THE MINIMAL LOG DISCREPANCY

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CONTENTS

INTRODUCTION

This note is a quick introduction to the *minimal log discrepancy*, a local invariant of *log varieties*. This fundamental invariant is ubiquitous in the birational classification of algebraic varieties. First introduced by Shokurov in connection to the termination of a sequence of flips, it has appeared in the local context of the classification of singularities, or the global context of Fujita's conjecture on adjoint linear systems. We present some of the basic open problems on minimal log discrepancies, and illustrate them with toric examples.

The plan of this note is as follows. In §1, we recall the construction of canonical models and discrepancies, and its logarithmic version. This seems to us the natural motivation for log varieties with *log canonical singularities*, since locally they are just open subsets of log canonical models. We give the rigorous definition of log varieties and minimal log discrepancies in §2, and present explicit combinatorial formulas for minimal log discrepancies of toric log varieties. We present some of the basic problems on minimal log discrepancies in §3, discuss their toric case and some methods, old and new.

1. BACKGROUND ON LOG CANONICAL MODELS

1-A. Canonical models, discrepancies. Let X be a complex projective manifold of general type, with *canonical divisor* K_X. The *canonical ring* $R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, mK_X)$ is expected to be finitely generated, and if it is, we would obtain a natural birational map

$$\Phi \colon X \dashrightarrow Y := \operatorname{Proj}(R(X, K_X)).$$

The birational model Y is called the *canonical model* of X. It depends only on the birational class of X and it has a canonical polarization, but it is singular in general. For example, Y may have some Du Val singularities in dimension two. The singularities that may appear on Y are called *canonical singularities*, introduced by Reid [26].

To get to the formal definition of canonical singularities, let us take a closer look at what Φ does for K_X. By Hironaka's resolution of singularities, there exist a *Hironaka hut*

$$
\begin{array}{ccc}
 & X' & \\
 {}^{f}\swarrow & & \searrow^{g} \\
X & \cdots\cdots\xrightarrow{\ \Phi\ }\cdots\cdots & Y
\end{array}
$$

that is X' is a projective manifold, f, g are birational morphisms and $\Phi = g \circ f^{-1}$. By definition, K_X is the divisor (ω) of zeros and poles of a non-zero top rational differential form $\omega \in \wedge^{\dim(X)}\Omega^1_X \otimes_{\mathbb{C}} \mathbb{C}(X)$. Denote $K_{X'} = (f^*\omega)$ and $K_Y = (g_* f^*\omega)$. The latter is a well defined Weil divisor, since Y is normal. Since X has no singularities, the divisor $A_f = K_{X'} - f^*(K_X)$ is effective and supported by the exceptional locus of f. Equivalently, the natural map $f_*\colon R(X', K_{X'}) \to R(X, K_X)$ is an isomorphism. In particular, $g\colon X' \to Y$ is the canonical model of X'. Since g is a morphism and $K_{X'}$ is a big divisor, it follows that there exists $m \in \mathbb{Z}_{\geq 1}$ such that mK_Y is a very ample divisor, and $A_g = \frac{1}{m}(mK_{X'} - g^*(mK_Y))$ is effective and supported by the exceptional locus of g. In particular, $g_*\colon R(X', K_{X'}) \to R(Y, K_Y)$ is also an isomorphism:

$$
\begin{array}{ccc}
 & R(X', K_{X'}) & \\
 {}^{\simeq}\swarrow & & \searrow^{\simeq} \\
R(X, K_X) & & R(Y, K_Y)
\end{array}
$$

Reid [26] called a normal germ $P \in Y$ a canonical singularity if A_g is well defined and effective, for a resolution of singularities $g\colon X' \to Y$. The coefficients of the $\mathbb{Q}$-divisor A_g are called *discrepancies*. To understand discrepancies in terms of the manifolds that we started with, we go back to our global setting and note that

$$
K_{X'} = g^*(K_Y) + A_g
$$

is a Zariski decomposition of $K_{X'}$, with positive part $g^*(K_Y)$ and fixed part A_g. Since $|mK_Y|$ defines a linear system free of base points, mA_g coincides with the fixed divisor of the linear system $|mK_{X'}|$. Finally, it turns out that $A_g - A_f$ is effective, and $f^*(K_X) = g^*(K_Y) + (A_g - A_f)$ is a Zariski decomposition of $f^*(K_X)$.

