U(0,1)$ distribution. Each player $i$ first observes $X_{il}=X_{i1}$ and chooses $A/R$. If $A(R)$ is chosen, the $X_{il}$ is accepted (rejected and the second r.v $X_{il}$ is observed). Player $i$’s score is

$$S_i(X_{1i},X_{2i}) = \begin{cases} x_{i1} & \text{if } X_{i1} = x_{i1} \\
\phi(x_{i1}, x_{i2}) & \text{if } X_{i1} \neq x_{i1}, \text{rejected, and } X_{i2} \text{is observed.} \end{cases}$$

We consider the cases

$$\phi(x_{i1}, x_{i2}) = x_{i2} - x_{i1}I(x_{i2} \geq x_{i1}) - (x_{i1}-x_{i2})I(x_{i1}+x_{i2} \geq 1) - \frac{1}{2}(x_{i1}+x_{i2})$$

(Keep-or-Exchange, Risky-Exchange, Showcase-Showdown, Competing Average, resp)

Player who gets the highest score is the winner. Each player wants $\Pr(he wins) \rightarrow \max$

Also we consider the two versions of the games;

Simultaneous-move version — Players are made indep. and are not known to his rivals.

Sequential-move version — Players move sequentially, i.e., I at first, II at second, and III at third. After I’s move, II and III are informed of $X_{il} = x_{i1}$ and I’s choice of $A/R$. After II’s move, III is informed of $X_{il} = x_{i2}$ and II’s choice of $A/R$. All players are intelligent, and each player should prepare for that any subsequent player must employ their optimal strategies.

In the present article, we use $\Gamma$= game, $C$= common, $EQ$= equilibrium, $S$= strategy, $V$= value, $D$= draw. Players 1, 2, 3 are sometimes written by I, II, III, resp.

2. Solutions to the Games. It is shown that games of $\Gamma^{(2)}SS$ and $\Gamma^{(2)}RE$ have the same solution (Ref[3]). Some other results are given below (Ref[1,2,3])
Solution to $\{^0\}KE$ (simult.-move); CEQS is "Choose A(R) if his first observ. is $\{(<)\}$
\[ a^* = \frac{1}{2}(\sqrt{5} - 1) \approx 0.618, \text{ i.e., a unique root in } (\mathcal{O}, I) \text{ of the equation } a^2 + a = 1."
CEQV = $\frac{1}{2}$.
In $\{^3\}KE$, the $a^*$, the equation and CEQV change to $a^* = 0.691, 2a^* = -1 - a^2 - a^3$ and $\frac{1}{3}$ resp.

Solution to $\{^0\}RE$ (simult.-move); CEQS is "Choose A(R), if his first observ. is $\{(<)\}$
\[ a^* = 0.544; \text{ i.e., a unique root in } (\mathcal{O}, I) \text{ of the equation } a^3 + a^2 + a = 1."
\[ \text{P(D)} = \frac{1}{4} a^* \approx 0.022. \]
P(W$_1$) = P(W$_2$) = $\frac{1}{2}(1-\text{P(D)}) = 0.489$.
In $\{^3\}RE$, $a^*$, the equation and the other result change to $a^* = 0.656, 2a^* + a^2 = -1 + a^2 + a$, P(D) = $\frac{1}{3} a^* \approx 0.010, \text{P(W)}_1$ =P(W$_2$) = P(W$_3$) = $\frac{1}{3}(1-\text{P(D)}) = 0.330$.

Solution to $\{^0\}RE$ (seq.-move); I's opt. str. is "Choose A(R) $X_{11} = x_1$, if $x_1 > \{<\}$ $x_0$
where $x_0 (\approx 0.570)$ is a unique root in $(\overline{\mathcal{O}}, 1)$ of $\overline{\mathcal{O}}^2 = 2x(1-3x)$" II's opt. str. is

"Choose A(R) $X_{21} = x_2$ if $x_2 > \{<\}$ $x_1$
where $x_1 (\approx 0.549)$ is a unique root in $(\overline{\mathcal{O}}, 1)$ of $\overline{\mathcal{O}}^2 = 4x - 2x + x^2$" II's opt. str. is

