Approximation Algorithm for Maximum Triangle Packing and Metric Maximum Clustering

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Abstract

This paper deals with the metric maximum clustering problem with given cluster sizes and the maximum triangle packing problem. For the former problem, Hassin and Rubinstein gave a randomized polynomial-time approximation algorithm achieving an expected ratio of $\frac{1}{2} - \frac{2}{k}$, where $k$ is the size of the smallest cluster. We improve the ratio to $\frac{1}{2} - \frac{2}{k} + \frac{1}{k(k-1)}$ and also derandomize it. For the latter problem, Hassin and Rubinstein gave a randomized polynomial-time approximation algorithm achieving an expected ratio of $\frac{2}{3} (1 - \epsilon)$. We improve the expected ratio of $\frac{187 + 320p}{347 + 640p} \cdot (1 - \epsilon)$ for any constant $\epsilon > 0$. Note that $p$ is close to 0.27.

1 Introduction

In the metric maximum clustering problem with given cluster sizes (METRIC MCP-GCS for short), we are given an edge-weighted complete graph $G = (V, E)$ and a sequence of positive integers $c_1, \ldots, c_p$ such that the edge weights are nonnegative and satisfy the triangle inequality and $\sum_{i=1}^{p} c_i = |V|$. The objective is to find a partition of $V$ into disjoint clusters of sizes $c_1, \ldots, c_p$ such that the total weight of edges whose endpoints belong to the same cluster is maximized. This problem has a lot of applications [9] and a number of approximation algorithms known have been given for it and its special cases [1,2,7,3,4,5]. In particular, Hassin and Rubinstein [5] gave a randomized polynomial-time approximation algorithm for METRIC MCP-GCS which achieves an expected ratio of $\frac{1}{2} - \frac{2}{k}$, where $k$ is the size of the smallest cluster. In this paper, we modify and derandomize their algorithm to obtain a polynomial-time approximation algorithm for METRIC MCP-GCS which achieves a ratio of $\frac{1}{2} - \frac{2}{k} + \frac{1}{k(k-1)}$. To our knowledge, our algorithm achieves the best ratio when $k$ is large.

A problem closely related to METRIC MCP-GCS is the maximum triangle packing problem (MTP for short). In this problem, we are given an edge-weighted complete graph $G = (V, E)$ such that the edge weights are nonnegative and $|V|$ is a multiple of 3. The objective is to find a partition of $V$ into $|V|/3$ disjoint subsets of size exactly 3 such that the total weight of edges whose endpoints belong to the same cluster is maximized. Obviously, if we do not require that the edge weights satisfy the triangle inequality in METRIC MCP-GCS, then MTP becomes a special case of METRIC MCP-GCS. Hassin and Rubinstein [4] gave a randomized polynomial-time approximation algorithm for MTP and claimed that their algorithm achieves an expected ratio of $\frac{88}{169} (1 - \epsilon)$ for any constant $\epsilon > 0$. However, the third author of this paper pointed out a flaw in their analysis and they [6] have corrected the ratio to $\frac{88}{169} (1 - \epsilon)$. In this paper, we modify their algorithm to obtain a polynomial-time approximation algorithm for MTP which achieves an expected ratio of $\frac{187 + 320p}{347 + 640p} \cdot (1 - \epsilon) > \frac{88}{169} (1 - \epsilon)$. Note that $p$ is close to 0.27.
2 Basic Definitions

Throughout the remainder of this paper, a graph means an undirected graph without parallel edges or self-loops whose edges each have a nonnegative weight.

Let $G$ be a graph. We denote the vertex set of $G$ by $V(G)$ and denote the edge set of $G$ by $E(G)$. The weight of $G$, denoted by $w(G)$, is the total weight of edges in $G$. We denote the weight of an edge $e \in E(G)$ by $w_G(e)$. For a subset $F$ of $E(G)$, we use $w_G(F)$ to denote the total weight of edges in $F$, and use $V(F)$ to denote the set of endpoints of the edges in $F$. The weight of a subgraph $H$ of $G$, denoted by $w_G(H)$, is $w_G(E(H))$. The degree of a vertex $v$ in $G$, denoted by $d_G(v)$, is the number of edges incident to $v$ in $G$.

