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移動中の感染と Phase-Compartmental Model

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1. Introduction

Transportation is considered as one of main factors that cause the outbreak of diseases, because we have a good example, SARS, which broke out with some infection in an airplane. If we remember correctly, there was one person infected with SARS and 9 people around the man were infected during transportation. SARS broke out with such kind of situation caused by transport-related infection. That may lead to importance to provides a mathematical groundwork for discussing the transport-related infection. In this paper, we propose a phase-compartmental model that can be basic, simple, and also mathematically tractable for the transport-related infection.

To consider the effect of transport-related infection, [1] proposed a two-city model where a population is divided into City 1 and City 2 with the same transportation rate. The model was the following:

$$
\begin{aligned}
S_1' &= a - \frac{\beta S_1 I_1}{S_1 + I_1} - bS_1 + dI_1 - \alpha S_1 + \alpha S_2 - \frac{\gamma \alpha S_2 I_2}{S_2 + I_2}, \\
I_1' &= \frac{\beta S_1 I_1}{S_1 + I_1} - (c + d + \alpha) I_1 + \alpha I_2 + \frac{\gamma \alpha S_2 I_2}{S_2 + I_2}, \\
S_2' &= a - \frac{\beta S_2 I_2}{S_2 + I_2} - bS_2 + dI_2 - \alpha S_2 + \alpha S_1 - \frac{\gamma \alpha S_1 I_1}{S_1 + I_1}, \\
I_1' &= \frac{\beta S_2 I_2}{S_2 + I_2} - (c + d + \alpha) I_2 + \alpha I_1 + \frac{\gamma \alpha S_1 I_1}{S_1 + I_1}.
\end{aligned}
$$

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S_1 and S_2 represent susceptibles in City 1 and City 2, respectively. I_1 and I_2 represent infectives in City 1 and City 2, respectively. For simplicity, the model assumed the same parameters between the two cities and the constant birth rate a for susceptibles. It was also assumed to be infection by the standard incidence shown here, death rates b and c for susceptibles and infectives, respectively, and recovering rate, d. But, there are some problems for modeling not so good on the transport-related infection. The transport-related infection was expressed in the last parts of the model, $\alpha S_2 - \frac{\gamma\alpha S_2 I_2}{S_2+I_2}$, $\alpha I_2 + \frac{\gamma\alpha S_2 I_2}{S_2+I_2}$, $\alpha S_1 - \frac{\gamma\alpha S_1 I_1}{S_1+I_1}$, and $\alpha I_1 + \frac{\gamma\alpha S_1 I_1}{S_1+I_1}$.

If we capture transport-related infection in a precise and strict way, we need some time span for transportation. Because, we use ordinary differential equations for modeling and we suppose to assume implicitly that the transportation occurs at an instantaneous time. It is clearly impossible to capture transport-related infection instantaneously. That's why, strictly speaking, some time span has to be considered for transportation.

Let τ denote the time span of transportation. Then, susceptibles S and infectives I in transportation are modeled as

$$S' = -\frac{\gamma SI}{S+I} \qquad I' = \frac{\gamma SI}{S+I}, \tag{1}$$

where γ is transport-related infection rate. It is natural to assume no birth and no death in transportation (for example, in airplanes). Solving these equations with initial data $\alpha S_i(t-\tau)$ and $\alpha I_i(t-\tau)$ tells us that there is too much approximation on transport-related infection in the model. In fact, when τ is equal to 0, it is easy to see that there is quite difference between the terms resulting from (1) and tranport-related infection terms given in the model. That is the point which should be improved in this paper.

2. Our model — a phase-compartmental model

Change the point of view for transport-related infection. Roughly speaking, it is one of natural ways to think that a population is divided into people who travel and people who do not travel. We now propose a model of population divided into traveling phase and non-traveling phase as follows:

$$
\begin{aligned}
S_1' &= -\frac{\gamma_1 S_1 I_1}{S_1 + I_1} + \alpha_S S_2 - \beta_S S_1, \\
I_1' &= \frac{\gamma_1 S_1 I_1}{S_1 + I_1} + \alpha_I I_2 - \beta_I I_1, \\
S_2' &= B(N) S_2 - \frac{\gamma_2 S_2 I_2}{S_2 + I_2} + \beta_S S_1 - \alpha_S S_2 + \mu I_2, \\
I_2' &= \frac{\gamma_2 S_2 I_2}{S_2 + I_2} + \beta_I I_1 - (\alpha_I + \mu + D) I_2.
\end{aligned} \tag{2}
$$