"Choose A(R) $X_{21} = x_2$ if $x_2 > \{<\}$ $x_1$
where $x_1 (\approx 0.549)$ is a unique root in $(\overline{\mathcal{O}}, 1)$ of $\overline{\mathcal{O}}^2 = 4x - 2x + x^2$ II's opt. str. is

Under which players decide their choices of A/R.

\[ \{^1\}^{1-11} \] means that I chooses $X_{11} = x_1$, II chooses $X_{21} = x_2$ and each player informs of his observed value to his opponent.

\[ \{^1\}^{0-11} \] means that I observes $X_{11} = x_1$, II observes $X_{21} = x_2$ and I informs his $x_1$ to II and II doesn't inform his $x_2$ to I.

It is clear that $\{^1\}^{0-0}$ is the information type discussed in Section II.

Some results are given as follows (Ref[4]).
We obtain \[ \text{P(D)} = \frac{1}{4}(\frac{1}{2}-1)^4 = 0.007, \text{P(W}_1) = \text{P(W}_2) = \frac{1}{2}(1-\text{P}(\text{D})) = 0.496. \]

• Solution to \( \mathbb{P}^	ext{K}E \) under \( \mathbb{I}^{[0-1]} \): I's opt. str. is "Choose A (R) \( X_{11} = x_1 \), if \( x_1 > ( < ) \) \( a^* = \sqrt{3/8} \approx 0.6124.\)  

<table>
<thead>
<tr>
<th>Condition</th>
<th>II's opt. Choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_i &lt; a^* )</td>
<td>( A(R) ), if ( x_2 &gt; ( &lt; ) ) ( b_2 )</td>
</tr>
<tr>
<td>( a^* &lt; x_1 &lt; x_2 )</td>
<td>( A )</td>
</tr>
<tr>
<td>( x_1 &gt; a^* \lor x_2 )</td>
<td>( R )</td>
</tr>
</tbody>
</table>

We obtain \( \text{P(W}_1) = 1 \rightarrow \text{P(W}_2) = \frac{1}{3} + \frac{1}{4}a^* \approx 0.4864. \)

4. 2-Player game of Continue-or-Stop. Player i observes \( X_{i1}, X_{i2}, \ldots, X_{in} \), sequentially one-by-one, and facing each \( X_{ij} \) chooses Cont/Stop. If Stop is chosen \( X_{ij} \) is accepted by player i and his play ends. If Cont. is chosen, \( X_{ij} \) is rejected and the next \( X_{ij+1} \) is observed and the game continues. Assume that \( \{ X_{ij} \} \) are i.i.d. with \( \mathcal{U}_{[c,1]} \) distribution. Score is

\[ S_i(X_{i1}, \ldots, X_{in}) = \begin{cases} X_{i1} & \text{if } X_{i1}, X_{i2}, \ldots, X_{ij} \text{ are rejected and stop at } X_{ij} \\ X_{ij+1} & \text{if } X_{ij}, X_{i2}, \ldots, X_{ij} \text{ are rejected and stop at } X_{ij+1} \end{cases} \]

Consider the situation where player i has \( n-1 \) decision thresholds \( \{ a_1, a_2, \ldots, a_{n-1} \} \) \( \geq 0 \), so that \( i \) chooses Stop (Cont.) if \( X_{ij} > ( < ) a_j \). Player who gets the higher score than his opponent is the winner. Each player aims \( \text{Pr} \left( \text{he wins} \right) \rightarrow \text{max}. \)

• Solution when \( n = 3 \): The common optimal thresholds are \( \left( \frac{b_1^0, b_2^0}{b_1^0, b_2^0} \right) \equiv (0.743, 0.654) \) where \( \left( \frac{b_1^0, b_2^0}{b_1^0, b_2^0} \right) \) is a unique root in \( (a,1)^2 \) of

\[ b_2^2 + b_2 - (b_1^{-1} + b_1) = 0 \]

\[ b_1^2 + (1+b_2)^{-1} b_1 - 1 = 0 \]

\[ \text{P(W}_1) = \text{P(W}_2) = \frac{1}{2}. \]

5. Remark. There are many open problems of interest around the topic—games of score showdown. Three among them are given below.