For a function $b$ mapping each vertex $v$ of $G$ to a nonnegative integer, a $b$-matching of $G$ is a subset $F$ of $E(G)$ such that each vertex $v$ of $G$ is incident to at most $b(v)$ edges in $F$; moreover, a maximum-weight $b$-matching of $G$ is a $b$-matching $M$ of $G$ such that $w_G(M) \geq w_G(M')$ for all $b$-matching $M'$ of $G$. When $b(v) \leq 1$ for all vertices $v$ of $G$, a $b$-matching of $G$ is called a matching of $G$. For a natural number $k$, a matching $M$ of $G$ is called a $k$-matching of $G$ if $|M| = k$, is called a maximum-weight $k$-matching of $G$ if $w_G(M) \geq w_G(M')$ for all $k$-matchings $M'$ of $G$, and is called a perfect matching of $G$ if $2|V(M)| \geq |V(G)| - 1$.

A cycle in $G$ is a connected subgraph of $G$ in which each vertex is of degree 2. A path in $G$ is either a single vertex of $G$ or a connected subgraph of $G$ in which exactly two vertices are of degree 1 and the others are of degree 2. The length of a cycle or path $C$, denoted by $|C|$, is the number of edges on $C$. A Hamiltonian cycle is a cycle $C$ with $V(C) = V(G)$. A cycle cover of $G$ is a subgraph $H$ of $G$ with $V(H) = V(G)$ in which each vertex is of degree 2.

For a sequence $c_1, \ldots, c_p$ of positive integers with $\sum_{i=1}^{p} c_i = |V(G)|$, a $(c_1, \ldots, c_p)$-clustering of $G$ is a partition of $V(G)$ into disjoint subsets (called clusters) of sizes $c_1, \ldots, c_p$, respectively. The weight of a $(c_1, \ldots, c_p)$-clustering $\{C_1, \ldots, C_p\}$ of $G$ is the total weight of edges $\{u, v\}$ of $G$ such that some cluster $C_i$ ($1 \leq i \leq p$) contains both $u$ and $v$. A triangle packing of $G$ is a $(3, \ldots, 3)$-clustering of $G$. Note that $G$ has a triangle packing if and only if $|V(G)|$ is a multiple of 3.

For a random event $A$, $\Pr[A]$ denotes the probability that $A$ occurs. For two random events $A$ and $B$, $\Pr[A \mid B]$ denotes the probability that $A$ occurs given the occurrence of $B$. For a random variable $X$, $\mathbb{E}[X]$ denotes the expected value of $X$.

3 Approximation Algorithm for Metric MCP-GCS

Throughout this section, fix an instance $(G, c_1, \ldots, c_p)$ of Metric MCP-GCS. Without loss of generality, we may assume that $c_1 \geq c_2 \geq \ldots \geq c_p$. Let $q = \lfloor \frac{p}{2} \rfloor$ and $S_{\text{odd}} = \{i \in \{1, \ldots, p\} \mid c_i \text{ is odd}\}$. We want to find a $(c_1, \ldots, c_p)$-clustering $\{C_1, \ldots, C_p\}$ of large weight.

The following algorithm is for this purpose and is a derandomization of Hassin and Rubinstein's algorithm [5]:

(1) Initialize $C_1 = \cdots = C_p = \emptyset$, $a_1 = 2\lceil \frac{q}{2} \rceil$, $\ldots$, $a_p = 2\lceil \frac{p}{2} \rceil$, $m_0 = 0$, $M_0 = \emptyset$.

(Comment: $S_{\text{odd}} = \{i \in \{1, \ldots, p\} \mid a_i = c_i - 1\}$.)

(2) For $j = 1, \ldots, q$ (in this order), perform the following steps:

(a) Let $r_j$ be the maximum $i \in \{1, \ldots, p\}$ with $a_i = a_1$.

(b) Let $m_j = m_{j-1} + r_j$.

(c) Compute a maximum $m_j$-matching $M_j$ of $G$ with $V(M_{j-1}) \subseteq V(M_j)$.

(Comment: By Lemma 2 in [5], this step can be done in polynomial time.)
(d) For each \(i \in \{1, \ldots, r_j\}\), decrease \(a_i\) by 2.

(3) Arbitrarily distribute the edges in \(M_1\) to \(C_1, \ldots, C_{r_1}\) so that each \(C_i\) (\(1 \leq i \leq r_1\)) receives (the endpoints of) exactly one edge in \(M_1\).

(4) For \(j = 2, \ldots, q\) (in this order), perform the following steps:

(a) Let \(U_j = V(M_j) - V(M_{j-1})\).

