数理解析研究所講究録
第 1552 巻 2007 年 129-133

弧状連結でない連結なコンパクト力学的不変集合

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1 Introduction

Strange attractors of some dynamical systems seem to be nonarcwisely connected ([1], [2]). The purpose of this paper is to prove this fact. We shall consider the 2-dimensional diffeomorphism T :

$$T(x,y)=(x',y'),\quad x'=\varphi(x,y)\,,y'=\psi(x,y)\,,$$

where $x,x',y,y'\in R$ and $\varphi(x,y)$ and $\psi(x,y)$ are once continuously differentiable with respect to x and y. In the following, $DT(P)$ denotes the Jacobi's matrix of T for $P=(x,y)$ and $|DT(P)|$ the Jacobian.
Our main theorem is the following.

Theorem 1
Assume that conditions (i) $\sim$ (iv) hold :

(i) $|DT(P)|<1$ *for* $P\in R^2$,

(ii) there exists a compact, simply connected set K *such that* $T(K)\subset K$,

(iii) there exists at least two distinct fixed points in K,

(iv) for one of the fixed points in K*, say* P_1*, the eigenvalues of* $DT(P_1)$*, say* λ_1 *and* λ_2*, satisfies that*

$$\lambda_1<-1<\lambda_2<0\,.$$

Under these conditions, the set $\bigcap_{n=0}^{\infty} T^n(K)$, *say* Ω, *is T-invariant and nonarcwisely connected.*

Remark 1 It is obvious that Ω is T-invariant, compact, connected and a null set and moreover that Ω is attractive to K, that is, for any P of K, $T^n(P)$ approaches Ω as n tends to infinity.
Remark 2 P_1 is said to be inversely unstable if (iv) holds.

2 Proof of Theorem 1

To the contrary, suppose that Ω is arcwisely connected. Since $P_1, P_2 \in \Omega$, there exists a simple, continuous arc γ joining P_1 and P_2 in Ω, that is, $\gamma \subset \Omega$. Since P_1 is inversely unstable, there exists a C^1-curve β containing P_1, which is the unstable manifold around P_1 (see [3, Theorem 5.1]). For convenience we shall take a small neighbourhood of P_1, say U, and show that $\gamma \cap U$ is identical to a part of β. In fact, if $\gamma \cap U$ is distinct from any part of β, $T(\gamma \cap U)$ must be located on another side of U with respect to β, because $\lambda_1 < 0$ and $\lambda_2 < 0$ (see Figure 1). Therefore γ and $T(\gamma)$ are distinct from each other. Since both γ and $T(\gamma)$ join P_1 and P_2, we can choose a simple closed curve C as parts of γ and $T(\gamma)$, whose interior is set to be D. Clearly the area of D, denoted by $|D|$, is positive. Since $C \subset \gamma \cup T(\gamma) \subset \Omega$, it follows that $C \subset T^n K$ for integers n, and hence $D \subset T^n K$, because $T^n K$ is simply connected. Thus $|D| < |T^n K|$. On the other hand, it follows from condition (i) that $|T^n K|$ tends to zero as n tends to infinity, and hence $|D| = 0$. This contradiction shows that $\gamma \cap U$ is identical to a part of β. Thus, $\gamma \cap U$ is located on one side of β with respect to P_1.

Now we can see that $T(\gamma \cap U)$ is located on another side of β with respect to P_1, because $\lambda_1 < -1$ (see Figure 2). Therefore γ and $T(\gamma)$ are distinct from each other, and hence we can choose a part of γ and $T(\gamma)$ as a simple closed curve. Thus by the same argument as above there arises a contradition. The proof is completed.

3 Duffing type equations

We shall consider the application of Theorem 1 to Duffing type equations :

$$\dot{x} = y \,, \; \dot{y} = -\varepsilon\lambda y - (1 + \varepsilon \cos 2t)x - ax^2 - x^3 \quad (\cdot = \frac{d}{dt}) \tag{1}$$

where ε, λ and a are positive constants. The Poincare mapping T for (1) is defined by $(x_2, y_2) = T(x_1, y_1)$:

$$x_2 = x(\pi, x_1, y_1) \,, \quad y_2 = y(\pi, x_1, y_1) \,,$$

where the pair of $x(t, x_1, y_1)$ and $y(t, x_1, y_1)$ is a solution of through (x_1, y_1) for $t = 0$.

Theorem 2
Assume that $a > \sqrt{2}$ and $0 < \lambda < \frac{1}{4}$. If ε is sufficiently small, then T has an invariant, compact, nonarcwisely connected set Ω. Moreover Ω is globally stable, that is, for any point P of R^2, $T^n(P)$ approaches Ω as n tends to infinity.

Proof First of all we shall show that conditions (i), (ii) and (iv) are statisfied. The appearance of positive damping term implies (i). We may prove that the null solution of (1) is inversely unstable, by the same argument as in [5, Lemma 2]. The existence of nontrivial π-periodic solutions follows from the perturbation theory for ε. In fact, when $\varepsilon = 0$, (1) is reduced to

$$\dot{x} = y \,, \quad \dot{y} = -x - ax^2 - x^3 \,,$$

which has the constant solution $x_1 = \frac{-a-\sqrt{a^2-2}}{2}$. Since the characteristic multiplier for x_1 is different from one, it follows that (1) has a π-periodic solution $x(t)$ for small ε, which is close to x_1. Now we shall prove (iii). The solutions of (1) is uniform-ultimately bounded [4], that is, there exists a disk D_0 such that for any disk D there is a positive number N such that $T^n(D) \subset D_0$ for $n \geq N$, where N may depend on D. Therefore there is a positive number m such that $T^m(D_0) \subset D_0$. By the famous fixed point theorem of L.E.J.Brower, there exists a point P_0 such that $T^m(P_0) = P_0$. We shall take a large disk $D_1 \supset D_0$ such that $D_1 \supset \bigcup_{k=0}^{m-1} \{T^k P_0\}$, which implies that $T(D_1) \cap D_1 \neq \emptyset$, and hence that $T^i(D_1) \cap T^{i+1}(D_1) \neq \emptyset$ for $i \geq 1$. Furthermore we may assume that