1-B. Log canonical models of open varieties. Let U be a complex quasi-projective manifold of general type, in the sense of Iitaka [13]. By Hironaka's resolution of singularities, there exists an open embedding $U \subset X$ such that X is a proper manifold, and the complement $X \setminus U = \sum_i E_i$ is a divisor with simple normal crossings. The general type assumption means that the *log canonical divisor* $K_X + \sum_i E_i$ is big. The *log canonical ring*

$$
R(X, K + \sum_i E_i) = \bigoplus_{m \in \mathbb{N}} H^0(X, m(K_X + \sum_i E_i))
$$

is independent of the choice of compactification, and in fact depends only on the (proper) birational class of U. This ring is expected to be finitely generated, and if it is, we would

obtain a natural birational map

$$\Phi \colon X \dashrightarrow Y := \mathrm{Proj}(R((X, K + \sum_i E_i)).$$

As before, we can find a Hironaka hut with the extra property that $\mathrm{Exc}(f) \cup (f^{-1})_*(\sum_i E_i)$ is a simple normal crossings divisor $\sum_{i'} E_{i'}$. Denote $B_Y = g_*(\sum_{i'} E_{i'})$. We can imitate the arguments in the compact case, and obtain isomorphisms

$$R(X', K_{X'} + \sum_{i'} E_{i'})$$

$$\stackrel{\simeq}{\swarrow} \qquad\qquad \stackrel{\simeq}{\searrow}$$

$$R(X, K_X + \sum_i E_i) \qquad\qquad\qquad R(Y, K_Y + B_Y)$$

Again, there exists $m \in \mathbb{Z}_{\geq 1}$ such that $m(K_Y + B_Y)$ is a very ample divisor, and we have Zariski decompositions $K_{X'} + \sum_{i'} E_{i'} = g^*(K_Y + B_Y) + A_g$ and $f^*(K_X + \sum_i E_i) = g^*(K_Y + B_Y) + (A_g - A_f)$. One can see that Φ^{-1} contracts no divisors of Y, and $\Phi_*(\sum_i E_i) = B_Y$. The pair (Y, B_Y) is log canonically polarized, and it's singularities are log canonical, as we will see shortly. The pair (Y, B_Y) is called the *log canonical model* of U.

1-C. Log canonical models of log manifolds.

Log manifolds provide the natural bridge between open and compact manifolds. By definition, they are pairs $(X, \sum_i b_i E_i)$, where X is nonsingular, the E_i's are nonsingular divisors intersecting transversely, and $b_i \in [0, 1] \cap \mathbb{Q}$ for all i. We call $\sum_i b_i E_i$ the boundary of the log manifold, and denote it by B. Suppose moreover that (X, B) is of log general type, that is the log canonical divisor $K_X + B$ is big. The log canonical ring $R(X, B) = \bigoplus_{m \in \mathbb{N}} H^0(X, m(K_X + B))$ is expected to be finitely generated, and if it is, we obtain a birational map

$$\Phi \colon X \dashrightarrow Y := \mathrm{Proj}(R(X, B)).$$

Again, we construct a Hironaka hut with the extra property that $\mathrm{Exc}(f) \cup (f^{-1})_*(\sum_i E_i)$ is a simple normal crossings divisor. Let $\cup_j F_j$ be the exceptional locus of f and denote $B_Y = g_*((f^{-1})_* B + \sum_j F_j)$. We imitate the previous argument, and obtain isomorphisms

$$R(X', K_{X'} + (f^{-1})_* B + \sum_j F_j)$$

$$\stackrel{\simeq}{\swarrow} \qquad\qquad \stackrel{\simeq}{\searrow}$$

$$R(X, K_X + B) \qquad\qquad\qquad R(Y, K_Y + B_Y)$$

Again, there exists $m \in \mathbb{Z}_{\geq 1}$ such that $m(K_Y + B_Y)$ is a very ample divisor, and we have Zariski decompositions

$$K_{X'} + (f^{-1})_* B + \sum_j F_j = g^*(K_Y + B_Y) + A_g$$

$$f^*(K_X + B) = g^*(K_Y + B_Y) + (A_g - A_f).$$

One can also see that Φ^{-1} contracts no divisors of Y, and $\Phi_*(B) = B_Y$. The birational model $\Phi \colon (X, B) \dashrightarrow (Y, B_Y)$ is called the *log canonical model* of (X, B). It is polarized by the *log canonical* $\mathbb{Q}$-*divisor* $K_Y + B_Y$, and its singularities are called *log canonical singularities*.

