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ON THE DEFINITION AND PROPERTIES OF SUPERPARABOLIC FUNCTIONS

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ABSTRACT. Superparabolic functions are defined as lower semicontinuous functions obeying the comparison principle. We discuss their Sobolev space properties and give sharp integrability exponents.

1. INTRODUCTION

The solutions of the partial differential equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)=\frac{\partial u}{\partial t}, \qquad 1<p<\infty, \tag{1.1}$$

form a similar basis for a nonlinear parabolic potential theory as the solutions of the heat equation do in the classical theory. Especially, the celebrated Perron method can be applied even in the nonlinear situation $p\neq 2$; see [8]. The equation is often called the *p-parabolic equation*, but is also known as the *evolutionary p-Laplace equation* and the *non-Newtonian filtration equation* in the literature. For the regularity theory we refer to [5]. See also Chapter 2 of [24].

In the parabolic potential theory, the so-called *p-superparabolic functions* are essential. They are defined as lower semicontinuous functions obeying the comparison principle with respect to the solutions of (1.1). The p-superparabolic functions are of actual interest also because they are *viscosity supersolutions* of (1.1), see [7]. Thus there is an alternative definition in the theory of viscosity solutions and our results automatically hold for the viscosity supersolutions. We should pay attention to the fact that, in their definition, the p-superparabolic functions are not required to have any derivatives, and, consequently, it is not evident how to directly relate them to the p-parabolic equation. Since the *weak supersolutions* of (1.1) have Sobolev derivatives, they constitute a more tractable class of functions. The reader should carefully distinguish between p-superparabolic functions and supersolutions. The objective of this expository article is to discuss Sobolev space properties of p-superparabolic functions. Indeed, we show that a p-parabolic

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function has spatial Sobolev derivatives with sharp local integrability bounds. We refer to the original articles [15] and [16] for the details. As an application, we study the Riesz measure associated with a p-parabolic function.

Our argument is based on a general principle and it applies to other equations as well. For the porous medium equation

$$\Delta(u^m) = \frac{\partial u}{\partial t}.$$

in the case $m \geq 1$, we refer to [14]. See [23] or Chapter 1 of [24] for the general theory of the porous medium equation. We have tried to keep our exposition as short as possible, omitting such generalizations. We have also deliberately decided to exclude the case $p < 2$. On the other hand, we think that some features might be interesting even for the ordinary heat equation, to which everything reduces when $p = 2$.

2. Weak supersolutions

We begin with some notation. In what follows, Q will always stand for an interval

$$Q = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n), \quad a_i < b_i, \quad i = 1, 2, \ldots, n,$$

in $\mathbf{R}^n$ and the abbreviations

$$Q_T = Q \times (0, T), \qquad Q_{t_1,t_2} = Q \times (t_1, t_2),$$

where $T > 0$ and $t_1 < t_2$, are used for the space-time boxes in $\mathbf{R}^{n+1}$. The parabolic boundary of Q_T is

$$(\overline{Q} \times \{0\}) \cup (\partial Q \times [0, T]).$$

Observe that the interior of the top $\overline{Q} \times \{T\}$ is not included. The parabolic boundary of a space-time cylinder $D_{t_1,t_2} = D \times (t_1, t_2)$, where $D \subset \mathbf{R}^n$, has a similar definition.

Let $1 < p < \infty$. In order to describe the appropriate function spaces, we recall that $W^{1,p}(Q)$ denotes the Sobolev space of functions $u \in L^p(Q)$ whose first distributional partial derivatives belong to $L^p(Q)$ with the norm

$$\|u\|_{W^{1,p}(Q)} = \|u\|_{L^p(Q)} + \|\nabla u\|_{L^p(Q)}.$$

The Sobolev space with zero boundary values, denoted by $W_0^{1,p}(Q)$, is the completion of $C_0^\infty(Q)$ in the norm $\|u\|_{W^{1,p}(Q)}$. We denote by $L^p(t_1, t_2; W^{1,p}(Q))$ the space of functions such that for almost every t, $t_1 \leq t \leq t_2$, the function $x \mapsto u(x,t)$ belongs to $W^{1,p}(Q)$ and

$$\int_{t_1}^{t_2} \int_Q \left(|u(x,t)|^p + |\nabla u(x,t)|^p\right) dx\, dt < \infty.$$

Notice that the time derivative u_t is deliberately avoided. The definition of the space $L^p(t_1, t_2; W_0^{1,p}(Q))$ is analogous.