(1). Let \( X_{ij}, Y_{ij}, i = 1,2, j = 1,2 \), are i.i.d. with joint p.d.f.

\[ f(x, y) = 1 + \frac{y}{2}(1-2x)(1-2y), \quad \forall (x, y) \in [0,1]^2, \]

with \( |Y| \leq 1 \). In the game \( \mathbb{P}^{(2)}E \) (simult.-move), the score is

\[ S(X_{ij}, Y_{i1}, X_{i2}, Y_{i2}) = \begin{cases} (X_{i1}, Y_{i1}) & \text{if } (X_{i1}, Y_{i1}) \text{ is accepted} \\ (X_{i2}, Y_{i2}) & \text{if } (X_{i2}, Y_{i2}) \text{ is rejected} \end{cases} \]
Define that \((x, y)\) is higher than \([\text{lower than, intermediate to}]\) \((x', y')\), if \((x, y) \geq [<] x'y2\). Player with score \((x, y)\) gets 1 \([-1, 0]\). Solve this zero-sum game.

(2). Sequential-move games \(\Gamma^{(3)}_R E\) and \(\Gamma^{(3)}_C A\) remain unsolved.

(3). 3-player games under various information types are of interest. \(\Gamma^{(3)}_K E, \Gamma^{(3)}_R E\) and \(\Gamma^{(3)}_C A\), under IIIIIIIII, II(00-10-11), II(00-01-11), etc. remain unsolved. The last one II(00-01-11), for example, means that each player observes his own \(X_{i}x = x_{i}x\), in addition I knows \(x_{2j}\), II knows \(x_{3j}\) and III knows \(x_{11}\).

REFERENCES


MULTISTAGE OPTIMAL STOPPING GAMES

1. Each Player has his Priority. Player I, II, and III observe \(X_{1}, X_{2}, \ldots , X_{n}\) with i.i.d. \(U[0,1]\) distribution sequentially one-by-one. They have their previously given priorities \(\langle p_{1}, p_{2}, p_{3} \rangle\).

Facing \(X_{j}\) each player chooses \(R/A\), independently of his rivals.

If only one player chooses A, he gets \(X_{j}\) with his priority, dropping out from the game thereafter, and the remaining two players continue their two-player game with the "revised" priorities. If two players choose A, one player selected according to the "revised" priorities gets \(X_{j}\), drops out from the game thereafter, and the remaining two players continue their 2-player game with the "revised" priorities. If the choices are A-A-A player i gets \(X_{j}\) with prob. \(p_{i}\) dropping out from the game, and the remaining two players continue their 2-player game with the revised" priorities. If the choices are R-R-R then \(X_{j}\) is rejected, the next \(X_{j+1}\) is observed and the subsequent 3-player game continues. Each player aims to maximize the ENV he can get ( N in ENV means net, i.e. no-observation-cost and no-discounting).

Let \(W_{n} (V_{n})\) be CEQV, for 3-player ( 2-player ) equal-priority game. Then the Opt. Eq. is

\[
(W_{n}, W_{n}, W_{n}) = E \left( \text{eq. val. } M_{4}(X) \right) \forall n \geq 1, \quad W_{0} = V_{0} = \theta
\]
The cases where player’s aim is \( Pr \) (he gets a r.v. better than opponent) are solved in \( \{\text{Re} \phi |_{j} \iota \} \),

\[
(V_n, V_n) = E\{\text{val. R}_{\iota}^{V_{n-1}, V_{n-1}} | V_{n-1}, x | U_{n-1}, x \}^{\iota} R_{\iota}^{A}
\]

where \( U_n = \frac{1}{2} (1 + U_{n-1}^{2}) \) (n \( \geq 1 \), \( U_0 = 0 \)).

Common opt. str. for each player is derived, and \( CEQ \) is computed as

\[
U_n = \frac{1}{2}, \frac{0.74/7}{0.8364}, \frac{0.5791}{0.5791}, \text{ for } n = 1, 4, 8, 12, \text{ resp.}
\]

\[
V_n = \frac{1}{4}, \frac{0.6466}{0.7182}, \frac{0.8361}{0.7182}, \text{ ibid}
\]

\[
W_n = \frac{1}{4} \frac{0.556}{0.7182}, \frac{0.7436}{0.7436}, \text{ ibid}
\]

The cases \( \langle p_1, p_2 \rangle = \langle i, 0 \rangle \) and \( \langle p_1, p_2, p_3 \rangle = \langle \frac{1}{2}, \frac{1}{2}, 0 \rangle \) are also solved (Ref.2, 4). Player with low priority stands at disadvantage, since he remains late in the game, and faces less offers.

