INTRODUCTION TO A THEORY OF $b$-FUNCTIONS

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We give an introduction to a theory of $b$-functions, i.e. Bernstein-Sato polynomials. After reviewing some facts from $D$-modules, we introduce $b$-functions including the one for arbitrary ideals of the structure sheaf. We explain the relation with singularities, multiplier ideals, etc., and calculate the $b$-functions of monomial ideals and also of hyperplane arrangements in certain cases.

1. $D$-modules.

1.1. Let $X$ be a complex manifold or a smooth algebraic variety over $\mathbb{C}$. Let $\mathcal{D}_{X}$ be the ring of partial differential operators. A local section of $\mathcal{D}_{X}$ is written as

$$\sum_{\nu \in \mathbb{N}^{n}} a_{\nu} \partial_{1}^{\nu_{1}} \cdots \partial_{n}^{\nu_{n}} \in \mathcal{D}_{X} \quad \text{with} \quad a_{\nu} \in \mathcal{O}_{X},$$

where $\partial_{i} = \partial / \partial x_{i}$ with $(x_{1}, \ldots, x_{n})$ a local coordinate system.

Let $F$ be the filtration by the order of operators i.e.

$$F_{p} \mathcal{D}_{X} = \left\{ \sum_{|\nu| \leq p} a_{\nu} \partial_{1}^{\nu_{1}} \cdots \partial_{n}^{\nu_{n}} \right\},$$

where $|\nu| = \sum \nu_{i}$. Let $\xi_{i} = \text{Gr}^{F}_{1} \partial_{i} \in \text{Gr}^{F}_{1} \mathcal{D}_{X}$. Then

$$\text{Gr}^{F} \mathcal{D}_{X} := \bigoplus_{p} \text{Gr}^{F}_{p} \mathcal{D}_{X} = \bigoplus_{p} \text{Sym}^{p} \Theta_{X} = \mathcal{O}_{X} [\xi_{1}, \ldots, \xi_{n}] \text{ locally},$$

(1.1.1)

$$\text{Spec}_{X} \text{Gr}^{F} \mathcal{D}_{X} = T^{*}X.$$

1.2 Definition. We say that a left $\mathcal{D}_{X}$-module $M$ is coherent if it has locally a finite presentation

$$\bigoplus \mathcal{D}_{X} \rightarrow \bigoplus \mathcal{D}_{X} \rightarrow M \rightarrow 0.$$

1.3. Remark. A left $\mathcal{D}_{X}$-module $M$ is coherent if and only if it is quasi-coherent over $\mathcal{O}_{X}$ and locally finitely generated over $\mathcal{D}_{X}$. (It is known that $\text{Gr}^{F} \mathcal{D}_{X}$ is a noetherian ring, i.e. an increasing sequence of locally finitely generated $\text{Gr}^{F} \mathcal{D}_{X}$-submodules of a coherent $\text{Gr}^{F} \mathcal{D}_{X}$-module is locally stationary.)

1.4. Definition. A filtration $F$ on a left $\mathcal{D}_{X}$-module $M$ is good if $(M, F)$ is a coherent filtered $\mathcal{D}_{X}$-module, i.e. if $F_{p} \mathcal{D}_{X} F_{q} M \subset M_{p+q}$ and $\text{Gr}^{F} M := \bigoplus_{p} \text{Gr}^{F}_{p} M$ is coherent over $\text{Gr}^{F} \mathcal{D}_{X}$.

1.5. Remark. A left $\mathcal{D}_{X}$-module $M$ is coherent if and only if it has a good filtration locally.

1.6. **Characteristic varieties.** For a coherent left $D_X$-module $M$, we define the characteristic variety $CV(M)$ by

\[(1.6.1) \quad CV(M) = \text{Supp} \text{Gr}^F M \subset T^* M,\]

taking locally a good filtration $F$ of $M$.

1.7. **Remark.** The above definition is independent of the choice of $F$. If $M = D_X/I$ for a coherent left ideal $I$ of $D_X$, take $P_i \in F_{k_i} I$ such that the $\rho_i := \text{Gr}^F_{k_i} P_i$ generate $\text{Gr}^F I$ over $\text{Gr}^F D_X$. Then $CV(M)$ is defined by the $\rho_i \in O_X[\xi_1, \ldots, \xi_n]$.

1.8. **Theorem** (Sato, Kawai, Kashiwara [39], Bernstein [2]). We have the inequality $\dim CV(M) \geq \dim X$. *(More precisely, $CV(M)$ is involutive, see [39].)*

1.9. **Definition.** We say that a left $D_X$-module $M$ is holonomic if it is coherent and $\dim CV(M) = \dim X$.

2. **De Rham functor.**

2.1. **Definition.** For a left $D_X$-module $M$, we define the de Rham functor $\text{DR}(M)$ by

\[(2.1.1) \quad M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega^\dim X_X \otimes_{\mathcal{O}_X} M,\]

where the last term is put at the degree 0. In the algebraic case, we use analytic sheaves or replace $M$ with the associated analytic sheaf $M^{\text{an}} := M \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\text{an}}$ in case $M$ is algebraic (i.e. $M$ is an $\mathcal{O}_X$-module with $\mathcal{O}_X$ algebraic).

2.2. **Perverse sheaves.** Let $D_c^b(X, C)$ be the derived category of bounded complexes of $\mathcal{O}_X$-modules $K$ with $\mathcal{H}^j K$ constructible. (In the algebraic case we use analytic topology for the sheaves although we use Zariski topology for constructibility.) Then the category of perverse sheaves $\text{Perv}(X, C)$ is a full subcategory of $D_c^b(X, C)$ consisting of $K$ such that

\[(2.2.1) \quad \dim \text{Supp} \mathcal{H}^{-j} K \leq j, \quad \dim \text{Supp} \mathcal{H}^{-j} \text{DK} \leq j,\]

where $\text{DK} := \mathcal{R}\text{Hom}(K, C[2 \dim X])$ is the dual of $K$, and $\mathcal{H}^j K$ is the $j$-th cohomology sheaf of $K$.

2.3. **Theorem** (Beilinson, Bernstein, Deligne [1]). $\text{Perv}(X, C)$ is an abelian category.

2.4. **Theorem** (Kashiwara). *If $M$ is holonomic, then $\text{DR}(M)$ is a perverse sheaf.*

**Outline of proof.** By Kashiwara [19], we have $\text{DR}(M) \in D_c^b(X, C)$, and the first condition of (2.2.1) is verified. Then the assertion follows from the commutativity of the dual $\mathcal{D}$ and the de Rham functor $\text{DR}$.