Next we give the definition of the weak (super)solutions.

Definition 2.1. Let Ω be an open set in $\mathbf{R}^{n+1}$ and suppose that $u \in L^p(t_1, t_2; W^{1,p}(Q))$ whenever $\overline{Q_{t_1,t_2}} \subset \Omega$. Then u is called a *solution* of (1.1) if

$$\int_{t_1}^{t_2} \int_Q \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) dx\, dt = 0 \tag{2.2}$$

whenever $\overline{Q_{t_1,t_2}} \subset \Omega$ and $\varphi \in C_0^\infty(Q_{t_1,t_2})$. If, in addition, u is continuous, then u is called *p-parabolic*. Further, we say that u is a *supersolution* of (1.1) if the integral (2.2) is non-negative for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. If this integral is non-positive instead, we say that u is a *subsolution.*

By parabolic regularity theory, the solutions are locally Hölder continuous after a possible redefinition on a set of measure zero, see [5] or [24]. In general, the time derivative u_t does not exist in Sobolev's sense. This is a principal, well-recognized difficulty with the definition. Namely, in proving estimates, we usually need a test function φ that depends on the solution itself, for example $\varphi = u\zeta$ where ζ is a smooth cutoff function. Then we cannot avoid that the forbidden quantity u_t shows up in the calculation of φ_t. In most cases, we can easily overcome this default by using an equivalent definition in terms of Steklov averages, as on pages 18 and 25 of [5] or in Chapter 2 of [24]. Alternatively, we can proceed using convolutions with smooth mollifiers as on pages 199–121 of [1].

3. Superparabolic functions

The supersolutions of the p-parabolic equation do not form a good closed class of functions. For example, consider the *Barenblatt solution* $\mathcal{B}_p : \mathbf{R}^{n+1} \to [0, \infty)$,

$$\mathcal{B}_p(x,t) = \begin{cases} t^{-n/\lambda} \left(c - \dfrac{p-2}{p} \lambda^{1/(1-p)} \left(\dfrac{|x|}{t^{1/\lambda}} \right)^{p/(p-1)} \right)_+^{(p-1)/(p-2)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $\lambda = n(p-2) + p$, $p > 2$, and the constant c is usually chosen so that

$$\int_{\mathbf{R}^n} \mathcal{B}_p(x,t)\, dx = 1$$

for every $t > 0$. In the case $p = 2$ we have the heat kernel

$$\mathcal{W}(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, & t > 0, \\ 0, \quad t \le 0. \end{cases}$$

The Barenblatt solution is a weak solution of (1.1) in the upper half space

$$\{(x,t) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n, t > 0\}$$

and it formally satisfies the weak form of the equation

$$\frac{\partial \mathcal{B}_p}{\partial t} - \operatorname{div}(|\nabla \mathcal{B}_p|^{p-2} \nabla \mathcal{B}_p) = \delta$$

in $\mathbf{R}^{n+1}$, where the right-hand side is Dirac's delta at the origin. In contrast with the heat kernel, which is strictly positive, the Barenblatt solution has a bounded support at a given instance $t > 0$. Hence the disturbancies propagate with finite speed when $p > 2$. The Barenblatt solution describes the propagation of the heat after the explosion of a hydrogen bomb in the atmosphere. This function was discovered in [3].

The Barenblatt solution is not a supersolution in an open set that contains the origin. It is the a priori integrability of $\nabla \mathcal{B}_p$ that fails. Indeed,

$$\int_{-1}^{1} \int_{Q} |\nabla \mathcal{B}_p(x,t)|^p \, dx \, dt = \infty,$$

where $Q = [-1,1]^n \subset \mathbf{R}^n$. In contrast, the truncated functions

$$\min(\mathcal{B}_p(x,t), \lambda),$$

belong to the correct space and are supersolutions in $\mathbf{R}^{n+1}$ for every $\lambda > 0$.

In order to include the Barenblatt solution in our exposition we define a class of supersolutions which is closed with respect to the increasing convergence. Indeed, the Barenblatt solution is a p-superparabolic function in $\mathbf{R}^{n+1}$ according to the following definition.