2. Log varieties, minimal log discrepancies

Log varieties with log canonical singularities are objects which locally are open subsets of canonical models of log manifolds of general type. For technical purposes, it is better to work in a slightly more general context, such as non-rational boundaries (to be able to take limits of log divisors), or even non-log canonical singularities (when "building a log canonical center" at a prescribed point).

Definition 2.1. A *log variety* (X, B) is a complex normal variety X endowed with an effective $\mathbb{R}$-Weil divisor $B = \sum_i b_i E_i$ such that $K_X + B$ is $\mathbb{R}$-Cartier.

Recall that the canonical divisor $K_X = (\omega)$ is the Weil divisor of zeros and poles of a non-zero top rational differential form ω (it depends on the choice of ω, but only up to linear equivalence). The E_i's are prime divisors and the b_i's are non-negative real numbers. The $\mathbb{R}$-Cartier assumption means that locally on X, $K_X + B$ equals a finite sum $\sum_i r_i(\varphi_i)$, where $r_i \in \mathbb{R}$ and $\varphi_i \in \mathbb{C}(X)^\times$.

Let now $\mu \colon X' \to X$ be birational morphism, and $E \subset X'$ a prime divisor. We use the same form to define the canonical class of X', that is $K_{X'} = (f^*\omega)$. The *log discrepancy* of (X, B) at E is defined as

$$a(E; X, B) = \mathrm{mult}_E(K_{X'} + E - \mu^*(K_X + B)) \in \mathbb{R}.$$

The log discrepancy depends only on the valuation that E induces on $\mathbb{C}(X)$. We call such valuations *geometric*, and denote $c_X(E) = \mu(E)$. For example, if E is a prime divisor in X, then $a(E; X, B) = 1 - \mathrm{mult}_E(B)$.

Definition 2.2. A log variety (X, B) has *log canonical singularities* if $a(E; X, B) \geq 0$ for every geometric valuation E of X.

Log canonicity involves all geometric valuations, but it may be checked at only finitely many valuations. Indeed, by Hironaka's resolution of singularities, we may find a birational morphism $\mu \colon X' \to X$ such that X' is nonsingular, and $(\mu^{-1})_*(\cup_i E_i) \cup \cup_j F_j$ is a simple normal crossings divisor, where $\mathrm{Exc}(\mu) = \cup_j F_j$. Then (X, B) is log canonical if and only if the $a(E_i; X, B) \geq 0$ for all i (that is $b_i \leq 1$ for all i) and $a(F_j; X, B) \geq 0$ for all j. If this is the case, the formula

$$K_{X'} + (\mu^{-1})_*B + \sum_j F_j = \mu^*(K_X + B) + \sum_j a(F_j; X, B)F_j.$$

becomes a Zariski decomposition of the log manifold of relative general type $(X', (\mu^{-1})_*B + \sum_j F_j) \to X$.

Example 2.3. Let X be a manifold, and $\sum_i E_i$ a simple normal crossings divisor. Then $(X, \sum_i b_i E_i)$ is a log variety if $b_i \geq 0$ for all i. It has log canonical singularities if and only if $b_i \in [0, 1]$ for all i.

Example 2.4. Let X be a toric variety and $X \setminus T = \cup_i E_i$ the complement of the torus. Then $(X, \sum_i E_i)$ is a log variety with log canonical singularities, and $K_X + \sum_i E_i = 0$.

Definition 2.5. The *minimal log discrepancy* of a log variety (X, B) at a Grothendieck point $\eta \in X$ is defined as

$$a(\eta; X, B) = \inf\{a(E; X, B); c_X(E) = \bar{\eta}\}$$

If (X, B) does not have log canonical singularities at η, then $a(\eta; X, B) = -\infty$. Otherwise, $a(\eta; X, B)$ is a non-negative real number. Again, it can be computed in finite time, on a log resolution $\mu\colon X' \to X$ such that $\mu^{-1}(\bar{\eta})$ is a divisor, and $\mu^{-1}(\bar{\eta}), (\mu^{-1})_*B, \mathrm{Exc}(\mu)$ are all suported by a simple normal crossings divisor. In particular, $na(P; X, B) \in \mathbb{Z}$ if $n(K + B)$ is a Cartier diivsor.