(b) Construct a complete bipartite graph \(B_j\) as follows: The vertex set of \(B_j\) is \(U_j \cup \{C_1, \ldots, C_{r_{j-1}}\}\). More precisely, the vertices on one side of \(B_j\) are exactly the vertices in \(U_j\) and the vertices on the other side of \(B_j\) are exactly the clusters \(C_1, \ldots, C_{r_{j-1}}\). The weight of each edge \((u, C_i)\) of \(B_j\) with \(u \in U_j\) and \(i \in \{1, \ldots, r_{j-1}\}\) is \(\sum_{v \in C_i} w_G(\{u, v\})\).

(c) Compute a maximum-weight \(b\)-matching \(N_j\) in \(B_j\), where \(b(u) = 1\) for each \(u \in U_j\) and \(b(C_i) = 2\) for each \(i \in \{1, \ldots, r_{j-1}\}\). (Comment: Since \(B_j\) is complete and \(r_j \geq r_{j-1}\), each \(C_i\) (\(1 \leq i \leq r_{j-1}\)) is incident to exactly two edges of \(N_j\).)

(d) For each edge \((u, C_i)\) in \(N_j\), add \(u\) to \(C_i\).

(e) Arbitrarily distribute those vertices in \(U_j\) not incident to an edge in \(N_j\) to \(C_{r_{j-1}+1}, \ldots, C_{r_j}\) so that each \(C_i\) (\(r_{j-1} + 1 \leq i \leq r_j\)) receives exactly two vertices.

(5) Arbitrarily distribute the vertices in \(V(G) - \bigcup_{1 \leq i \leq p} V(C_i)\) to the sets \(C_i\) with \(i \in S_{\text{odd}}\) so that each such set \(C_i\) receives exactly one vertex.

(6) Output \(C_1, \ldots, C_p\).

**Lemma 3.1** Let \(Apx\) be the weight of the clustering \(C_1, \ldots, C_p\) output by the algorithm. Then, \(Apx \geq 2 \sum_{j=1}^{q-1} w_G(M_j)\).

Consider an optimal clustering \(O_1, \ldots, O_p\) for \((G, c_1, \ldots, c_p)\). Let \(Opt\) be the weight of this clustering. For each \(i \in \{1, \ldots, p\}\) such that \(c_i\) is odd, we choose a vertex \(t_i \in O_i\) such that \(\sum_{u \in O_i - \{t_i\}} w_G(\{t_i, u\}) \leq \sum_{u \in O_i - \{t_i\}} w_G(\{v, u\})\) for all \(v \in O_i\).

The following lemma is the key for us to improve the ratio obtained by Hassin and Rubinstein's algorithm:

**Lemma 3.2** \(\sum_{i=1}^{p} \sum_{u \in O_i - \{t_i\}} w_G(\{t_i, u\}) \leq \frac{k}{k} Opt\), where \(k = \min\{c_1, \ldots, c_p\}\).

For each \(i \in \{1, \ldots, p\}\), let \(O'_i = O_i\) if \(c_i\) is even, and let \(O'_i = O_i - \{t_i\}\) otherwise. Let \(Opt' = \sum_{i=1}^{p} \sum_{(u, v) \leq O'_i} w_G(\{u, v\})\).

**Lemma 3.3** \(Opt' \leq 4 \sum_{j=1}^{q-1} w_G(M_j) + OPT'\), where \(k\) is as in Lemma 3.2.

**Theorem 3.4** There is a polynomial-time approximation algorithm for metric MCP-GCS that achieves a ratio of at least \(\frac{1}{k} - \frac{2}{k} + \frac{1}{k(k-1)}\).
4 An Approximation Algorithm

Throughout this section, fix an instance $G$ of MTP and an arbitrary constant $\epsilon > 0$. Moreover, fix a maximum-weight triangle packing $\text{Opt}$ of $G$.

To compute a triangle packing of large weight, Hassin and Rubinstein's algorithm [4] (H&R-algorithm for short) starts by computing a maximum-weight cycle cover $\mathcal{C}$ of $G$. It then breaks each cycle $C \in \mathcal{C}$ with $|C| > \frac{1}{\epsilon}$ into cycles of length at most $\frac{1}{\epsilon}$. This is done by removing a set $F$ of edges on $C$ with $w_G(F) \leq \epsilon w_G(C)$ and then adding one edge between each resulting path. In this way, the length of each cycle in $\mathcal{C}$ becomes short, namely, is at most $\frac{1}{\epsilon}$. H&R-algorithm then uses $\mathcal{C}$ to compute three triangle packings $P_1, ..., P_3$ of $G$, and further outputs the packing whose weight is maximum among the three.

