Approximation for extinction probability of the contact process based on the Gröbner basis

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Abstract. In this note we give a new method for getting a series of approximations for the extinction probability of the one-dimensional contact process by using the Gröbner basis.

1 Introduction

Let $X = \{0, 1\}^{\mathbb{Z}^d}$ denote a configuration space, where $\mathbb{Z}^d$ is the $d$-dimensional integer lattices. The contact process $\{\eta_t : t \geq 0\}$ is an $X$-valued continuous-time Markov process. The model was introduced by Harris in 1974 [1] and is considered as a simple model for the spread of a disease with the infection rate $\lambda$. In this setting, an individual at $x \in \mathbb{Z}^d$ for a configuration $\eta \in X$ is infected if $\eta(x) = 1$ and healthy if $\eta(x) = 0$. The formal generator is given by

$$\Omega f(\eta) = \sum_{x \in \mathbb{Z}^d} c(x, \eta)[f(\eta^x) - f(\eta)],$$

where $\eta^x \in X$ is defined by $\eta^x(y) = \eta(y) (y \neq x)$, and $\eta^x(x) = 1 - \eta(x)$. Here for each $x \in \mathbb{Z}^d$ and $\eta \in X$, the transition rate is

$$c(x, \eta) = (1 - \eta(x)) \times \lambda \sum_{y : |y-x|=1} \eta(y) + \eta(x),$$
with $|x| = |x_1| + \cdots + |x_d|$. In particular, the one-dimensional contact process is

\[
\begin{align*}
001 & \rightarrow 011 \quad \text{at rate } \lambda, \\
100 & \rightarrow 110 \quad \text{at rate } \lambda, \\
101 & \rightarrow 111 \quad \text{at rate } 2\lambda, \\
1 & \rightarrow 0 \quad \text{at rate } 1.
\end{align*}
\]

Let $Y = \{ A \subset \mathbb{Z}^d : |A| < \infty \}$, where $|A|$ is the number of elements in $A$. Let $\xi_t^A (\subset \mathbb{Z}^d)$ denote the state at time $t$ of the contact process with $\xi_0^A = A$. There is a one-to-one correspondence between $\xi_t^A (\subset \mathbb{Z}^d)$ and $\eta \in \mathcal{X}$ such that $x \in \xi_t^A$ if and only if $\eta_t(x) = 1$. For any $A \in Y$, we define the extinction probability of $\Lambda$ by $\lambda^{\downarrow \infty} P(\xi^A = \emptyset)$. Define $\nu_{\lambda}(A) = \nu_{\lambda}\{ \eta : \eta(x) = 0 \text{ for any } x \in \Lambda \}$, where $\nu_{\lambda}$ is an invariant measure of the process starting from a configuration: $\eta(x) = 1$ ($x \in \mathbb{Z}^d$) and is called the upper invariant measure. In other words, let $\delta_1 S(t)$ denote the probability measure at time $t$ for initial probability measure $\delta_t$ which is the pointmass $\eta \equiv i(i = 0, 1)$. Then $\nu_{\lambda} = \lim_{t \to \infty} \delta_1 S(t)$. Then self-duality of the process implies that $\nu_{\lambda}(A) = \lim_{t \to \infty} P(\xi_t^A = \emptyset)$. The correlation identities for $\nu_{\lambda}(A)$ can be obtained as follows:

**Theorem 1.1** For any $A \in Y$,

$$
\lambda \sum_{x \in A} \sum_{y : |y - x| = 1} \left[ \nu_{\lambda}(A \cup \{y\}) - \nu_{\lambda}(A) \right] + \sum_{x \in A} \left[ \nu_{\lambda}(A \setminus \{x\}) - \nu_{\lambda}(A) \right] = 0.
$$

From now on we consider the one-dimensional case. We introduce the following notation:

$$
\nu_{\lambda}(\circ) = \nu_{\lambda}(\{0\}), \quad \nu_{\lambda}(\circ\circ) = \nu_{\lambda}(\{0, 1\}), \quad \nu_{\lambda}(\circ \times \circ) = \nu_{\lambda}(\{0, 2\}), \ldots.
$$