2.5. **Example.** $\text{DR}(\mathcal{O}_X) = C_X[\dim X]$.

2.6. **Direct images.** For a closed immersion $i : X \to Y$ such that $X$ is defined by $x_i = 0$ in $Y$ for $1 \leq i \leq r$, define the direct image of left $D_X$-modules $M$ by

\[i_+ M := M[\partial_1, \ldots, \partial_r].\]
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(Globally there is a twist by a line bundle.) For a projection \( p : X \times Y \to Y \), define

\[
p_+ M = Rp_* DR_X(M).
\]

In general, \( f_+ = p_+ i_+ \) using \( f = pi \) with \( i \) graph embedding. See [4] for details.

2.7. Regular holonomic \( \mathcal{D} \)-modules. Let \( M \) be a holonomic \( \mathcal{D}_X \)-module with support \( Z \), and \( U \) be a Zariski-open of \( Z \) such that \( DR(M)|_U \) is a local system up to a shift. Then \( M \) is regular if and only if there exists locally a divisor \( D \) on \( X \) containing \( Z \setminus U \) and such that \( M(*D) \) is the direct image of a regular holonomic \( \mathcal{D} \)-module 'of Deligne-type' (see [11]) on a desingularization of \( (Z, Z \cap D) \), and \( \text{Ker}(M \to M(*D)) \) is regular holonomic (by induction on \( \dim \text{Supp} \, M \)).

Note that the category \( M_{rh}(\mathcal{D}_X) \) of regular holonomic \( \mathcal{D}_X \)-modules is stable by subquotients and extensions in the category \( M_h(\mathcal{D}_X) \) of holonomic \( \mathcal{D}_X \)-modules.

2.8. Theorem (Kashiwara-Kawai [24], [22], Mebkhout [28]).

(i) The structure sheaf \( \mathcal{O}_X \) is regular holonomic.

(ii) The functor \( DR \) induces an equivalence of categories

\[
\text{DR} : M_{rh}(\mathcal{D}_X) \sim \text{Perv}(X, \mathbb{C}).
\]

(See [4] for the algebraic case.)

3. \( b \)-Functions.

3.1. Definition. Let \( f \) be a holomorphic function on \( X \), or \( f \in \Gamma(X, \mathcal{O}_X) \) in the algebraic case. Then we have

\[
\mathcal{D}_X[s]f^s \subset \mathcal{O}_X[\frac{1}{f}]f^s \quad \text{where } \partial_i f^s = s(\partial_i f)f^{s-1},
\]

and \( b_f(s) \) is the monic polynomial of the least degree satisfying

\[
b_f(s)f^s = P(x, \partial, s)f^{s+1} \quad \text{in } \mathcal{O}_X[\frac{1}{f}]f^s,
\]

with \( P(x, \partial, s) \in \mathcal{D}_X[s] \). Locally, it is the minimal polynomial of the action of \( s \) on

\[
\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}.
\]

We define \( b_{f,x}(s) \) replacing \( \mathcal{D}_X \) with \( \mathcal{D}_{X,x} \).

3.2. Theorem (Sato [38], Bernstein [2], Bjork [3]). The \( b \)-function exists at least locally, and exists globally in the case \( X \) affine variety with \( f \) algebraic.

3.3. Observation. Let \( i_f : X \to \bar{X} := X \times \mathbb{C} \) be the graph embedding. Then there are canonical isomorphisms

\[
\bar{M} := i_f \mathcal{O}_X = \mathcal{O}_X[\partial_t] \delta(f - t) = \mathcal{O}_{X \times \mathbb{C}}[\frac{1}{t}] / \mathcal{O}_{X \times \mathbb{C}},
\]

where the action of \( \partial_t \) on \( \delta(f - t) \) is given by

\[
\partial_t \delta(f - t) = - (\partial_t f) \partial \delta(f - t).
\]

Moreover, \( f^s \) is canonically identified with \( \delta(f - t) \) setting \( s = -\partial_t t \), and we have a canonical isomorphism as \( \mathcal{D}_X[s] \)-modules

\[
\mathcal{D}_X[s]f^s = \mathcal{D}_X[s] \delta(f - t).
\]
3.4. V-filtration. We say that $V$ is a filtration of Kashiwara-Malgrange if $V$ is exhaustive, separated, and satisfies for any $\alpha \in \mathbb{Q}$:

(i) $V^\alpha \widehat{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\widehat{M}$.
(ii) $tV^\alpha \widehat{M} \subset V^{\alpha+1} \widehat{M}$ and $= \mathcal{D}_X[\alpha(\mathcal{D}_X[s]f^\ell)]$ holds for $\alpha \gg 0$.
(iii) $\partial t V^\alpha \widehat{M} \subset V^{\alpha-1} \widehat{M}$.
(iv) $\partial t - \alpha$ is nilpotent on $Gr^\alpha \widehat{M}$.

If it exists, it is unique.

3.5. Relation with the b-function. If $X$ is affine or Stein and relatively compact, then the multiplicity of a root $\alpha$ of $b_f(s)$ is given by the minimal polynomial of $s - \alpha$ on

$$\text{Gr}^\alpha \mathcal{D}_X[s]f^\ell/\mathcal{D}_X[s]f^{\ell+1},$$

using $\mathcal{D}_X[s]f^\ell = \mathcal{D}_X[s]\delta(f-t)$ with $s = -\partial t$.

Note that $V^\alpha \widehat{M}$ and $\mathcal{D}_X[s]f^{\ell+i}$ are 'lattices' of $\widehat{M}$, i.e.

$$V^\alpha \widehat{M} \subset \mathcal{D}_X[s]f^{\ell+i} \subset V^\beta \widehat{M} \quad \text{for} \quad \alpha \gg i \gg \beta,$$

and $V^\alpha \widehat{M}$ is an analogue of the Deligne extension with eigenvalues in $[\alpha, \alpha+1)$. The existence of $V$ is equivalent to the existence of $b_f(s)$ locally.

3.6. Theorem (Kashiwara [21], [23], Malgrange [27]). The filtration $V$ exists on $\widehat{M} := i_{f+}M$ for any holonomic $\mathcal{D}_X$-module $M$.

3.7. Remarks. (i) There are many ways to prove this theorem, since it is essentially equivalent to the existence of the $b$-function (in a generalized sense). One way is to use a resolution of singularities and reduce to the case where $CV(M)$ has normal crossings, if $M$ is regular.