$T^m(D_1) \subset D_1$ for the previous m, and hence setting $E = \bigcup_{i=0}^{m-1} T^i(D_1)$, we can see that $T(E) \subset E$. Letting J_i be the boundary of $T^i(D_1)$ for $0 \le i \le m-1$, we shall apply [6, Theorem9.1] in order that the infinite component $R^2 - E$ has for boundary a Jordan curve J contained in $\bigcup_{i=0}^{m-1} J_i$. Letting K be the interior of J, we can see that $T(K) \subset K$, because $K \supset E \supset T(E) \supset T(J)$. Thus, Theorem 1 guarantees that $\bigcap_{n=0}^{\infty} T^n(K)$, say Ω, is nonarcwisely connected. Now, let P be any point $P \in R^2$. Since $T^n(P)$ remains in D_0 for large n and since $D_0 \subset D_1 \subset E \subset K$, it follows that $T^n(P)$ remains in K for large n, which implies that $T^n(P)$ approaches Ω as n tends to infinity. The proof is completed.

Finally we shall treat the Duffing equation, which describes the dynamics of electric current of some electric circuits,

$$\dot{x} = y\,, \quad \dot{y} = -ky - x^3 + B_0 + B\cos t\,, \tag{2}$$

where k, B_0 and B are positive constants. It is difficult to prove the existence of inversely unstable periodic solutions for this system ; the experimental results of [1] suggests that the existence of inversely unstable periodic solutions implies the nonarcwise connectedness of the attractor.

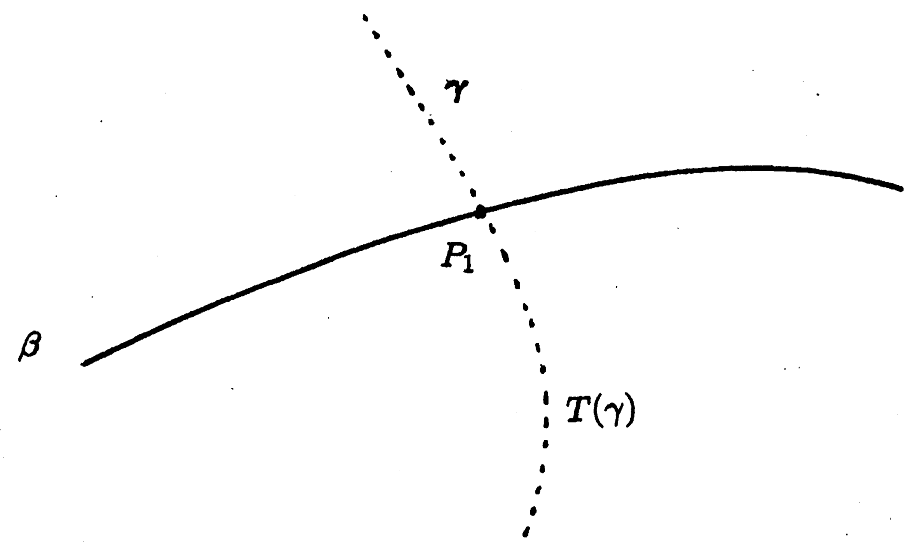

Figure 1

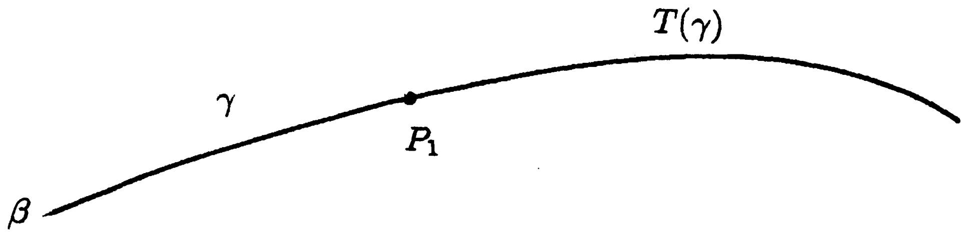

Figure 2

References

[1] Y.Ueda, Random phenomena resulting from nonlinearity in the system described by Duffing's equation, Int.J.Non-linear Mechanics, Vol.20, No.5/6, pp.481-491, 1985.

[2] M.Henon, A two-dimensional mapping with a strange attractor, Commun.math.phys.50, 69-77, 1976.

[3] P.Hartman, Ordinary Differential Equations, John Wiley and Sons, Inc, p.239, 1973.

[4] F.Nakajima, Nonlinear Mathieu equations I, Gakuto International series, Mathematical sciences and applications, Nonlinear waves, Vol.10, pp.353-359, 1997.

[5] F.Nakajima, Bifurcation of nonsymmetric solutions for some Duffing equations, Bull.Austral.Math.Soc., Vol.60, pp.119-128, 1999.

[6] S.Lefchetz, Differential Equations : Geometric Theory, Second edition, Interscience Publications, New York, pp.370-372.