The cases where player’s aim is Pr. (he gets a r.v. better than opponent) are solved in Ref.2. In the 2-player No-Information case, for example, let us

Define state \((i, y)\) to mean that (1) both players remain in the game, and (2) players currently face the r.v. \( Y_i = y \). Let \( V(i, y) \) be the value of the game in state \((i, y)\), for the \( n \)-problem. Note that \( n \) is fixed throughout, players should choose \( A-A \) in state \((n, y)\) and hence draw of the game cannot occur.

Then the Opt. Eq. in state \((i, y)\) is

\[
V(i, y) = \text{val. } R_{\iota}^{M_{i+1}, 1-g(i, y)} A_{\iota}^{g(i, y) (p-F) g(i, y) + F}
\]

\[
\sum_{i=1}^{n} V(i, y) = p \in \{0, 1\}, \forall y \in \{2, 3, \ldots, n\}
\]

where \( \mu_{i+1} = \sum_{y=1}^{n} V(i, y) \), and

\[
g(i, y) = P_{y} \{ i: \text{it has absolute rank } y, \text{ in state } (i, y) \}
\]

\[
= \prod_{j=1}^{n} (1 - \frac{y}{i+j}) = \left( \frac{1}{y} \right) = \left( \frac{i}{j} \right) = (i_{y} (n))
\]
The solution is: \( \text{Opt. str. pair in state } (x, y) \) is
\[
R \rightarrow R, \text{ } R \rightarrow A, \text{ } A \rightarrow A, \text{ } \text{if } 0 < g(y) < \overline{v}_{n1}, \text{ } \overline{v}_{n} < g(y) < \frac{1}{2}, \text{ } \frac{1}{2} < g(y) < 1, \text{ resp.}
\]
The winning prob for \( n=10 \) are computed downward in \( i \) until reaching \( m=V(i, 1) \). We obtain
\[
P_r(I \text{ wins }) \approx 0.578, 0.619, \text{ for } p=\frac{3}{4}, 1, \text{ resp.}
\]

3. Committee's Selection

I and II observe \((X_j, Y_j), j = 1, \ldots, n, \) i.i.d. with \( U_{Cq,j} \)
distribution sequentially one-by-one, and each player chooses \( R \rightarrow A \). \( X_j \) \( (Y_j) \) is I's
(II's) evaluation of \( j \)-th applicant's ability (Ref.[9]). The Opt. Eq. is
\[
(u_n, v_n) = E[R, v_n, M_n(X, Y)], M_n(x, y) = R
\]
where
\[
u_n = \frac{1}{2} \left\{pu_{n-1} + \overline{v}(2x_{n-1} - 1) \nu_{n-1} + 1 \right\}
\]
It is shown that \( u_n \uparrow u_{\infty}, v_n \uparrow v_{\infty}, \) and \( (u_n, v_n) \) is a unique root in
\((0, 1)^2\)
\[
u = \frac{\sqrt{1 - pu}}{\sqrt{1 - pu} + \sqrt{p_u}}
\]
Convergence is quick:
\[
u_{10} = 0.6592, 0.6867, 0.7530, 0.8611, \text{ for } p=0.5, 0.6, 0.8, 1.0, \text{ resp.}
\]
Various interesting open problems arise. 1. If \( X_j \) \( (Y_j) \) is the ability of management
(foreign language), then \( X_j \) and \( Y_j \) are not independent. 2. The case where players aim
ENV of \((X_j \mid Y_j \geq A) \rightarrow \max, \) (Ref.[11]). 3. 3-player game where \((X_j,Y_j,Z_j)\) is
observed.