(ii) The filtration $V$ is indexed by $\mathbb{Q}$ if $M$ is quasi-unipotent.

3.8. Relation with vanishing cycle functors. Let $\rho : X_t \to X_0$ be a 'good' retraction (using a resolution of singularities of $(X, X_0)$), where $X_t = f^{-1}(t)$ with $t \neq 0$ sufficiently near 0. Then we have canonical isomorphisms

$$\psi_f C_X = R\rho_* C_{X_t}, \quad \varphi_f C_X = \psi_f C_X/C_{X_0},$$

where $\psi_f C_X, \varphi_f C_X$ are nearby and vanishing cycle sheaves, see [13].

Let $F_x$ denote the Milnor fiber around $x \in X_0$. Then

$$\mathcal{H}^i(\psi_f C_X)_x = H^i(F_x, C), \quad (\mathcal{H}^i \varphi_f C_X)_x = \widetilde{H}^i(F_x, C).$$

For a $\mathcal{D}_X$-module $M$ admitting the $V$-filtration on $\widehat{M} = i_{f+}M$, we define $\mathcal{D}_X$-modules

$$\psi_f M = \bigoplus_{0 \leq \alpha \leq 1} \text{Gr}^\alpha \widehat{M}, \quad \varphi_f M = \bigoplus_{0 \leq \alpha < 1} \text{Gr}^\alpha \widehat{M}.$$

3.9. Theorem (Kashiwara [23], Malgrange [27]). For a regular holonomic $\mathcal{D}_X$-module $M$, we have canonical isomorphisms

$$\text{DR}_X \psi_f(M) = \psi_f \text{DR}_X(M)[-1],$$
$$\text{DR}_X \varphi_f(M) = \varphi_f \text{DR}_X(M)[-1].$$
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and \( \exp(-2\pi i\partial_t t) \) on the left-hand side corresponds to the monodromy \( T \) on the right-hand side.

3.10. Definition. Let

\[
R_f = \{ \text{roots of } b_f(-s) \},
\]
\[
\alpha_f = \min R_f,
\]
\[
m_\alpha : \text{the multiplicity of } \alpha \in R_f.
\]
(Similarly for \( R_{f,x} \), etc. for \( b_{f,x}(s) \).)

3.11. Theorem (Kashiwara [20]). \( R_f \subset \mathbb{Q}_{>0} \).

(This is proved by using a resolution of singularities.)

3.12. Theorem (Kashiwara [23], Malgrange [27]).

(i) \( e^{-2\pi R_f} = \{ \text{the eigenvalues of } T \text{ on } H^j(F_x, \mathbb{C}) \text{ for } x \in X_0, j \in \mathbb{Z} \} \),

(ii) \( m_\alpha \leq \min \{ i \mid N^i \psi_{f,\lambda} C_X = 0 \} \) with \( \lambda = e^{-2\pi i \alpha} \),

where \( \psi_{f,\lambda} = \ker(T_\epsilon - \lambda) \subset \psi_f \), \( N = \log T_u \) with \( T = T_s T_u \).

(This is a corollary of the above Theorem (3.9) of Kashiwara and Malgrange.)

4. Relation with other invariants.

4.1. Microlocal \( b \)-function. We define \( \tilde{R}_f, \tilde{m}_\alpha, \tilde{\alpha}_f \) with \( b_f(s) \) replaced by the microlocal (or reduced) \( b \)-function

\[
(4.1.1) \quad \tilde{b}_f(s) := b_f(s)/(s + 1).
\]

This \( \tilde{b}_f(s) \) coincides with the monic polynomial of the least degree satisfying

\[
(4.1.2) \quad \tilde{b}_f(s) \delta(f - t) = \tilde{P} \partial_t^{-1} \delta(f - t) \quad \text{with } \tilde{P} \in \mathcal{D}_X[s, \partial_t^{-1}].
\]

Put \( n = \dim X. \) Then

4.2. Theorem. \( \tilde{R}_f \subset [\tilde{\alpha}_f, n - \tilde{\alpha}_f] \), \( \tilde{m}_\alpha \leq n - \tilde{\alpha}_f - \alpha + 1 \).

(The proof uses the filtered duality for \( \varphi_f \), see [35].)

4.3. Spectrum. We define the spectrum by \( \text{Sp}(f, x) = \sum \alpha n_{\alpha} t^\alpha \) with

\[
(4.3.1) \quad n_{\alpha} := \sum_j (-1)^{j-n+1} \dim \text{Gr}_p^\alpha \tilde{F}_j(F_x, \mathbb{C})_\lambda,
\]

where \( p = [n - \alpha], \lambda = e^{-2\pi i \alpha} \), and \( F \) is the Hodge filtration (see [12]) of the mixed Hodge structure on the Milnor cohomology, see [44]. We define

\[
(4.3.2) \quad E_f = \{ \alpha \mid n_{\alpha} \neq 0 \} \quad \text{(called the exponents)}.
\]

4.4. Remarks. (i) If \( f \) has an isolated singularity at the origin, then \( \tilde{\alpha}_{f,x} \) coincides with the minimal exponent as a corollary of results of Malgrange [26], Varchenko [45], Scherk-Steenbrink [41].

(ii) If \( f \) is weighted-homogeneous with an isolated singularity at the origin, then by Kashiwara (unpublished)

\[
(4.4.1) \quad \tilde{R}_f = E_f, \quad \max \tilde{R}_f = n - \tilde{\alpha}_f, \quad \tilde{m}_\alpha = 1 (\alpha \in \tilde{R}_f).
\]
If $f = \sum_i x_i^2$, then $\overline{\alpha}_f = n/2$ and this follows from the above Theorem (4.2).

By Steenbrink [42], we have moreover

$$(4.4.2) \quad \text{Sp}(f, x) = \prod_i (t - t^{w_i})/(t^{w_i} - 1),$$

where $(w_1, \ldots, w_n)$ is the weights of $f$, i.e. $f$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $\sum_i w_im_i = 1$.

4.5. Malgrange's formula (isolated singularities case). We have the Brieskorn lattice [5] and its saturation defined by

$$(4.5.1) \quad H''_f = \Omega^n_{x,x}/df \wedge d\Omega^n_{x,x}^{-2}, \quad \tilde{H}''_f = \sum_{i \geq 0} (t\partial_t)^i H''_f \subset H''_f[t^{-1}].$$

These are finite $\mathbb{C}\{t\}$-modules with a regular singular connection.