$P_1$ is computed from $\mathcal{C}$ by a deterministic subroutine. Its weight is large when the total weight of edges in those cycles $C \in \mathcal{C}$ with $|C| = 3$ is large compared to the weight of $\mathcal{C}$. Here, instead of detailing how to compute $P_1$, we just mention the following result:

**Lemma 4.1** [4] Let $\alpha \cdot w_G(\mathcal{C})$ be the total weight of edges in those cycles $C \in \mathcal{C}$ with $|C| = 3$. Then, $w_G(P_1) \geq \frac{1 + \alpha}{2} \cdot w_G(\mathcal{C}) \geq \frac{1 + \alpha}{2} (1 - \epsilon) \cdot w_G(\text{Opt}).$

$P_2$ is also computed from $\mathcal{C}$ by a deterministic subroutine. Its weight is large when the total weight of those edges $\{u, v\}$ such that some cluster in $\text{Opt}$ contains both $u$ and $v$ and some cycle in $\mathcal{C}$ contains both $u$ and $v$ is large compared to the weight of $\mathcal{C}$. Here, instead of detailing how to compute $P_2$, we just mention the following result:

**Lemma 4.2** [4] Let $\beta \cdot w_G(\text{Opt})$ be the total weight of those edges $\{u, v\}$ such that some cluster in $\text{Opt}$ contains both $u$ and $v$ and some cycle in $\mathcal{C}$ contains both $u$ and $v$. Then, $w_G(P_2) \geq \beta \cdot w_G(\text{Opt}).$

Unlike $P_1$ and $P_2$, $P_3$ is computed from $\mathcal{C}$ by a complicated randomized subroutine. In Section 4.1, we substantially modify their subroutine, obtaining a new randomized subroutine for computing $P_3$. In Section 4.2, we analyze the approximation ratio achieved by the new algorithm.

4.1 Computation of $P_3$

Throughout this subsection, let $p$ be the smallest real number satisfying the inequality $\frac{27}{24}p^2 - \frac{9}{10}p^3 \geq \frac{27}{32}$; the reason why we select $p$ in this way will become clear in Lemma 4.8. Note that $p$ is close to 0.27 and hence $p < \frac{1}{2}$. Let $C_1, ..., C_r$ be the cycles in $\mathcal{C}$. Consider the following randomized subroutine which computes $P_3$ from $\mathcal{C}$ as follows:

1. Compute a maximum-weight $b$-matching $M_1$ in a graph $G_1$, where
   - $V(G_1) = V(G)$,
   - $E(G_1)$ consists of those $\{u, v\} \in E(G)$ such that $u$ and $v$ belong to different cycles in $\mathcal{C}$, and
   - $b(v) = 2$ for each $v \in V(G_1)$.

2. In parallel, for each cycle $C_i$ in $\mathcal{C}$, process $C_i$ by performing the following steps:
   a. Initialize $R_i$ to be the empty set.
(b) If \( |C_i| = 3 \), then for each edge \( e \) of \( C_i \), add \( e \) to \( R_i \) with probability \( p \).

(Comment: \( E[w_G(R_i)] = (1 - p) \cdot w_G(C_i) \). Moreover, each vertex of \( C_i \) is incident to exactly one edge of \( R_i \) with probability \( 2p(1 - p) \). Furthermore, each vertex of \( C_i \) is incident to exactly two edges of \( R_i \) with probability \( p^2 \). Thus, each vertex of \( C_i \) is incident to at least one edge of \( R_i \) with probability \( 2p - p^2 \).)

(c) If \( |C_i| \geq 4 \), then perform the following steps:

i. Choose one edge \( e_1 \) from \( C_i \) uniformly at random.

ii. Starting at \( e_1 \) and going clockwise around \( C_i \), label the other edges of \( C_i \) as \( e_2, ..., e_c \) where \( c \) is the number of edges in \( C_i \).

iii. Add the edges \( e_j \) with \( j \equiv 1 \pmod{4} \) and \( j \leq c - 3 \) to \( R_i \).

(Comment: \( R_i \) is a matching of \( C_i \) and \( |R_i| = \lfloor \frac{|C_i|}{4} \rfloor \).

iv. If \( c \equiv 1 \pmod{4} \), then add \( e_{c-1} \) to \( R_i \) with probability \( \frac{1}{4} \).