S_1 and I_1 represent susceptibles and infectives in traveling phase, respectively. On the other hand, S_2 and I_2 represent susceptibles and infectives in non-traveling phase, respectively. Note that α_S, β_S, α_I, and β_I are not the transportation rates but the phase-changing rates of population. α_S and β_S are parameters representing the changing rates of susceptibles between traveling phase and non-traveling phase. Also, α_I and β_I are parameters representing the changing rates of infectives between the two phases. γ_1 and γ_2 are infection rates in traveling-phase and non-traveling-phase, respectively.

We assume no birth and no death in traveling phase because it is natural to think that nobody has a baby and nobody dies, for example, in an airplane. This is a quite different point from well known compartmental population models (*i.e.* geographically divided compartment models). For non-traveling phase, however, we have to consider a population growth rate $B(N)$. The growth rate $B(N)$ is assumed to be differentiable and have the density dependence as the derivative of $B(N)$ is negative. Furthermore, $B(N)$ is assumed to be expressed as $B = B^+ - B^-$ where B^+ and B^- are positive functions of N, which is some technical assumption but has little restriction on a biological sense. Also, we consider the death rate and recovery of infectives in non-traveling phase, expressed by D and μ, respectively (We do not consider the disease recovery in traveling phase, which should be neglected as no birth and no death are assumed in traveling).

It may be thought that our model (2) has the same framework as ever well-known compartment models including the population model mentioned before [1-3, and references cited therein]. But, we notice that this model is not that kind of compartment population models. The model mentioned before is a city-compartment model, that is, a geographically divided population model. On the other hand, our model proposed here is a phase-qualitatively divided population model, such as traveling phase and non-traveling phase. We call the model as 'phase-compartment' model.

3. Result — basic reproduction ratio

Basic reproduction ratio is a key concept in considering epidemiological models. In order to find the basic reproduction ratio of our phase-compartment model(2), we use a method established by van den Driessche and Watmough [2]. To do this, we need several important procedures. Actually we can successfully check and confirm that those procedures are satisfied (which are not mentioned here). And then, we obtain the basic reproduction ratio R_0 as follows:

$$R_0 = \frac{\gamma_1(\alpha_I + \mu + D) + \gamma_2\beta_I + \sqrt{[\gamma_1(\alpha_I + \mu + D) + \gamma_2\beta_I]^2 - 4\gamma_1\gamma_2\beta_I(\mu + D)}}{2\beta_I(\mu + D)} \quad (3)$$

Most surprising thing is that there are no parameters related to changing rates of susceptibles. This implies that the susceptibles travel does not have any influence on whether the disease will spread or not. That is the difference between an intuitive point and mathematical result, which we have never known unless we do mathematical modeling.

Illustrating R_0 with a figure gives us Figure 1. Horizontal axis $gamma_1$ is the infection rate in traveling, and vertical axis $gamma_2$ is the infection rate in non-traveling. Curve in red, which is express by

$$\gamma_2 = \frac{\gamma_1(\alpha_I + \mu + D) - \beta_I(\mu + D)}{\gamma_1 - \beta_I}$$

plays a role of threshold for the disease spread. In fact, there can be no spread of disease inside of the red curve and can be spread of disease outside. It is mathematically cleared that the danger of the disease spread increases if infectious people who travel increase, because of α_I in denominators and β_I in numerators.

4. Discussion

A phase-compartmental model was proposed as basic, simple, and also mathemotaically tractable model for discussing the transport-related infection. From our result we understand two things. First, the basic reproduction ratio R_0 does not depend on the parameters corresponding to changing rates of susceptibles, which is interesting point since we have difference between intuition and mathematics. Second, R_0 suggests that restricting travel of infected individuals is important for controlling disease spread (which is obvious and within our intuition). And also our result actually partially generalizes and realizes results of [1] (not shown here in detail).