4.6. Theorem (Malgrange [26]). The reduced $b$-function $\tilde{b}_f(s)$ coincides with the minimal polynomial of $-\partial t$ on $\tilde{H}''_f/t\tilde{H}''_f$.

(The above formula of Kashiwara on $b$-function (4.4.1) can be proved by using this together with Brieskorn’s calculation.)

4.7. Asymptotic Hodge structure (Varchenko [45], Scherk-Steenbrink [41]). In the isolated singularity case we have

$$(4.7.1) \quad F^pH^{n-1}(F_x, C)_\lambda = \text{Gr}^\alpha_v H''_f,$$

using the canonical isomorphism

$$(4.7.2) \quad H^{n-1}(F_x, C)_\lambda = \text{Gr}^\alpha_v H''_f[t^{-1}],$$

where $p = [n - \alpha], \lambda = e^{-2\pi i\alpha}$, and $V$ on $H''_f[t^{-1}]$ is the filtration of Kashiwara and Malgrange.

(This can be generalized to the non-isolated singularity case using mixed Hodge modules.)

4.8. Reformulation of Malgrange's formula. We define

$$(4.8.1) \quad \tilde{F}^pH^{n-1}(F_x, C)_\lambda = \text{Gr}^\alpha_v \tilde{H}''_f,$$

using the canonical isomorphism (4.7.2), where $p = [n - \alpha], \lambda = e^{-2\pi i\alpha}$. Then

$$(4.8.2) \quad \tilde{m}_\alpha = \text{the minimal polynomial of } N \text{ on } \text{Gr}^\alpha_v \tilde{F}^pH^{n-1}(F_x, C)_\lambda.$$

4.9. Remark. If $f$ is weighted homogeneous with an isolated singularity, then

$$(4.9.1) \quad \tilde{F} = F, \quad \tilde{R}_f = E_f \text{ (by Kashiwara)}.$$

If $f$ is not weighted homogeneous (but with isolated singularities), then

$$(4.9.2) \quad \tilde{R}_f \subset \bigcup_{k \in \mathbb{N}} (E_f - k), \quad \tilde{\alpha}_f = \min \tilde{R}_f = \min E_f.$$

4.10. Example. If $f = x^5 + y^4 + x^3y^2$, then

$$E_f = \left\{ \frac{i}{5} + \frac{j}{4} : 1 \leq i \leq 4, 1 \leq j \leq 3 \right\}, \quad \tilde{R}_f = E_f \cup \left\{ \frac{11}{20} \right\} \setminus \left\{ \frac{31}{20} \right\}.$$
More generally, if \( f = g + h \) with \( g \) weighted homogeneous and \( h \) is a linear combination of monomials of higher degrees, then \( E_f = E_g \) but \( \tilde{R}_f \neq \tilde{R}_g \) if \( f \) is a non trivial deformation.

**4.11. Relation with rational singularities [34].** Assume \( D := f^{-1}(0) \) is reduced. Then \( D \) has rational singularities if and only if \( \tilde{\alpha}_f > 1 \). Moreover, \( \omega_D/\rho_*\omega_{\tilde{D}} \cong F_{1-n}\varphi_f\mathcal{O}_X \), where \( \rho: \tilde{D} \to D \) is a resolution of singularities.

In the isolated singularities case, this was proved in 1981 (see [31]) using the coincidence of \( \tilde{\alpha}_f \) and the minimal exponent.

**4.12. Relation with the pole order filtration [34].** Let \( P \) be the pole order filtration on \( \mathcal{O}_X(*D) \), i.e. \( P_i = \mathcal{O}_X((i+1)D) \) if \( i \geq 0 \), and \( P_i = 0 \) if \( i < 0 \). Let \( F \) be the Hodge filtration on \( \mathcal{O}_X(*D) \). Then \( F_i \subset P_i \) in general, and \( F_i = P_i \) on a neighborhood of \( x \) for \( i \leq \tilde{\alpha}_{f,x} - 1 \).

(For the proof we need the theory of microlocal b-functions [35].)

**4.13. Remark.** In case \( X = \mathbb{P}^n \), replacing \( \tilde{\alpha}_{f,x} \) with \( [(n-r)/d] \) where \( r = \text{dim} \text{Sing} \ D \) and \( d = \text{deg} \ D \), the assertion was obtained by Deligne (unpublished).

5. Relation with multiplier ideals.

**5.1. Multiplier ideals.** Let \( D = f^{-1}(0) \), and \( \mathcal{J}(X, \alpha D) \) be the multiplier ideals for \( \alpha \in \mathbb{Q} \), i.e.

\[
\mathcal{J}(X, \alpha D) = \rho_*\omega_{\tilde{X}/X}(-\sum_i[\alpha m_i]\tilde{D}_i),
\]

where \( \rho: (\tilde{X}, \tilde{D}) \to (X, D) \) is an embedded resolution and \( \tilde{D} = \sum_i m_i\tilde{D}_i := \rho^*D \).

There exist jumping numbers \( 0 < \alpha_0 < \alpha_1 < \cdots \) such that

\[
\mathcal{J}(X, \alpha_j D) = \mathcal{J}(X, \alpha D) \neq \mathcal{J}(X, \alpha_{j+1} D) \quad \text{for} \quad \alpha_j \leq \alpha < \alpha_{j+1}.
\]

Let \( V \) denote also the induced filtration on

\[
\mathcal{O}_X \subset \mathcal{O}_X[\partial_t]\delta(f-t).
\]

**5.2. Theorem** (Budur, S. [10]). If \( \alpha \) is not a jumping number,

\[
\mathcal{J}(X, \alpha D) = V^\alpha \mathcal{O}_X.
\]

For \( \alpha \) general we have for \( 0 < \epsilon \ll 1 \)

\[
\mathcal{J}(X, \alpha D) = V^{\alpha+\epsilon} \mathcal{O}_X, \quad V^\alpha \mathcal{O}_X = \mathcal{J}(X, (\alpha - \epsilon)D).
\]

Note that \( V \) is left-continuous and \( \mathcal{J}(X, \alpha D) \) is right-continuous, i.e.

\[
V^\alpha \mathcal{O}_X = V^{\alpha-\epsilon} \mathcal{O}_X, \quad \mathcal{J}(X, \alpha D) = \mathcal{J}(X, (\alpha + \epsilon)D).
\]

The proof of (5.2) uses the theory of bifiltered direct images [32], [33] to reduce the assertion to the normal crossing case.

As a corollary we get another proof of the results of Ein, Lazarsfeld, Smith and Varolin [16], and of Lichtin, Yano and Kollár [25]:
5.3. Corollary.