Example 2.6. For a nonsingular point $P \in X$, $a(P; X) = \dim(X)$.

2-A. Examples of minimal log discrepancies. *Toric log varieties*, log varieties (X, B) such that X is a toric variety and B is supported by the complement of the torus, are a special class of log varieties for which minimal log discrepancies can be easily computed. We only consider here $\mathbb{Q}$-factorial, log canonical toric germs of log varieties

$$P \in (X, B) = (T_N \, \mathrm{emb}(\sigma), \sum_{i=1}^{d} b_i H_i).$$

They are in one-to-one correspondence with the following data:

- $\sigma = \{x \in \mathbb{R}^d; x_1, \ldots, x_d \geq 0\}$.
- $N \subset \mathbb{R}^d$ is a lattice, containing $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ as primitive vectors.
- $(b_1, \ldots, b_d) \in [0, 1]^d$.

The following basic facts provide lots of examples of minimal log discrepancies:

(a) $a(\eta_{H_i}; X, B) = 1 - b_i$.

(b) Let $x \in N^{prim} \cap \sigma$ be a primitive vector. Then x defines a barycentric subdivision Δ_x of σ, and the exceptional locus of the induced birational map $T_N \, \mathrm{emb}(\Delta_x) \to T_N \, \mathrm{emb}(\sigma)$ is a prime divisor E_x. Then $a(E_x; X, B) = \sum_{i=1}^{d}(1 - b_i)x_i$.

(c) Log resolutions exists in the toric category. Therefore minimal log discrepancies can be computed using only valuations E_x as in (b).

(d) The point P is the unique fixed point of the torus action. Its minimal log discrepancy is computed as follows

$$a(P; X, B) = \min\{\sum_{i=1}^{d}(1 - b_i)x_i; x \in N \cap \mathrm{int}(\sigma)\}.$$

(e) Let $P \in C \subset X$ be the toric cycle corresponding to a face $\tau \prec \sigma$. The minimal log discrepancy at its generic point is

$$a(\eta_C; X, B) = \min\{\sum_{i=1}^{d}(1 - b_i)x_i; x \in N \cap \mathrm{relint}(\tau)\}.$$

(f) The global minimal log discrepancy $a(X, B)$ is defined as the smallest log discrepancy of (X, B). It is computed as follows

$$a(X, B) = \min\{\sum_{i=1}^{d}(1 - b_i)x_i; x \in N \cap \sigma \setminus 0\}.$$

(g) In all minimums above, it suffices to consider only the finitely many lattice points $x \in N \cap [0, 1]^d$.

Example 2.7. Suppose $N = \mathbb{Z}^d$, that is $X = \mathbb{C}^d$ and the H_i's are the coordinate hyperplanes. For the cycle $C : (x_1 = \cdots = x_s = 0)$, we have $a(\eta_C; X, B) = s - b_1 - \cdots - b_s$.

Example 2.8. Suppose $B = 0$. Since σ is fixed, only the lattice N varies.

(i) Take $N = \mathbb{Z}^2 + \mathbb{Z}(\frac{1}{q}, \frac{q-1}{q})$, for some integer $q \geq 2$. The surface germ $P \in X$ is a A_{q-1}-singularity. We compute $N \cap (0,1]^2 = \{(\frac{k}{q}, \frac{q-k}{q}); 1 \leq k \leq q-1\} \cup \{(1,1)\}$, so $a(P; X) = 1$.

(ii) Take $N = \mathbb{Z}^2 + \mathbb{Z}(\frac{1}{k}, \frac{1}{k})$, for some positive integer k. As above, we compute $a(P; X) = \frac{2}{k}$.

(iii) Take $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{q}(1, p, q - p)$, where p, q are integers with $1 \leq p \leq q - 1, \gcd(p, q) = 1$. Then $P \in X$ is a terminal 3-fold singularity, with $a(P; X) = 1 + \frac{1}{q}$.