Since the result of this present work is just at a starting point, there are many future works. First one is to clear the stability for an endemic equilibrium, which is also a key concept for understanding the disease spread, and also permanence, which is one of important properties. Second one is to generalize, that is, make the model more detail and more realistic. For example,a two-city model with traveling phase should be considerd in order to understand the effect of the transport-related infection in more detail way. After having clear answer about these works, I will complete analysis the most generalized systems with arbitrary n cities and m kinds of traveling phase, which is left for final future work.

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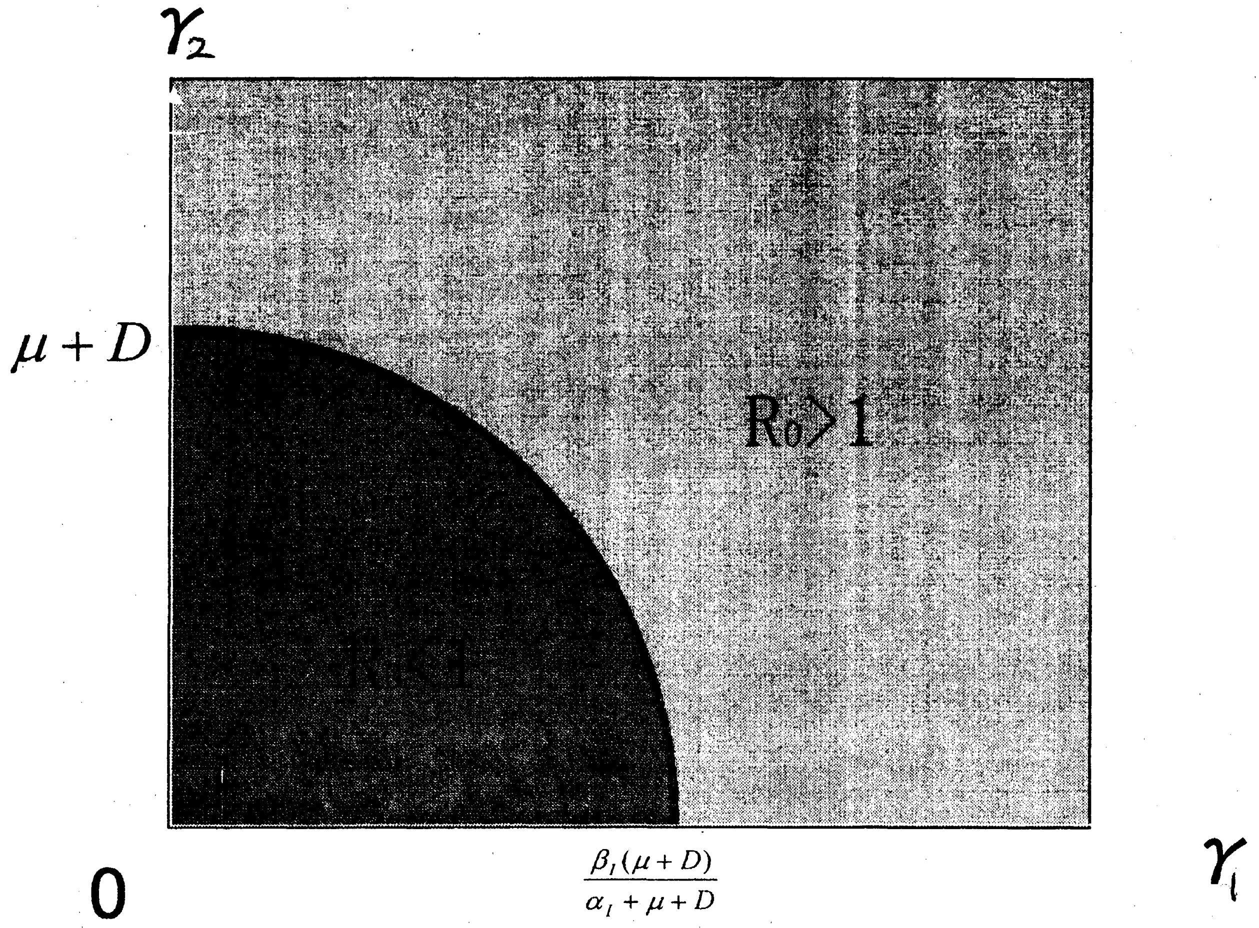

Figure 1