

Homogeneous Epidemic Systems in the Stable Population*

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Abstract

In this paper, we develop a new approach to deal with asymptotic behavior of the age-structured homogeneous epidemic systems. For homogeneous systems, there is no attracting nontrivial equilibrium, instead we have to examine existence and stability of persistent solutions. Assuming that the host population dynamics is described by the stable population model, we rewrite the basic system into a system of ratio age distribution, which is the age profile divided by the stable age profile. If the host population has the stable age profile, the ratio age distribution system is reduced to a normalized system. Then we prove a linearized stability principle that the local stability [instability] of steady states of the normalized system implies the orbital stability [instability] of corresponding persistent solutions of the original homogeneous system.

Keywords: homogeneous epidemic system, stable population, age profile, linearized stability principle.

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1 Introduction: Homogeneous epidemic systems

First let us introduce a very general formulation for age-structured epidemic systems. Suppose that the host population is a closed age-structured population divided into n subpopulations (compartments), each of which corresponds to an epidemic i -state, for example, susceptible, exposed, infective, removed, vaccinated state and so on. The state of such a population is described by means of age density functions $U_j(t, a)$, where t denotes time, a stands for age

*This paper is a revised version of the first half of Inaba [13]. We have removed the assumption used in [13] that the host population has a positive Malthusian parameter.

and j indicates the j -th epidemic state. The dynamics of this model is described by the system of equations

$$\begin{aligned} \frac{\partial U_j(t, a)}{\partial t} + \frac{\partial U_j(t, a)}{\partial a} &= -\mu_j(a)U_j(t, a) + \sum_{k=1}^n g_{jk}(a, U(t))U_k(t, a), \\ U_j(t, 0) &= \sum_{k=1}^n \int_0^\omega m_{jk}(a)U_k(t, a)da, \\ U_j(0, a) &= U_{0j}(a), \end{aligned} \quad (1.1)$$

where ω denotes the upper bound of age, U_{0j} is the initial data. The function μ_j denotes the natural death rate of the j -th state population and $m_{jk}(a)$ is the birth rate of j -th newborns by k -th individuals. The function $g_{ij}(a, U)$, $i \neq j$ denotes the density-dependent *force of transition* from j -th state to i -th state and g_{jj} is defined as

$$g_{jj}(a, U) := - \sum_{i \neq j} g_{ij}(a, U).$$

We assume that $U_j(t, \cdot) \in L_+^1(0, \omega)$ since U_j represents a density. Moreover each age-specific fertility function m_{ij} is assumed to be a bounded nonnegative function with a compact support $[a_1, a_2]$, where $0 < a_1 < a_2 < \omega$ and $[a_1, a_2]$ is called the reproductive age period.

Let us define the survival function $\ell_j(a)$ at j -th state by

$$\ell_j(a) := \exp\left(-\int_0^a \mu_j(\sigma)d\sigma\right). \quad (1.2)$$

In order to guarantee $\ell_j(\omega) = 0$, we assume that μ_j is a locally integrable nonnegative function satisfying

$$\int_0^\omega \mu_j(\sigma)d\sigma = +\infty. \quad (1.3)$$

Moreover we assume that the incident rate of death is essentially bounded:

$$\sup_{a \in [0, \omega]} \mu_j(a)\ell_j(a) < +\infty. \quad (1.4)$$

Then it is reasonable to restrict the state space of age density functions such that $\mu_j(a)U_j(t, a)$ becomes integrable with respect to age a . For this purpose, we assume that the state space of age-density function U_j is given by

$$X_j := \left\{ \phi \in L_+^1(0, \omega) : \frac{\phi(\cdot)}{\ell_j(\cdot)} \in L_+^1(0, \omega) \right\}, \quad (1.5)$$

where its norm is given by $\|\phi\|_{X_j} = \|\phi(\cdot)/\ell_j(\cdot)\|_{L^1}$. If there is no differential (natural) mortality among epidemic states, then each state space of age-density functions is given by the same set as $\{U_j \in L_+^1(0, \omega) : U_j(\cdot)/\ell(\cdot) \in L_+^1(0, \omega)\}$.

Then the age-structured epidemic system (1.1) can be formulated as a semi-linear dynamical system as follows:

$$\begin{aligned}\frac{dU(t)}{dt} &= AU(t) + F(U), \\ U(0) &= U_0 \in E_+, \end{aligned} \tag{1.6}$$

where $E = \prod_{j=1}^n X_j$ is a Banach space of state vectors and E_+ is its positive cone, $U(t) = (U_1(t, \cdot), U_2(t, \cdot), \dots, U_n(t, \cdot))^{\text{T}1}$ is the state vector at time t , and the operator $A : D(A) \subset E \rightarrow E$ is the population operator, given by

$$(AU)(a) = (-dU_1/da - \mu_1(a)U_1, \dots, -dU_n/da - \mu_n(a)U_n)^{\text{T}}.$$

The domain $D(A)$ is given as follows:

$$D(A) = \left\{ U \in E : \frac{U_j}{\ell_j} \in AC[0, \omega], U(0) = \int_0^\omega M(a)U(a)da \right\},$$

where $AC[0, \omega]$ denotes the set of absolutely continuous functions, $M(a) := (m_{ij}(a))_{1 \leq i, j \leq n}$ is a $n \times n$ fertility matrix, and the nonlinear term $F(U)$ is given as

$$F(U)(a) = G(a, U)U, \tag{1.7}$$

where $G(a, U) = (g_{ij}(a, U))_{1 \leq i, j \leq n}$ is a $n \times n$ transition matrix.