Definition 3.1. A function $v : \Omega \to (-\infty, \infty]$ is called *p-superparabolic* if

(i) v is lower semicontinuous,
(ii) v is finite in a dense subset of Ω,
(iii) v satisfies the following comparison principle on each subdomain $D_{t_1,t_2} = D \times (t_1, t_2)$ with $D_{t_1,t_2} \subset\subset \Omega$: if h is p-parabolic in D_{t_1,t_2} and continuous in $\overline{D_{t_1,t_2}}$ and if $h \le v$ on the parabolic boundary of D_{t_1,t_2}, then $h \le v$ in D_{t_1,t_2}.

It follows immediately from the definition that, if u and v are p-superparabolic functions, so are their pointwise minimum $\min(u, v)$ and $u+\alpha$, $\alpha \in \mathbf{R}$. Observe that $u+v$ and αu are not superparabolic in general. This is well in accordance with the corresponding properties of supersolutions. In addition, the class of superparabolic functions is closed with respect to to the increasing convergence, provided the limit function is finite in a dense subset. We also mention that it is enough to compare in the space-time boxes Q_{t_1,t_2} in the definition of the p-superparabolic function.

The reader should carefully distinguish between the supersolutions and the p-superparabolic functions. Notice that a p-superparabolic function is defined at every point in its domain, but supersolutions are defined only up to a set of measure zero. The semicontinuity is an essential assumption. On the other hand, supersolutions have Sobolev derivatives with respect to the spatial variable and they satisfy a differential inequality in a weak sense. By contrast, no differentiability is assumed in the definition of a p-superparabolic function. The only tie to the differential equation is through the comparison principle.

There is a relation between supersolutions and p-superparabolic functions. Supersolutions satisfy the comparison principle and, roughly speaking, the supersolutions are p-superparabolic, provided the issue about lower semicontinuity is properly handled. This is not a serious issue, since every supersolution has a lower semicontinuous representative. In particular, a lower semicontinuous supersolution is p-superparabolic. On the other hand, as we shall see later the truncations $\min(v, \lambda)$, $\lambda \in \mathbf{R}$, of a p-superparabolic function v are supersolutions and hence a p-superparabolic function can be approximated by an increasing sequence of supersolutions.

The obstacle problem in the calculus of variations is a basic tool in the study of the p-superparabolic functions. In order to bypass some technical difficulties related to the time derivative u_t, we use the regularized equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left((|\nabla u|^2 + \varepsilon^2)^{(p-2)/2} \nabla u\right), \tag{3.2}$$

which does not degenerate at the critical points where $\nabla u = 0$. Here ε is a real parameter. The solutions of (3.2) are smooth, provided $\varepsilon \neq 0$. The smoothness in the case $\varepsilon \neq 0$ follows from a standard parabolic regularity theory described in [17]. See also [18]. In the case $\varepsilon = 0$ the equation (3.2) reduces to the true p-parabolic equation (1.1). See Chapter 2 of [24].

Let $\psi \in C^\infty(\mathbf{R}^{n+1})$ and consider the class $\mathcal{F}_\psi$ of all functions $w \in C(\overline{Q}_T)$ such that $w \in L^p(0, T; W^{1,p}(Q))$, $w = \psi$ on the parabolic boundary of Q_T and $w \geq \psi$ in Q_T. The function ψ is an obstacle and also prescribes the boundary values.

The following existence theorem is essential for us.

Lemma 3.3. *There is a unique $w \in \mathcal{F}_\psi$ such that*

$$\int_0^T \int_Q \left((|\nabla w|^2 + \varepsilon^2)^{(p-2)/2} \nabla w \cdot \nabla(\phi - w) + (\phi - w)\frac{\partial \phi}{\partial t} \right) dx\, dt \geq \frac{1}{2} \int_Q |\phi(x,T) - w(x,T)|^2\, dx$$

for all smooth functions ϕ in the class $\mathcal{F}_\psi$. In particular, w is a continuous supersolution of (3.2). Moreover, in the open set $\{w > \psi\}$ the function w is a solution of (3.2). In the case $\varepsilon \neq 0$ we have $w \in C^\infty(Q_T)$.

Proof. The existence can be shown as in the proof of Theorem 3.2 in [2]. The continuity of solution follows from a rather standard parabolic regularity theory. □

Let w_ε denote the solution of (3.2) with $\varepsilon \neq 0$ and let v denote the one with $\varepsilon = 0$. We keep the obstacle ψ fixed and let $\varepsilon \to 0$ in (3.2). The question is about the convergence of the solutions of the obstacle problems: Do the w_ε's converge to v in some sense?