(i) \{Jumping numbers \leq 1\} \subset R_f, see [16].
(ii) \(\alpha_f = \text{minimal jumping number}, \) see [25].

Define \(\alpha'_{f,x} = \min_{y \neq x} \{\alpha_{f,y}\}\). Then

5.4. Theorem. If \(\xi f = f\) for a vector field \(\xi\), then

\[(5.4.1) \quad R_f \cap (0, \alpha'_{f,x}) = \{\text{Jumping numbers}\} \cap (0, \alpha'_{f,x}).\]

(This does not hold without the assumption on \(\xi\) nor for \([\alpha'_{f,x}, 1])\.)

For the constantness of the jumping numbers under a topologically trivial deformation of divisors, see [14].

6. \(b\)-Functions for any subvarieties.

6.1. Let \(Z\) be a closed subvariety of a smooth \(X\), and \(f = (f_1, \ldots, f_r)\) be generators of the ideal of \(Z\) (which is not necessarily reduced nor irreducible). Define the action of \(t_j\) on

\[O_X \left[ \left[ \frac{1}{f_1, \ldots, f_r} \right] \right] [s_1, \ldots, s_r] \prod_i f_i^{s_i},\]

by \(t_j(s_i) = s_i + 1\) if \(i = j\), and \(t_j(s_i) = s_i\) otherwise. Put \(s_{i,j} := s_i t_i^{-1} t_j\), \(s = \sum_i s_i\).

Then \(b_f(s)\) is the monic polynomial of the least degree satisfying

\[(6.1.1) \quad b_f(s) \prod_i f_i^{s_i} = \sum_{k=1}^{r} P_k t_k \prod_i f_i^{s_k},\]

where \(P_k\) belong to the ring generated by \(\mathcal{D}_X\) and \(s_{i,j}\).

Here we can replace \(\prod_i f_i^{s_i}\) with \(\prod_i \delta(t_i - f_i)\), using the direct image by the graph of \(f : X \rightarrow \mathbb{C}^r\). Then the existence of \(b_f(s)\) follows from the theory of the \(V\)-filtration of Kashiwara and Malgrange. This \(b\)-function has appeared in work of Sabbah [30] and Gyoja [18] for the study of \(b\)-functions of several variables.

6.2. Theorem (Budur, Mustata, S. [8]). Let \(c = \text{codim}_X Z\). Then \(b_Z(s) := b_f(s - c)\) depends only on \(Z\) and is independent of the choice of \(f = (f_1, \ldots, f_r)\) and also of \(r\).

6.3. Equivalent definition. The \(b\)-function \(b_f(s)\) coincides with the monic polynomial of the least degree satisfying

\[(6.3.1) \quad b_f(s) \prod_i f_i^{s_i} \in \sum_{|c| = 1} \mathcal{D}_X[s] \prod_{c_i < 0} \left( \frac{1}{s_i} \right) \prod_i f_i^{s_i + c_i},\]

where \(c = (c_1, \ldots, c_r) \in \mathbb{Z}^r\) with \(|c| := \sum_i c_i = 1\). Here \(\mathcal{D}_X[s] = \mathcal{D}_X[s_1, \cdots, s_r]\).

This is due to Mustata, and is used in the monomial ideal case. Note that the well-definedness does not hold without the term \(\prod_{c_i < 0} \left( \frac{1}{s_i} \right)\).

We have the induced filtration \(V\) by

\[O_X \subset i_f+O_X = O_X[\partial_1, \ldots, \partial_r] \prod_i \delta(t_i - f_i).\]

6.4. Theorem (Budur, Mustata, S. [8]). If \(\alpha\) is not a jumping number,

\[(6.4.1) \quad \mathcal{J}(X, \alpha Z) = \mathcal{V}^\alpha O_X.\]
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For \( \alpha \) general we have for \( 0 < \varepsilon \ll 1 \)

\[(6.4.2) \quad J(X, \alpha Z) = V^{\alpha + \varepsilon} \mathcal{O}_X, \quad V^{\alpha} \mathcal{O}_X = J(X, (\alpha - \varepsilon)Z).\]

6.5. Corollary (Budur, Mustață, S. [8]). We have the inclusion

\[(6.5.1) \quad \{\text{Jumping numbers}\} \cap [\alpha_f, \alpha_f + 1) \subset R_f.\]

6.6. Theorem (Budur, Mustață, S. [8]). If \( Z \) is reduced and is a local complete intersection, then \( Z \) has only rational singularities if and only if \( \alpha_f = r \) with multiplicity 1.

7. Monomial ideal case.

7.1. Definition. Let \( a \subset C[x] := C[x_1, \ldots, x_n] \) a monomial ideal. We have the associated semigroup defined by

\[\Gamma_a = \{u \in \mathbb{N}^n \mid x^u \in a\}.\]

Let \( P_a \) be the convex hull of \( \Gamma_a \) in \( \mathbb{R}^n_{\geq 0} \). For a face \( Q \) of \( P_a \), define

\[M_Q : \text{the subsemigroup of } \mathbb{Z}^n \text{ generated by } u - v \text{ with } u \in \Gamma_a, v \in \Gamma_a \cap Q.\]

\[M'_Q = v_0 + M_Q \text{ for } v_0 \in \Gamma_a \cap Q \text{ (this is independent of } v_0).\]

For a face \( Q \) of \( P_a \) not contained in any coordinate hyperplane, take a linear function with rational coefficients \( L_Q : \mathbb{R}^n \rightarrow \mathbb{R} \) whose restriction to \( Q \) is 1. Let

\[V_Q : \text{the linear subspace generated by } Q.\]
\[e = (1, \ldots, 1).\]
\[R_Q = \{L_Q(u) \mid u \in (e + (M_Q \setminus M'_Q) \cap V_Q\}.\]
\[R_a = \{\text{roots of } \partial a (-s)\}.\]

7.2. Theorem (Budur, Mustață, S. [9]). We have \( R_a = \bigcup_Q R_Q \) with \( Q \) faces of \( P_a \) not contained in any coordinate hyperplanes.

Outline of the proof. Let \( f_j = \prod_i x_i^{a_{1,j}} \), \( \ell_i(s) = \sum_j a_{i,j}s_j \). Define

\[g_c(s) = \prod_{c_i < 0}(s_i^{c_i}) \prod_{c_i > 0}(s_i)^{(\ell_i(s))}.\]

Let \( I_a \subset C[s] \) be the ideal generated by \( g_c(s) \) with \( c \in \mathbb{Z}^r, \sum_i c_i = 1 \). Then

7.3. Proposition (Mustață). The \( b \)-function \( b_a[s] \) of the monomial ideal \( a \) is the monic generator of \( C[s] \cap I_a \), where \( s = \sum_i s_i \).