(iv) Take $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{2q}(1, q, 1 + q)$, with $q \geq 1$. Then $P \in X$ is a 3-fold singularity with $a(P; X) = 1 + \frac{1}{q}$. This germ has the minimal log discrepancy of a terminal singularity, but it's not terminal, since it is not an isolated singularity. The singular locus of X is $C_2 : (x_1 = x_3 = 0)$, and $a(\eta_{C_2}; X) = \frac{2}{q}$.

Minimal log discrepancies of toric varieties are related to lattice-point-free convex bodies. To see this, consider the simplex $\Delta = \{x \in \mathbb{R}^d; x_1, \ldots, x_d \geq 0, \sum_{i=1}^{d}(1 - b_i)x_i \leq 1\}$. Then $a(P; X, B) = \inf\{t \in \mathbb{R}_{\geq 0}; N \cap \text{int}(t\Delta) \neq \emptyset\}$.

3. PROBLEMS ON MINIMAL LOG DISCREPANCIES

Minimal log discrepancies originate in the problem of the termination of log flips: starting with a given log variety, can we perform log flips infinitely many times? Log flips are surgery operations which preserve codimension 1 cycles, and improve the singularities of higher codimensional cycles. As a measure of this improvement, log discrepancies may only increase after a log flip, and some of them increase strictly. This has been the heuristic behind the termination of a sequence of log flips, and it lead Shokurov [27] to question the existence of an infinite increasing sequence of minimal log discrepancies.

First, we fix a log variety (X, B), and investigate the set of minimal log discrepancies of all cycles of X [4]. The basic formula $a(\eta_C; X, B) = a(P; X, B) - \dim(C)$, for a general closed point P on a cycle $C \subset X$, shows that closed points contain the essential information. Consider now the minimal log discrepancy $a(P; X, B)$ as a function on the set of closed points $P \in X$. This function has a finite image, and in particular the set of minimal log discrepancies of all cycles of X is *finite*. Moreover, the level sets $\{P \in X; a(P; X, B) \leq t\}$ $(t \geq 0)$ are constructible. Simple examples, such as a Du Val singularity $P \in X$, with $a(x; X) = 2$ for $x \neq P$, and $a(P; X) = 1$, suggest that these level sets are in fact closed.

Conjecture 3.1 ([3]). *The minimal log discrepancy $a(P; X, B)$ is lower semi-continuous as a function on the closed points P of X.*

This behaviour is confirmed in several special cases: a) $\dim(X) \leq 3$ [3, 4]; b) (X, B) is a toric log variety [4]; c) X is a local complete intersection [11, 10]. Also, it is equivalent to the inequality $a(P; X, B) \leq a(\eta_C; X, B) + 1$, for every closed point on a curve in X [4].

Now consider the general case, when log flips change the log variety (X, B) in codimension at least 2. The coefficients of the boundary are preserved, so we may assume that they belong to a given finite set. More generally, let $\mathcal{B} \subset [0, 1]$ be a set satisfying the descending chain condition $(\mathcal{B} = \{1 - \frac{1}{n}; n \geq 1\} \cup \{1\}$ is a typical example), and define

$$\text{Mld}(d, \mathcal{B}) = \{a(P; X, B); \dim(X) = d, \text{ coefficients of } B \text{ belong to } \mathcal{B}\}.$$

The set $\mathrm{Mld}(1,\mathcal{B}) = \{1 - b; b \in \mathcal{B}\}$ clearly satisfies the ascending chain condition.

Conjecture 3.2 (Shokurov [27]). *The following properties hold:*

(1) $\mathrm{Mld}(d,\mathcal{B})$ *satisfies the ascending chain condition.*

(2) $a(P;X,B) \leq \dim(X)$. *Moreover, if* $a(P;X,B) > \dim(X) - 1$, *then* $P \in X$ *is a nonsingular point and* $a(P;X,B) = \dim(X) - \mathrm{mult}_P(B)$.

(3) *Assume* $\mathcal{B} \cap [0, 1 - \frac{1}{n}]$ *is a finite set for every* $n \geq 2$. *Then the accumulation points of* $\mathrm{Mld}(d,\mathcal{B})$ *are included in* $\mathrm{Mld}(d - 1,\mathcal{B}')$, *for a suitable set* $\mathcal{B}'$.