4. Bounded superparabolic functions

Supersolutions and p-superparabolic functions are often identified in the literature, even though this is not strictly speaking correct, as the Barenblatt solution shows. However, we show that there are no other locally bounded p-superparabolic functions than supersolutions. Indeed, locally bounded p-superparabolic functions have Sobolev derivatives with respect to the spatial variable and we can substitute them in (2.2). This is the content of the following theorem.

Theorem 4.1. *Let $p \geq 2$. Suppose that v is a locally bounded p-superparabolic function in an open set $\Omega \subset \mathbf{R}^{n+1}$. Then the Sobolev derivative*

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right)$$

exists and the local integrability

$$\int_{t_1}^{t_2} \int_Q |\nabla v|^p\, dx\, dt < \infty$$

holds for each $\overline{Q} \times [t_1, t_2] \subset \Omega$. *Moreover, we have*

$$\int_{t_1}^{t_2} \int_Q \left(|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi - v \frac{\partial \varphi}{\partial t} \right) dx\, dt \geq 0$$

whenever $\varphi \in C_0^\infty(Q \times (t_1, t_2))$ *with* $\varphi \geq 0$.

Thus the variational inequality (2.2) is at our disposal for bounded functions. For a proof, see Theorem 1.1 in [15].

In the case $p = 2$, the proof of Theorem 4.1 can be extracted from the linear representation formulas. Then all superparabolic functions can be represented in terms of the heat kernel. For $p > 2$ the principle of superposition is not available. Instead we use an obstacle problem to construct supersolutions which approximate a given p-superparabolic function.

Theorem 4.2. *Suppose that* v *is a* p*-superparabolic function in* Ω *and let* $\overline{Q}_{t_1,t_2} \subset \Omega$. *Then there is a sequence of supersolutions*

$$v_k \in C(\overline{Q}_{t_1,t_2}) \cap L^p(t_1, t_2; W^{1,p}(Q)), \quad k = 1, 2, \ldots,$$

of (1.1) *such that* $v_1 \leq v_2 \leq \cdots \leq v$ *and* $v_k \to v$ *pointwise in* Q_{t_1,t_2} *as* $k \to \infty$. *If, in addition,* v *is locally bounded in* Ω, *then the Sobolev derivative* ∇v *exists and* $\nabla v \in L^p_{\mathrm{loc}}(\Omega)$.

For a proof we refer to Lemma 4.2 of [15].

This theorem could be taken as a characterization of p-superharmonicity. Indeed, if we have an increasing sequence of continuous supersolutions and the limit function is finite in dense subset, then the limit function is p-superparabolic. Moreover, if the limit function is bounded, then it is a supersolution. Let us mention that u_t can be interpreted as an object, for example, in the theory of J.-L. Lions et consortes, see [21], but using this approach does not seem to give a class of functions which is closed under bounded increasing convergence. For example, the function $u : \mathbf{R}^{n+1} \to \mathbf{R}$,

$$u(x,t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is a supersolution and it can easily be approximated by an increasing sequence of continuous supersolutions which only depend on the time variable. However, the time derivative of u does not belong to the dual of the parabolic Sobolev space.

5. Unbounded superparabolic functions

The Barenblatt solution clearly shows that the class of p-superparabolic functions contains more than supersolutions. The following slightly unexpected result shows that all p-superparabolic functions have Sobolev derivatives. For bounded p-superparabolic functions this follows from Theorem 4.1, but our objective is to study unbounded p-superparabolic functions and provide a sharp result for them.

Theorem 5.1. *Let $p \geq 2$. Suppose that v is a p-superparabolic function in an open set Ω in $\mathbf{R}^{n+1}$. Then $v \in L^q_{\mathrm{loc}}(\Omega)$ for every q with $0 < q < p - 1 + p/n$. Moreover, the Sobolev derivative*

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)$$

exists and the local integrability

$$\int_{t_1}^{t_2} \int_Q |\nabla v|^q \, dx \, dt < \infty$$

holds for each $\overline{Q} \times [t_1, t_2] \subset \Omega$ whenever $0 < q < p - 1 + 1/(n+1)$.

For a proof see Theorem 1.1 in [16].

Observe, that the integrability exponent for the gradient of a p-superparabolic function v is strictly smaller than p. However, since ∇v is locally integrable to the power $p - 1$, we can formally insert it in (2.2). Unfortunaly this approach is difficult to use in practise, since if we want to use a test function which depends on v, the integrand in (2.2) may fail to be integrable. A similar question for solutions has been studied in [13] and [12].