(Comment: \( R_i \) remains to be a matching of \( C_i \). Moreover, \( E[|R_i|] = \frac{|C_i|}{4} + 1 \cdot \frac{1}{4} = \frac{|C_i|}{4} \).

v. If \( c \equiv 2 \pmod{4} \), then add \( e_{c-1} \) to \( R_i \) with probability \( \frac{1}{4} \).

(Comment: \( R_i \) remains to be a matching of \( C_i \). Moreover, \( E[|R_i|] = \frac{|C_i|}{4} - 1 \cdot \frac{1}{4} = \frac{|C_i|}{4} \).

vi. If \( c \equiv 3 \pmod{4} \) and \( c > 3 \), then add \( e_{c-2} \) to \( R_i \) with probability \( \frac{3}{4} \).

(Comment: \( R_i \) remains to be a matching of \( C_i \). Moreover, \( E[|R_i|] = \frac{|C_i|}{4} - 1 \cdot \frac{3}{4} = \frac{|C_i|}{4} \).

(3) Let \( R = R_1 \cup \cdots \cup R_r \).

(Comment: If \( |C_i| = 3 \), then \( \Pr[e \in R_i] = p \) for every edge \( e \) of \( C_i \). If \( |C_i| \geq 4 \), then \( E[|R_i|] = \frac{|C_i|}{4} \) by the comments on Step 2(c)iv through 2(c)vi. Moreover, each edge of \( C_i \) with \( |C_i| \geq 4 \) is added to \( R_i \) with the same probability. Thus, if \( |C_i| \geq 4 \), then \( \Pr[e \in R_i] = \frac{1}{4} \) for every edge \( e \) of \( C_i \), and hence each vertex of \( C_i \) is incident to at least one edge of \( R \) with probability \( \frac{1}{4} \).)

(4) Let \( M_2 \) be the set of all edges \( \{u, v\} \in M_1 \) such that both \( u \) and \( v \) are of degree 0 or 1 in graph \( \mathcal{C} - R \). Let \( G_2 \) be the graph \( (V(G), M_2) \).

(5) For each odd cycle \( C \) of \( G_2 \), select one edge uniformly at random and delete it from \( G_2 \).

(6) Partition the edge set of \( G_2 \) into two matchings \( N_1 \) and \( N_2 \).

(7) For each edge \( e \) of \( G_2 \) which alone forms a connected component of \( G_2 \), add \( e \) to the matching \( N_i(i \in \{1, 2\}) \) which does not contain \( e \).

(8) Select \( M \) from \( N_1 \) and \( N_2 \) uniformly at random.

(Comment: \( M \) is a matching of the graph \( (V(G), M_1) \).)

(9) Let \( \mathcal{C}' \) be the graph obtained from graph \( \mathcal{C} - R \) by adding the edges in \( M \).

(Comment: Each connected component of \( \mathcal{C}' \) is a path or cycle. Moreover, each cycle \( K \) in \( \mathcal{C}' \) may be a triangle or not. If \( K \) is a triangle, then it must be a triangle in \( \mathcal{C} \). On the other hand, if \( K \) is not a triangle, then it must contain at least two edges in \( M \).)

(10) Classify the cycles \( C \) of \( \mathcal{C}' \) into three types: superb, good, or ordinary. Here, \( C \) is superb if \( |C| = 3 \); \( C \) is good if \( |C| = 6 \), \( |E(C) \cap M| = 2 \), and there are triangles \( C_i \) and \( C_j \) in \( \mathcal{C} \).
such that $|E(C_i) \cap E(C)| = 2$ and $|E(C_j) \cap E(C)| = 2$; $C$ is ordinary if it is neither good nor superb.

(11) For each ordinary cycle $C$ in $C'$, choose one edge in $E(C) \cap M$ uniformly at random and delete it from $C'$.

(12) For each good cycle $C$ in $C'$, change $C$ back to two triangles in $C$ as follows: Delete the two edges of $M \cap E(C)$ from $C$ and then close each of the two resulting paths (of length 2) by adding the edge between its endpoints.  

(Comment: Because of the maximality of $C$, this step does not decrease $w_G(C')$.)

(13) If $C'$ has at least one path component, then connect the path components of $C'$ into a single cycle $Y$ by adding some edges of $G$, and further break $Y$ into paths each of length 2 by removing a set $F$ of edges from $Y$ with $w_G(F) \leq \frac{1}{3} \cdot w_G(Y)$.