This conjecture was confirmed for surfaces [28, 1], and toric log varieties [7, 5]. By the classification of terminal 3-fold singularities, $\mathrm{Mld}(3,\{0\}) \cap (1,+\infty) = \{1 + \frac{1}{q}; q \geq 1\} \cup \{3\}$ [17, 22]. Also, (2) holds if X is a local complete intersection [11, 10]. Recently, Shokurov [31] reduced the termination of a sequence of log flips to the lower semi-continuity and ascending chain condition of minimal log discrepancies.

Another interesting problem, called precise inversion of adjunction, is to compare minimal log discrepancies under adjunction.

Conjecture 3.3 (Shokurov [29], Kollár [18]). *Let* $P \in S \subset (X,B)$ *be the germ of a log variety and a normal prime divisor* S *with* $\mathrm{mult}_S(B) = 1$. *By adjunction, we have* $(K_X + B)|_S = K_S + B_S$. *Then* $a(P;X,B) = a(P;S,B_S)$.

This formula is useful in inductive arguments in the log category. It follows from the Log Minimal Model Program if $a(P;X,B) \leq 1$ [18], and it holds if X is a local complete intersection [11, 10].

Another interesting local question posed by Shokurov is the relationship between minimal log discrepancy and the index of a singularity. Suppose $P \in X$ is the germ of a d-fold with log canonical singularities. If $nK_X \sim 0$ and Conjecture 3.2.(2) holds, then the minimal log discrepancy can take at most finitely many values: $a(P;X) \in \{0, \frac{1}{n}, \ldots, \frac{nd}{n}\}$. Conversely, does there exists an integer n, depending only on d and $a(P;X)$, such that $nK_X \sim 0$? The answer is positive if $d = 2$ (Shokurov, unpublished). Also, suppose $a(P;X) = 0$. If $d = 2$, then $n \in \{1,2,3,4,6\}$ [29]. If $d = 3$, then $\varphi(n) \leq 20$ and $n \neq 60$, where φ is the Euler number [14]. See also [12] for a higher dimensional reduction to a global problem on Calabi-Yau varieties in one dimension less.

Minimal log discrepancies also appear in global contexts, such as Fujita's Conjecture on adjoint linear systems. Another global problem is to bound Fano varieties in terms of its minimal log discrepancies.

Conjecture 3.4 (Alexander and Lev Borisov [6]-Alexeev [2]). *Let* $\epsilon \in (0,1]$ *and* $d \in \mathbb{Z}_{\geq 1}$. *Then log Fano d-folds, with log discrepancies at least* ϵ, *form a bounded family.*

This conjecture is known in several cases: a) X is toric [6]; b) X nonsingular [19]; c) $d = 2$ [2]; d) $d = 3$, $\epsilon = 1$ [16, 20]; e) $d = 3$, and the index of K_X is fixed [9].

3-A. Toric case. In the assumptions and notations of § 2-A, we illustrate some of the local problems on minimal log discrepancies. For *lower semi-continuity*, it is enough to see that $a(P;X,B) \leq a(\eta_C;X,B) + 1$ for a torus-invariant curve $P \in C$. Suppose C corresponds to the face $\tau = \sigma \cap (x_d = 0)$. There exists $(x',0) \in N^{prim} \cap \mathrm{relint}(\sigma)$ such that $a(\eta_C;X,B) = \sum_{i=1}^{d-1}(1 - b_i)x_i$. Then $(x',1) \in N \cap \mathrm{int}(\sigma)$ and there exists $x \in N^{prim}$

and a positive integer $m \geq 1$ with $mx = (x', 1)$. We have

$$a(E_x; X, B) \leq ma(E_x; X, B) = \sum_{i=1}^{d-1}(1 - b_i)x'_i + 1 - b_d \leq a(\eta_C; X, B) + 1.$$

Therefore $a(P; X, B) \leq a(\eta_C; X, B) + 1$.