The Barenblatt solution shows that these critical integrability exponents for a p-superparabolic function and its gradient are optimal. A direct calculation reveals that the Barenblatt solution does not attain these exponents. There is a difference compared to the stationary case. The corresponding critical exponents related to the elliptic equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

are larger, see [19]. The fundamental solution for the stationary case is

$$v(x) = \begin{cases} c|x|^{(p-n)/(p-1)}, & 1 < p < n, \\ -c \log |x|, & p = n, \end{cases}$$

in $\mathbf{R}^n$, but the Barenblatt solution is far more intricate.

In the case $p = 2$, the proof of Theorem 5.1 can be extracted from the linear representation formulas. Instead we approximate the given

p-superparabolic function v by $v_j = \min(v, j)$, $j = 1, 2, \ldots$ A priori estimates for the approximants are derived through variational inequalities and these estimates are passed over to the limit.

The proof consists of three steps depending on the exponents. First, an iteration based on a test function used by [4] and [9] in the elliptic case implies that v is locally integrable to any exponent q with $0 < q < p-2$. Second, the passage over $p-2$ requires an iteration taking the influence of the time variable into account. This procedure reaches all exponents q with $0 < q < p-1$. Third, a more sophisticated arrangement of the estimates is needed to bound the quantity involving integrals over time slices. Finally, Sobolev's inequality yields the correct critical exponents for the function and its Sobolev derivative. In other words, there are two different iteration methods involved in the proof.

In the elliptic case, the proof can also be based on Moser's iteration technique, see [19]. Moser's technique applies also in the parabolic case, if we already know that v is locally integrable to a power $q > p-2$. However, we have not been able to settle the passage over $p-2$ in the parabolic case by using merely Moser's approach and hence we present an alternative proof of the passage. Moser's approach for a doubly nonlinear equation has been studied in [11]. In a remarkable recent paper [6], the intrinsic parabolic Harnack inequality is studied by a different method for equations of p-parabolic type.

6. The Riesz measure of a superparabolic function

Finally we briely study the Riesz measure associated with a p-superparabolic function. Classical Riesz decomposition theorem states that every superharmonic function can be locally represented as a Newton potential of a Radon measure up to a harmonic function. For a nonlinear elliptic result with the Wolff potential, we refer to [9]. Riesz measures play an essential role in the boundary regularity for the Dirichlet problem in [10] and hence it is a very interesting question to study possible extensions to the parabolic case.

If u is a weak supersolution of (1.1) in Ω, then by a partition of unity we see that

$$(Au, \varphi) = \iint_\Omega \left(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \right) dx\, dt \geq 0$$

for every non-negative $\varphi \in C_0^\infty(\Omega)$. From this we conclude that the operator

$$Au = \frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u)$$

is a nonnegative functional on $C_0^\infty(\Omega)$ and by the Riesz representation theorem, there is a unique Radon measure μ on Ω such that $Au = \mu$. In other words,

$$\iint_\Omega \Big(|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi - u \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt = \iint_\Omega \varphi \, d\mu$$

for every $\varphi \in C_0^\infty(\Omega)$.

Next we show that that the same holds for p-superparabolic functions. Observe, that some caution is needed here, since, in general, a p-superparabolic function does not belong to the correct Sobolev space. However, by Theorem 5.1, we have $|\nabla u| \in L^{p-1}_{\mathrm{loc}}(\Omega)$ and exactly in the same way as for supersolutions we can conclude the following theorem.

Theorem 6.1. *Let v be a p-superparabolic function in Ω. Then there is a unique Radon measure μ on Ω such that*

$$\frac{\partial v}{\partial t} - \operatorname{div}(|\nabla v|^{p-2} \nabla v) = \mu, \tag{6.2}$$

that is,

$$\iint_\Omega \Big(|\nabla v|^{p-2} \nabla v \cdot \nabla \varphi - v \frac{\partial \varphi}{\partial t} \Big)\, dx\, dt = \iint_\Omega \varphi \, d\mu$$

for every $\varphi \in C_0^\infty(\Omega)$.

The measure μ in Theorem 6.1 is called the *Riesz measure* associated with v. In view of Theorem 6.1, it is natural to ask wheter the converse holds. More precisely, if μ is a finite Radon measure on a bounded open set Ω, does there exist a p-superharmonic function v in Ω, which satisfies (6.2). In the elliptic case this question has been studied in [9]. See also [4] and [22].

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