Using this, Theorem (7.2) follows from elementary computations.

7.4. Case \( n = 2 \). Here it is enough to consider only 1-dimensional \( Q \) by (7.2). Let \( Q \) be a compact face of \( P_a \) with \( \{v^{(1)}, v^{(2)}\} = \partial Q \), where \( v^{(i)} = (v_{1}^{(i)}, v_{2}^{(i)}) \) with \( v_1^{(1)} < v_1^{(2)}, v_2^{(1)} > v_2^{(2)} \). Let

\[G : \text{the subgroup generated by } u - v \text{ with } u, v \in Q \cap \Gamma_a.\]
\[v^{(3)} \in Q \cap \mathbb{N}^2 \text{ such that } v^{(3)} - v^{(1)} \text{ generates } G.\]
\[S_Q = \{(i, j) \in \mathbb{N}^2 \mid i < v_1^{(3)}, j < v_2^{(1)}\}.\]
$S_{Q}^{[1]} = S \cap M_{Q}'$, $S_{Q}^{[0]} = S_{Q} \setminus S_{Q}^{[1]}$.

Then

$$R_{Q} = \{L_{Q}(u + e) - k \mid u \in S_{Q}^{[k]} (k = 0, 1)\}.$$ 

In the case $Q \subset \{x = m\}$, we have $R_{Q} = \{i/m \mid i = 1, \ldots, m\}$.

7.5. **Examples.** (i) If $a = (x^{a_{1}}, xy^{b})$, with $a, b \geq 2$, then

$$R_{a} = \left\{ \frac{(b - 1)i + (a - 1)j}{ab - 1} \mid 1 \leq i \leq a, 1 \leq j \leq b \right\}.$$

(ii) If $a = (xy^{5}, x^{3}y^{2}, x^{5}y)$, then $S_{Q}^{[1]} = \emptyset$ and

$$R_{a} = \left\{ \frac{5}{13}, \frac{i}{13} (7 \leq i \leq 17), \frac{19}{13}, \frac{j}{6} (3 \leq j \leq 9) \right\}.$$

(iii) If $a = (xy^{5}, x^{3}y^{2}, x^{4}y)$, then $S_{Q}^{[1]} = \{(2, 4)\}$ for $\partial Q = \{(1, 5), (3, 2)\}$ with $L_{Q}(v_{1}, v_{2}) = (3v_{1} + 2v_{2})/13$, and

$$R_{a} = \left\{ \frac{i}{13} (5 \leq i \leq 17), \frac{j}{5} (2 \leq j \leq 6) \right\}.$$

Here $19/13$ is shifted to $6/13$.

7.6. **Comparison with exponents.** If $n = 2$ and $f$ has a nondegenerate Newton polygon with compact faces $Q$, then by Steenbrink [43]

$$E_{f} \cap (0, 1] = \bigcup_{Q} E_{Q}^{\leq 1} \quad \text{with} \quad E_{Q}^{\leq 1} = \{L_{Q}(u) \mid u \in \{0\} \cup Q \cap \mathbb{Z}_{>0}^2\},$$

where $\{0\} \cup Q$ is the convex hull of $\{0\} \cup Q$. Here we have the symmetry of $E_{f}$ with center 1.

7.7. **Another comparison.** If $a = (x_{1}^{a_{1}}, \ldots, x_{n}^{a_{n}})$, then

$$R_{a} = \left\{ \sum_{i} p_{i} / a_{i} \mid 1 \leq p_{i} \leq a_{i} \right\}.$$

On the other hand, if $f = \sum_{i} x_{i}^{a_{i}}$, then

$$\tilde{R}_{f} = E_{f} = \left\{ \sum_{i} p_{i} / a_{i} \mid 1 \leq p_{i} \leq a_{i} - 1 \right\}.$$

8. Hyperplane arrangements.

8.1. Let $D$ be a central hyperplane arrangement in $X = \mathbb{C}^{n}$. Here, central means an affine cone of $Z \subset \mathbb{P}^{n-1}$. Let $f$ be the reduced equation of $D$ and $d := \deg f > n$. Assume $D$ is not the pull-back of $D' \subset \mathbb{C}^{n'} (n' < n)$.

8.2. **Theorem.** (i) $\max R_{f} < 2 - \frac{1}{d}$. (ii) $m_{1} = n$.

Proof of (i) uses a partial generalization of a solution of Aomoto's conjecture due to Esnault, Schechtman, Viehweg, Terao, Varchenko ([17], [40]) together with a generalization of Malgrange's formula (4.8) as below:
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8.3. Theorem (Generalization of Malgrange's formula) [36]. There exists a pole order filtration \( P \) on \( H^{n-1}(F_0, C)_\lambda \) such that if \( (\alpha + \mathbb{N}) \cap R'_f = \emptyset \), then

\begin{equation}
\alpha \in R'_f \iff \text{Gr}^p_{\lambda} H^{n-1}(F_0, C)_\lambda \neq 0,
\end{equation}

with \( p = [n - \alpha], \lambda = e^{-2\pi i \alpha} \), where \( R'_f = \bigcup_{x \neq 0} R_{f,x} \).

This reduces the proof of (8.2)(i) to

\begin{equation}
P^i H^{n-1}(F_0, C)_\lambda = H^{n-1}(F_0, C)_\lambda,
\end{equation}

for \( i = n - 1 \) if \( \lambda = 1 \) or \( e^{2\pi i /d} \), and \( i = n - 2 \) otherwise.