For *precise inversion of adjunction*, suppose $B = \sum_{i=1}^{d-1} b_i H_i + H_d$. Then $S = H_d$ is the toric variety T_{N_d} emb(σ_d), where $\sigma_d = \{x \in \mathbb{R}^{d-1}; x_1, \ldots, x_{d-1} \geq 0\}$ and $N_d = \{x \in \mathbb{R}^{d-1}; \exists t \in \mathbb{R}, (x, t) \in N\}$. To bring this to the normal form in § 2-A, note that there are positive integers $n_1, \ldots, n_{d-1}$ such that $\frac{1}{n_i}(0, \ldots, \overset{i}{1}, \ldots, 0)$ are primitive vectors of N_d^{prim}. Then $S = T_{N'}$ emb(σ'), where $N' = \{x' \in \mathbb{R}^{d-1}; (n_1 x'_1, \ldots, n_{d-1}x'_{d-1}) \in N_d\}$ and σ' is the usual positive cone. Let $H'_1, \ldots, H'_{d-1}$ the torus invariant prime divisors of S. The key observation is that the log canonical divisor $K + B = \sum_{i=1}^{d-1} -(1 - b_i)H_i$ is independent of H_d. It follows that the boundary of S induced by adjunction is $B_S = \sum_{i=1}^{d-1}(1 - \frac{1-b_i}{n_i})H'_i$, and the equality $a(P; X, B) = a(P; S, B_S)$ is clear.

Finally, for the *ascending chain condition*, assume by contradiction that we have a strictly increasing sequence $a^1 < a^2 < a^3 < \cdots$, where $a^n = a(P^n; T_{N^n}$ emb$(\sigma))$ for $n \geq 1$. For simplicity, we assume that the boundary is zero, so only the lattice changes. We may find $x^n \in (0, 1]^d \cap N^n$ such that $a^n = \sum_{i=1}^d x_i^n$. In particular, $a^n \leq d$ for all n. Consider now the strictly increasing sequence of open sets

$$U^n = \{x \in (0, +\infty)^d; \sum_{i=1}^d x_i < a^n\}.$$

By [21], $G^n = \{x \in \mathbb{R}^d; U^n \cap (\mathbb{Z}^d + \mathbb{Z}x) = \emptyset\}$ is the union of finitely many closed subgroups containing $\mathbb{Z}^d$ (the Flatness Theorem of Khinchin [15] gives an alternative proof). We have $G^n \supsetneq G^{n+1}$ since $a^n < a^{n+1}$ and $x^n \in G^n \setminus G^{n+1}$, so we obtain a strictly decreasing sequence of finite unions of closed subgroups containing $\mathbb{Z}^d$. This is impossible, since the set of finite unions of closed subgroups containing $\mathbb{Z}^d$ satisfies the descending chain condition.

3-B. Methods.

The toric case (see also [23, 8]) suggests that behind the ascending chain condition of minimal log discrepancies lies a deeper fact, *the boundedness of singularities with minimal log discrepancy bounded away from zero*. Some log canonical singularities are classified in low dimension, but in general we could only expect general structure theorems and boundedness results in terms of minimal log discrepancies. For example, Du Val singularities are classified as follows: A_n, D_n, E_6, E_7, E_8. From the above point of view, Du Val singularities are nothing but surface singularities having minimal log discrepancy at least 1, and they come in two types: a 1-dimensional series with two components (A and D), and a 0-dimensional series (E).

The known method for bounding germs $P \in X$ is to study the singularities at P of the linear systems $|mK|$ ($m < 0$), and reduce this local problem to the global problem of bounding log Fano or log Calabi-Yau varieties in one dimension less [30, 25]. Given that minimal log discrepancies are actually invariants objects of general type, as §1 suggests, it also seems natural to investigate the singularities at P of the linear systems $|mK|$ ($m > 0$), and relate germs with log canonical models in one dimension less.

THE MINIMAL LOG DISCREPANCT

Finally, it is likely that minimal log discrepancies can be understood from several points of view: analytic, birational, motivic or p-adic. The motivic interpretation of minimal log discrepancies is known in the case when the canonical divisor is Q-Cartier [24, 32]. As for the analytic side, the description of log discrepancies as the coefficients of a Zariski decomposition suggests an interpretation of minimal log discrepancies in terms of Lelong numbers. For example, the bound of Conjecture 3.2.(2) is equivalent to the following problem. Suppose X is a projective manifold of general type which admits a Zariski decomposition $K_X = P + F$ such that the fixed part F has a support with simple normal crossings. Then some coefficient of F is at most $\dim(X)$.

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