We first consider a simple n-round poker. Each of two players I and II receives a hand \( x \) and \( y \), respectively, in \([0,1]\), according to a uniform distribution, and chooses one of two alternatives Reject or Accept. If choice-pair is R-R, the game proceeds to the next round and both players are dealt new hands \( x \) and \( y \). If the choice-pair is A-A showdown occurs and the game ends with I's reward \( sgn(x-y) \). If players choose different choices, then arbitration comes in, and forces them to take the same choices as I's (II's) with probability \( p \). This zero-sum game is played in \( n \)-rounds, and player I(II) aims to maximize(minimize) the expected reward to I.

Let \( \phi_n(x)(\psi_n(y)) \) be the probability that player I (II) chooses A on the hand \( x(y) \). Also let \( v_n \) be the value (for I) of the \( n \)-round game. Then we have

\[
v_n = \max_{\phi_n} \min_{\psi_n} E_{x,y}[(\bar{\phi}_n(x), \phi_n(x))M_n(x,y)(\bar{\psi}_n(y), \psi_n(y))^T]
\]

where

\[
M_n(x,y) = R \begin{pmatrix} v_{n-1} & sgn(x-y) + \rho v_{n-1} \\ \rho sgn(x-y) + \sigma v_{n-1} & sgn(x-y) \end{pmatrix}
\]

The solution is

\[
\phi_n(x) = I(x > a_n), \quad \psi_n(y) = I(y > \bar{a}_n), \quad v_n = 2 a_{n+1} - 1
\]

where \( \{a_n\} \) is determined by

\[
a_{n+1} = a_n + \frac{1}{2} \left( \frac{p a_n^2 - f}{a_n} \right) \quad (n > 1), \quad a_1 = \frac{1}{2}
\]

We obtain

\[
a_n \uparrow a_\infty = \sqrt{p/(1+p+\sqrt{p})}, \quad v_n \uparrow 2 a_\infty - 1 = (\sqrt{p}+\sqrt{f})/(\sqrt{p}+\sqrt{f})
\]

The bilateral-move version, when \( p = 1/2 \), has an interesting solution. Let \( w_n \) be the value of the \( n \)-stage game. \( w_n < 0 \), since I must move first and inevitably gives some information about his true hand. We find that

\[
w_\infty \downarrow w_\infty = -1 + \sqrt{3}/(1+\sqrt{3}) \approx -0.1197
\]

where \( g = \sqrt{2}/(1+\sqrt{3}) \approx 0.618 \), the golden bisection number.

Also we find that disadvantage disappears when \( p = 0.6 \), i.e.

\[
w_n = 0, \quad \forall n > 1. \quad (Ref.[10]).
\]

3-player High-Hand-Wins poker, under simple-majority rule and with \( B > 0 \), is also an interesting open problem.

4. Odd-Man-Wins and Odd-Man-Out. In the three-player two-choice games there often appear Odd-Man and Even-Men. What is the reasonable partition among players of each \( X_j \)?

Let \( v_\infty(w_\infty) \) be CEQV of \( n \)-stage Odd-Man-Wins (Odd-Man-Out). Then the payoff matrices \( M_n(X) \) are
Each player must think about: (1) He wants to become the odd-man (an even-man), when the game is Odd-Man-Wins (Odd-Man-Out), especially when he faces a very large $X_j$, and (2) Since $X_j$ is a random variable, he can expect a larger one may come up in the future.

It is shown that (Ref. [6]), $v_n \uparrow v_\infty \doteq 0.205$ and $w_n \downarrow w_\infty \doteq 0.160$. So, multi-stage play yields each player a merit of size $V_\infty \doteq \frac{1}{6}$ and a demerit of size $\frac{1}{6} - w_\infty \doteq 0.044$.

The extension to the many-player two-choice simple-majority games is an interesting open problem.

The case where the odd-man has priority $p$, and the even-men has priority $\sqrt{2}/2$ is solved in Ref. [7]. When $p = \sqrt{3}$, $\text{CEQV} = \frac{1}{3} \mu_n$, where $\{\mu_n\}$ is the Moser's sequence. When $p = 0 \ (1)$, the game reduces to Odd-Man-Out (Odd-Man-Wins).

REFERENCES


In some of these, names of co-author(s) are omitted.

* 3-26-4 Midorigaoka, Toyonaka, Osaka, 560-0002 Japan.
Fax: +81-6-6856-2314 E-mail: minorus@tect.zaq.ne.jp