8.4. Construction of the pole order filtration \( P \). Let \( U = P^{n-1} \setminus Z \), and \( F_0 = f^{-1}(0) \subset C^n \). Then \( F_0 = \tilde{U} \) with \( \pi : \tilde{U} \to U \) a \( d \)-fold covering ramified over \( Z \). Let \( L^{(k)} \) be the local systems of rank 1 on \( U \) such that \( \pi_* C = \bigoplus_{0 \leq i < d} L^{(k)} \) and \( T \) acts on \( L^{(k)} \) by \( e^{-2\pi ik/d} \). Then

\begin{equation}
H^j(U, L^{(k)}) = H^j(F_0, C)_{e(-k/d)},
\end{equation}

and \( P \) is induced by the pole order filtration on the meromorphic extension \( L^{(k)} \) of \( L^{(k)} \otimes_C \mathcal{O}_U \) over \( P^{n-1} \), see [15], [36], [37]. This is closely related to:

8.5. Solution of Aomoto's conjecture ([17], [40]). Let \( Z_i \) be the irreducible components of \( Z \) \( (1 \leq i \leq d) \), \( g_i \) be the defining equation of \( Z_i \) on \( P^{n-1} \setminus Z_d \) \( (i < d) \), and \( \omega := \sum \alpha_i \omega_i \) with \( \omega_i = dg_i/g_i, \alpha_i \in C \). Let \( \nabla \) be the connection on \( \mathcal{O}_U \) such that \( \nabla u = du + \omega \wedge u \). Set \( \alpha_d = -\sum \alpha_i \). Then \( H^*_{\text{DR}}(U, (\mathcal{O}_U, \nabla)) \) is calculated by

\begin{equation}
(A^*_\alpha, \omega \wedge) \quad \text{with} \quad A^*_\alpha = \sum C \omega_{i_1} \wedge \ldots \wedge \omega_{i_p},
\end{equation}

if \( \sum_{Z_i \supset L} \alpha_i \notin N \setminus \{0\} \) for any dense edge \( L \subset Z \) (see (8.7) below). Here an edge is an intersection of \( Z_i \).

For the proof of (8.2)(ii) we have

8.6. Proposition. \( N^{n-1} \psi_{f,\lambda} C \neq 0 \) if \( \text{Gr}^W_{2n-2} H^{n-1}(F_z, C)_\lambda \neq 0 \).

(Indeed, \( N^{n-1} : \text{Gr}^W_{2n-2} \psi_{f,\lambda} C \xrightarrow{\sim} \text{Gr}^W_0 \psi_{f,\lambda} C \) by the definition of \( W \), and the assumption of (8.6) implies \( \text{Gr}^W_{2n-2} \psi_{f,\lambda} C \neq 0 \).)

Then we get (8.2)(ii), since \( \omega_{i_1} \wedge \ldots \wedge \omega_{i_{n-1}} \neq 0 \) in \( \text{Gr}^W_{2n-2} H^{n-1}(P^{n-1} \setminus Z, C) = \text{Gr}^W_{2n-2} H^{n-1}(F_z, C)_{\lambda} \).

8.7. Dense edges. Let \( D = \bigcup_i D_i \) be the irreducible decomposition. Then \( L = \bigcap_{i \in I} D_i \) is called an edge of \( D \) \( (I \neq \emptyset) \).

We say that an edge \( L \) is dense if \( \{D_i/L \mid D_i \supset L\} \) is indecomposable. Here \( C^n \supset D \) is called decomposable if \( C^n = C'^n \times C''^n \) such that \( D \) is the union of the pull-backs of \( C'^n \). \( C''^n \) with \( n', n'' \neq 0 \).

Set \( m_L = \# \{D_i \mid D_i \supset L\} \). For \( \lambda \in C \), define

\[\mathcal{D}\mathcal{E}(D) = \{\text{dense edges of } D\}, \quad \mathcal{D}\mathcal{E}(D, \lambda) = \{L \in \mathcal{D}\mathcal{E}(D) \mid \lambda^{m_L} = 1\}\]
We say that $L$, $L'$ are strongly adjacent if $L \subset L'$ or $L \supset L'$ or $L \cap L'$ is non-dense. Let
\begin{equation*}
m(\lambda) = \max\{|S| \mid S \subset \mathcal{D}(D, \lambda) \text{ such that any } L, L' \in S \text{ are strongly adjacent}\}.
\end{equation*}

8.8. Theorem [37]. $m_{\alpha} \leq m(\lambda)$ with $\lambda = e^{-2\pi i \alpha}$.

8.9. Corollary. $R_{f} \subset \bigcup_{L \in \mathcal{D}(D)} Zm_{L}^{-1}$.

8.10. Corollary. If $\text{GCD}(m_{L}, m_{L'}) = 1$ for any strongly adjacent $L, L' \in \mathcal{D}(D)$, then $m_{\alpha} = 1$ for any $\alpha \in R_{f} \backslash Z$.

Theorem 2 follows from the canonical resolution of singularities $\pi : (\tilde{X}, \tilde{D}) \to (\mathbb{P}^{n-1}, D)$ due to [40], which is obtained by blowing up along the proper transforms of the dense edges. Indeed, $\text{mult} \tilde{D}(\lambda)_{\text{red}} \leq m(\lambda)$, where $\tilde{D}(\lambda)$ is the union of $\tilde{D}_{i}$ such that $\lambda^{\tilde{m}_{i}} = 1$ and $\tilde{m}_{i} = \text{mult}_{\tilde{D}_{i}} \tilde{D}$.

8.11. Theorem (Mustaţă [29]). For a central arrangement,
\begin{equation*}
\mathcal{J}(X, \alpha D) = I_{0}^{k} \text{ with } k = [d\alpha] - n + 1 \text{ if } \alpha < \alpha'_{f},
\end{equation*}
where $I_{0}$ is the ideal of 0 and $\alpha'_{f} = \min_{x \neq 0} \{\alpha_{f,x}\}$.

(This holds for the affine cone of any divisor on $\mathbb{P}^{n-1}$, see [36].)

8.12. Corollary. We have $\dim F^{n-1} H^{n-1}(F_{0}, C)_{e(-k/d)} = \binom{k-1}{n-1}$ for $0 < \frac{k}{d} < \alpha'_{f}$, and the same holds with $F$ replaced by $P$.

8.13. Corollary. $\alpha_{f} = \min(\alpha'_{f}, \frac{n}{d}) < 1$.

(Note that $\alpha_{f}$ coincides with the minimal jumping number.)

8.14. Generic case. If $D$ is a generic central hyperplane arrangement, then
\begin{equation*}
b_{f}(s) = (s + 1)^{n-1} \prod_{j=n}^{2d-2} (s + \frac{i}{d})
\end{equation*}
by U. Walther [46] (except for the multiplicity of $-1$). He uses a completely different method.

Note that Theorems (8.2) and (8.8) imply that the left-hand side divides the right-hand side of (8.14.1), and the equality follows using also (8.12).