By Theorem 1.1, we obtain

**Corollary 1.2**

1. $2\lambda \nu_{\lambda}(\circ\circ) - (2\lambda + 1) \nu_{\lambda}(\circ) + 1 = 0$,
2. $\lambda \nu_{\lambda}(\circ \circ \circ) - (\lambda + 1) \nu_{\lambda}(\circ\circ) + \nu_{\lambda}(\circ) = 0$,
3. $2\lambda \nu_{\lambda}(\circ \circ \circ) + \nu_{\lambda}(\circ \times \circ) - (2\lambda + 3) \nu_{\lambda}(\circ \circ \circ) + 2\nu_{\lambda}(\circ\circ) = 0$,
4. $\lambda \nu_{\lambda}(\circ \circ \times \circ) - (2\lambda + 1) \nu_{\lambda}(\circ \times \circ) + \lambda \nu_{\lambda}(\circ \circ \circ) + \nu_{\lambda}(\circ) = 0$. 

The detailed discussion concerning results in this section can be seen in Konno [2, 3]. If we regard \( \lambda, \nu_{\lambda}(o), \nu_{\lambda}(oo), \nu_{\lambda}(o \circ o), \ldots \) as variables, then the left hand sides of the correlation identities by Theorem 1.1 are polynomials of degree at most two. In the next section, we give a new procedure for getting a series of approximations for extinction probabilities based on the Gröbner basis by using Corollary 1.2. As for the Gröbner basis, see [4], for example.

2 Our results

Put \( x = \nu_{\lambda}(o), \ y = \nu_{\lambda}(oo), \ z = \nu_{\lambda}(o \circ o), \ w = \nu_{\lambda}(o \times o), \ s = \nu_{\lambda}(o o oo), \ u = \nu_{\lambda}(o o \circ o) \). Let \(<\) denote the lexicographic order with \( \lambda < x < y < w < z < u < s \). For \( m = 1, 2, 3 \), let \( I_{m} \) be the ideals of a polynomial ring \( \mathbb{R}[x_{1}, x_{2}, \ldots, x_{n(m)}] \) over \( \mathbb{R} \) as defined below. Here \( x_{1} = \lambda, x_{2} = x, x_{3} = y, x_{4} = z, x_{5} = w, x_{6} = s, x_{7} = u \) and \( n(1) = 3, n(2) = 4, n(3) = 7 \).

2.1 First approximation

We consider the following ideal based on Corollary 1.2 (1):

\[
I_{1} = (2\lambda y - 2\lambda x - x + 1, \ y - x^{2}) \subset \mathbb{R}[\lambda, x, y].
\]

Here \( y - x^{2} \) corresponds to the first (or mean-field) approximation: \( \nu_{\lambda}^{(1)}(oo) = (\nu_{\lambda}^{(1)}(o))^{2} \). Then

\[
G_{1} = \{(x - 1)(2\lambda x - 1), \ y - x^{2}\}
\]

is the reduced Gröbner basis for \( I_{1} \) with respect to \(<\). Therefore the solution except a trivial one \( x(= y) = 1 \) is \( x = \nu_{\lambda}^{(1)}(o) = 1/(2\lambda) \). Remark that the trivial solution means that the invariant measure is \( \delta_{0} \). From this, we obtain the first approximation of the density of the particle, \( \rho_{\lambda} = \hat{E}_{\nu_{\lambda}}(\eta(x)) \), as follows:

\[
\rho_{\lambda}^{(1)} = 1 - \nu_{\lambda}^{(1)}(o) = \frac{2\lambda - 1}{2\lambda},
\]

for any \( \lambda \geq 1/2 \). This result gives the first lower bound \( \lambda_{c}^{(1)} \) of the critical value \( \lambda_{c} \) of the one-dimensional contact process, that is, \( \lambda_{c}^{(1)} = 1/2 \leq \lambda_{c} \). However it should be noted that the inequality is not proved in our approach. The estimated value of \( \lambda_{c} \) is about 1.649.
2.2 Second approximation

Consider the following ideal based on Corollary 1.2 (1) and (2):

\[ I_2 = \langle 2\lambda y - 2\lambda x - x + 1, \lambda z - \lambda y - y + x, xz - y^2 \rangle \subset \mathbb{R}[\lambda, x, y, z]. \]