(14) Let $P_3$ be the $(3, \ldots, 3)$-clustering of $G$ induced by the connected components of $C'$. More precisely, the clusters in $P_3$ one-to-one correspond to the vertex sets of the connected components of $C'$.

Lemma 4.3 For each $e \in M_1$, $\Pr[e \in M | e \in M_2] \geq \frac{9}{20}$.

Lemma 4.4 For each edge $e \in M$ such that at least one endpoint of $e$ does not appear on a triangle in $C$, $e$ survives the deletion in Step 11 with probability at least $\frac{3}{4}$.

Lemma 4.5 For each $e \in M_1$ such that neither endpoint of $e$ appears on a triangle in $C$, $e$ is contained in $C'$ immediately after Step 11 with probability at least $\frac{27}{320}$.

Lemma 4.6 For each $e \in M_1$ such that exactly one endpoint of $e$ appears on a triangle in $C$, $e$ is contained in $C'$ immediately after Step 11 with probability at least $\frac{27}{320}$.

Lemma 4.7 Suppose that $e = \{u_1, v_1\}$ is an edge in $M$ such that both $u_1$ and $v_1$ appear on triangle in $C$ and both $u_1$ and $v_1$ are incident to exactly one edge in $R$. Then, the probability that $e$ is contained in $C'$ immediately after Step 11 is at least $\frac{3}{4}$.

Lemma 4.8 For each $e \in M_1$ such that both endpoints of $e$ appear on triangles in $C$, $e$ is contained in $C'$ immediately after Step 11 with probability at least $\frac{27}{320}$.

4.2 Analysis of the Approximation Ratio

By the comment on Step 3, the expected total weight of the edges of $C$ remaining in $C'$ immediately after Step 11 is at least

\[
\left((1-p)\alpha + \frac{3}{4}(1-\alpha)\right)w_G(C) = \left(\frac{3}{4} - (p - \frac{1}{4})\alpha\right)w_G(C) \geq \left(\frac{3}{4} - (p - \frac{1}{4})\alpha\right)(1 - e)w_G(\text{Opt}).
\]

Moreover, by Lemmas 4.5 through 4.8, the expected total weight of edges of $M_1$ remaining in $C'$ immediately after Step 11 is at least $\frac{27}{320}w_G(M_1)$. Furthermore, by the construction of $M_1$, $w_G(M_1)$ is larger than or equal to the total weight of those edges $\{u, v\}$ such that some cluster in $\text{Opt}$ contains both $u$ and $v$ but no cycle in $C$ contains both $u$ and $v$. So, $w_G(M_1) \geq (1 - \beta)w_G(\text{Opt})$. Now, since $w_G(P_3)$ is at least $\frac{2}{3}$ of the total weight of edges in $C'$ immediately after Step 11, we have
\[ \mathcal{E}[w_G(P_3)] \geq \frac{2}{3} \left( \frac{3}{4} - (p - \frac{1}{4}) \alpha \right) (1 - \epsilon) w_G(\text{Opt}) + \frac{2}{3} \cdot \frac{27}{320} (1 - \beta) w_G(\text{Opt}) \]  

\[ = \left( \frac{89}{160} - \frac{1}{2} \epsilon - \frac{2}{3} (p - \frac{1}{4}) (1 - \epsilon) \alpha - \frac{9}{160} \beta \right) w_G(\text{Opt}). \]  

(3.1)

(3.2)

So, by Lemma 4.1 and 4.2, we have

\[ \frac{3}{4} (p - \frac{1}{4}) w_G(P_1) + \frac{9}{160} w_G(P_2) + w_G(P_3) \geq \frac{187 + 320p - (320p + 160) \epsilon}{480} \cdot w_G(\text{Opt}). \]

Therefore, the weight of the best packing among \( P_1, P_2, \) and \( P_3 \) is at least

\[ \frac{187 + 320p - (320p + 160) \epsilon}{640p + 347} \cdot w_G(\text{Opt}) \geq \frac{187 + 320p}{347 + 640p} \cdot (1 - \epsilon) w_G(\text{Opt}). \]

In summary, we have proven the following theorem:

**Theorem 4.9** For any constant \( \epsilon > 0 \), there is a polynomial-time randomized approximation algorithm for MTP that achieves an expected ratio of \( \frac{187 + 320p}{347 + 640p} \cdot (1 - \epsilon) \).

**References**