8.15. Explicit calculation. Let $\alpha = k/d$, $\lambda = e^{-2\pi i \alpha}$ for $k \in \{1, \ldots, d\}$. If $\alpha \geq \alpha'_{f}$, we assume there is $I \subset \{1, \ldots, d-1\}$ such that $|I| = k - 1$, and the condition of [40]
\begin{equation*}
\sum_{Z_{i} \supset L} \alpha_{i} \not\in \mathbb{N} \setminus \{0\} \text{ for any dense edge } L \subset Z,
\end{equation*}
is satisfied for
\begin{equation*}
\alpha_{i} = 1 - \alpha \text{ if } i \in I \cup \{d\}, \text{ and } -\alpha \text{ otherwise}.
\end{equation*}

Let $V(I)$ be the subspace of $H^{n-1} A_{c}$ generated by \[\omega_{i_{1}} \wedge \cdots \wedge \omega_{i_{n-1}} \text{ for } \{i_{1}, \ldots, i_{n-1}\} \subset I.\]

8.16. Theorem. Let $\alpha = k/d$, $\lambda = e^{-2\pi i \alpha}$ for $k \in \{1, \ldots, d\}$. Then
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(a) If \( k = d - 1 \) or \( d \), then \( \alpha \in R_f, \alpha + 1 \notin R_f \).
(b) If \( \alpha < \alpha_f \), then \( \alpha \in R_f \leftrightarrow k \geq d \).
(c) If \( \binom{k-1}{n-1} \) \( < \dim H^{n-1}(F_0, C)_{\lambda} \), then \( \alpha + 1 \notin R_f \).
(d) If \( \alpha < \alpha_f \), \( \alpha \notin R_f + \mathbb{Z} \) and \( \binom{k-1}{n-1} = \chi(U) \), then \( \alpha + 1 \notin R_f \).
(e) If \( \alpha \geq \alpha_f \) and \( V(I) \neq 0 \), then \( \alpha \in R_f \).
(f) If \( \alpha \geq \alpha_f \) and \( V(I) = H^{n-1}A_{\alpha}^{*} \), then \( \alpha + 1 \notin R_f \).

8.17. Theorem [37]. Assume \( n = 3 \), \( \text{mult}_z Z \leq 3 \) for any \( z \in Z \subset \mathbb{P}^2 \), and \( d \leq 7 \). Let \( \nu_3 \) be the number of triple points of \( Z \), and assume \( \nu_3 \neq 0 \). Then

\[
(8.17.1) \quad b_f(s) = (s + 1) \prod_{i=2}^{4}(s + \frac{1}{3}) \prod_{j=3}^{r}(s + id),
\]

with \( r = 2d - 2 \) or \( 2d - 3 \). We have \( r = 2d - 2 \) if \( \nu_3 < d - 3 \), and the converse holds for \( d < 7 \). In case \( d = 7 \), we have \( r = 2d - 3 \) for \( \nu_3 > 4 \), however, for \( \nu_3 = 4 \), \( r \) can be both \( 2d - 2 \) and \( 2d - 3 \).

8.18. Remarks. (i) We have \( \nu_3 < d - 3 \) if and only if

\[
(8.18.1) \quad \chi(U) = \frac{(d-2)(d-3)}{2} - \nu_3 > \frac{(d-3)(d-4)}{2} = \left( \frac{d-3}{2} \right).
\]

(ii) By (8.4.1) we have \( \chi(U) = h^2(F_0, C)_{\lambda} - h^1(F_0, C)_{\lambda} \) if \( \lambda^d = 1 \) and \( \lambda \neq 0 \).

(iii) Let \( \nu'_{i} \) be the number of \( i \)-ple points of \( Z' := Z \cap C^2 \). Then by [6]

\[
(8.18.2) \quad b_0(U) = 1, \quad b_1(U) = d - 1, \quad b_2(U) = \nu'_2 + 2\nu'_3,
\]

8.19. Examples. (i) For \( (x^2 - 1)(y^2 - 1) = 0 \) in \( \mathbb{C}^2 \) with \( d = 5 \), (8.17.1) holds with \( r = 7 \), and \( 8/5 \notin R_f \). In this case we do not need to take \( I \), because \( (d - 2)/d = 3/5 < \alpha_f = 2/3 \). We have \( b_1(U) = b_2(U) = 4 \) and \( h^2(F_0, C)_{\lambda} = \chi(U) = 1 \) if \( \lambda^5 = 1 \) and \( \lambda \neq 1 \). So \( j/5 \in R_f \) for \( 3 \leq j \leq 7 \) by (a), (b), (c), and \( 8/5 \notin R_f \) by (d).

(ii) For \( (x^2 - 1)(y^2 - 1)(x + 1)(y + 1) = 0 \) in \( \mathbb{C}^2 \) with \( d = 6 \), (8.17.1) holds with \( r = 9 \), and \( 10/6 \notin R_f \). In this case we have \( b_1(U) = 5, b_2(U) = 6, \chi(U) = 2, h^1(F_0, C)_{\lambda} = 1, h^2(F_0, C)_{\lambda} = 5 \) for \( \lambda = e^{\pm 2\pi i/3} \). Then \( 4/6 \in R_f \) by (e) and \( 10/6 \notin R_f \) by (f), where \( I^c \) corresponds to \( (x + 1)(y + 1) = 0 \). For other \( j/6 \), the argument is the same as in (i).

(iii) For \( (x^2 - y^2)(x^2 - 1)(y^2 + 1) = 0 \) in \( \mathbb{C}^2 \) with \( d = 6 \), (8.17.1) holds with \( r = 10 \), and \( 10/6 \in R_f \). In this case we have \( b_1(U) = 5, b_2(U) = 9, \chi(U) = 5, h^1(F_0, C)_{\lambda} = 0, h^2(F_0, C)_{\lambda} = 5 \) for \( \lambda = e^{\pm 2\pi i/3} \). Then \( 4/6 \in R_f \) by (e) and \( 10/6 \in R_f \) by (c), where \( I^c \) corresponds to \( (x + 1)(y + 2) = 0 \).

(iv) For \( (x^2 - y^2)(x^2 - 1)(y^2 - 1) = 0 \) in \( \mathbb{C}^2 \) with \( d = 7 \), (8.17.1) holds with \( r = 11 \), and \( 12/7 \notin R_f \). In this case we have \( b_1(U) = 6, b_2(U) = 9, \chi(U) = 4, h^2(F_0, C)_{\lambda} = 4 \) if \( \lambda^7 = 1 \) and \( \lambda \neq 1 \). Then \( 5/7 \in R_f \) by (e) and \( 12/7 \notin R_f \) by (f), where \( I^c \) corresponds to \( (x + 1)(y + 1) = 0 \). Note that \( 5/7 \) is not a jumping number.
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