Here \( xz - y^2 \) corresponds to the second (or pair) approximation: \( \nu_\lambda^{(2)}(o) \nu_\lambda^{(2)}(oo) = (\nu_\lambda^{(2)}(oo))^2 \). Then

\[
G_2 = \{(x - 1)((2\lambda - 1)x - 1), 1 + 2\lambda(y - x) - x,
- y - yx + 2x^2, -z - y(2 + y) + 4x^2\}
\]

is the reduced Gröbner basis for \( I_2 \) with respect to \( \prec \). Therefore the solution except a trivial one \( x(= y = z) = 1 \) is \( x = \nu_\lambda^{(2)}(o) = 1/(2\lambda - 1) \). As in a similar way of the first approximation, we get the second approximation of the density of the particle:

\[
\rho_\lambda^{(2)} = \frac{2(\lambda - 1)}{2\lambda - 1},
\]

for any \( \lambda \geq 1 \). This result implies the second lower bound \( \lambda_{c}^{(2)} = 1 \). We should remark that if we take

\[ I'_2 = \langle 2\lambda y - 2\lambda x - x + 1, \lambda z - \lambda y - y + x, y - x^2, z - x^3 \rangle \subset \mathbb{R}[\lambda, x, y, z], \]

then we have

\[
G'_2 = \{z - 1, y - 1, x - 1\}
\]

is the reduced Gröbner basis for \( I'_2 \) with respect to \( \prec \). Here \( y - x^2 \) and \( z - x^3 \) correspond to an approximation: \( \nu_\lambda^{(2')}(oo) = (\nu_\lambda^{(2')}(o))^2 \) and \( \nu_\lambda^{(2')}(o \circ o) = (\nu_\lambda^{(2')}(o))^3 \), respectively. Then we have only trivial solution: \( x = y = z = 1 \).

2.3 Third approximation

Consider the following ideal based on Corollary 1.2 (1)–(4):

\[ I_3 = \langle 2\lambda y - 2\lambda x - x + 1, \lambda z - \lambda y - y + x, 2\lambda s + w - (2\lambda + 3)z + 2y, \lambda u - (2\lambda + 1)w + \lambda z + x, y s - z^2, x u - y w \rangle \subset \mathbb{R}[\lambda, x, y, z, w, s, u]. \]
Here $y = z^2$ and $x = yw$ correspond to the third approximation: $
u^{(3)}_{\lambda}(\circ \circ) \nu^{(3)}_{\lambda}(\circ \circ \circ) = (\nu^{(3)}_{\lambda}(\circ \circ \circ))^2$ and $
u^{(3)}_{\lambda}(\circ) \nu^{(3)}_{\lambda}(\circ \circ \circ) = \nu^{(3)}_{\lambda}(\circ \circ \circ) \nu^{(3)}_{\lambda}(\circ \circ \circ)$, respectively. Then

$$G_3 = \{(x - 1)((12\lambda^3 - 5\lambda - 1)x^2 - 2\lambda(2\lambda + 3)x - \lambda + 1), \ldots\}$$

is the reduced Gröbner basis for $I_3$ with respect to $<$. Therefore the solution except a trivial one $x = 1$ is $x = \nu^{(3)}_{\lambda}(\circ) = (\lambda(2\lambda + 3) + \sqrt{D})/(12\lambda^3 - 5\lambda - 1)$, where $D = 16\lambda^4 + 4\lambda^2 + 4\lambda + 1$. Then we obtain the third approximation of the density of the particle:

$$\rho^{(3)}_{\lambda} = \frac{4\lambda(3\lambda^2 - \lambda - 3)}{12\lambda^3 - 2\lambda^2 - 8\lambda - 1 + \sqrt{D}},$$

for any $\lambda \geq (1 + \sqrt{37})/6$. This result corresponds to the third lower bound $\lambda^{(3)}_c = (1 + \sqrt{37})/6 \approx 1.180$.

## 3 Summary

We obtain the first, second, and third approximations for the extinction probability, the density of the particle, and the lower bound of the one-dimensional contact process by using the Gröbner basis with respect to a suitable term order. These results coincide with results given by the Harris lemma (more precisely, the Katori-Konno method, see [3]) or the BFKL inequality [5] (see also [3]). As we saw, the generators of $I_m$ in Section 2 have degree at most two in $x_1, x_2, \ldots$, such as $2\lambda y - 2\lambda x - x + 1$, $ys - z^2$ in the case of $I_3$. We expect that this property will lead to get the higher order approximations of the process (and other interacting particle systems having a similar property) effectively.

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## References