The transition rate from the susceptible class to the exposed or infected class is called the *force of infection*. For many classical epidemic models, the force of infection is the only density-dependent transition rate (which depends on the density of infectious population) and other transition rates are assumed to be given age-dependent functions or constants.

To ensure the existence and uniqueness of a solution, we assume local Lipschitz continuity, that is, there exists an increasing function $k(r)$ such that $\|F(U) - F(V)\| \leq k(r)\|U - V\|$ for all $U, V \in \{U \in E_+ : \|U\| \leq r\}$. Under the above assumption, it can be shown that the epidemic system (1.1) has a unique global mild solution $S(t)u_0$ such that $S(t)(E_+) \subset E_+$ and if $u_0 \in E_+ \cap D(A)$, $S(t)u_0$ becomes a classical solution. In general the evolution system (1.1) can be seen as a nonlinear extension of a demographic *multistate stable population model* [11], which has been extensively studied by many authors ([17], [19], [9], [4]).

In this paper we focus on the special case in which the basic system (1.1) defines a *homogeneous dynamical system*. A dynamical system can be called *homogeneous* if its solution operator is homogeneous, that is, $S(t)(\alpha U) = \alpha S(t)(U)$ for all $U \in E_+$ and for all $\alpha > 0$. For the semilinear Cauchy problem (1.6), if F is a homogeneous of degree one nonlinear continuous operator:

$$F(\alpha U) = \alpha F(U), \quad \forall U \in E_+, \forall \alpha > 0, \tag{1.8}$$

¹T denotes the transpose of the vector.

then (1.6) becomes a homogeneous dynamical system. In the following we call the age-structured multistate model (1.1) the *homogeneous epidemic system* if it describes an epidemic compartment model and F satisfies the homogeneity condition (1.7)-(1.8) with $G(a, \alpha U) = G(a, U)$.

The assumption of homogeneity has been widely used in the epidemic modeling to reflect the saturation of contact number in large scale populations ([3], [6], [18]). In general, the force of infection (λ) can be seen as a product of three factors, the transmission probability of infectious agents per one contact (β), the number of contacts per capita and per unit time ($C(N)$) and the fraction of the population that is infectious (I/N). That is,

$$\lambda \approx \beta \times C(N) \times \frac{I}{N}, \quad (1.9)$$

where I denotes the size of the infectious population and N is the size of the total population. Though it is natural to suppose that the number of contacts $C(N)$ is an increasing function of the population size N , for many cases it is also believed that $C(N)$ levels off at high population densities ([1], [18], [14]). Then it would be reasonable to adopt the simplified assumption that C is constant if the host population size is so large that $C(N)$ is essentially equal to its limiting value ($C(N)$ is assumed to be an increasing function bounded above). In such a case, λ becomes homogeneous of degree zero with respect to the scale of population densities, which leads to the homogeneous epidemic system.

If U^* is an equilibrium solution of the system (1.6) with homogeneity condition (1.8), then so is cU^* for any $c > 0$. Then there is no possibility of an attracting nontrivial equilibrium, which is very different from the classical epidemic models whose force of infection is a linear functional of the infected population density. So our interest focuses on the existence and stability of persistent solutions defined as follows: The solution of (1.6) with (1.8) is called a persistent solution if it has the form as $e^{r^*t}U^*$ where r^* is a constant and $U^* \in E$. Then it follows from the homogeneous nonlinearity that a (biologically meaningful) persistent solution exists if and only if the nonlinear eigenvalue problem

$$AU^* + F(U^*) = r^*U^*, \quad U^* \in E_+, \quad (1.10)$$

has a solution.

The homogeneous dynamical system has been studied by several authors who have already developed some approaches to show stability results of persistent solutions in the homogeneous (epidemic) dynamical systems ([5], [3], [6], [7], [8], [20], [21], [10]). However, in fact the above nonlinear eigenvalue problem is very difficult to solve except for some special cases. For the homogeneous epidemic system, the most important special case such that the eigenvalue problem (1.10) could be solved is the case that the host population dynamics is independent from the epidemic and described by the stable population model.

In order to consider the solvable case, we adopt the following additional assumption:

Assumption 1.1 *There is no differential mortality, $\mu(a) = \mu_j(a)$ ($1 \leq j \leq n$) is the common age-specific natural death rate. Moreover there is no differential fertility, so there exists a state-independent age-specific fertility function $f(a)$ such that $m_{ij}(a) = k_i f(a)$, where k_i denotes the proportion of newborns whose epidemic state is i and $\sum_{i=1}^n k_i = 1$, $k_i \geq 0$.*

Note that under the Assumption 1.1 we can escape from the regulation problem, that is, the effects of the epidemic on the demographic vital rates. Yet demography influences the disease dynamics. For example, the host population structure and the Malthusian parameter could affect the basic reproduction ratio ([2], [12]).

If the Assumption 1.1 is satisfied, it is easy to see that the age density function of the host population, denoted by $P(t, a) := \sum_{i=1}^n U_i(t, a)$ satisfies the stable population model:

$$\begin{aligned} \frac{\partial P(t, a)}{\partial t} + \frac{\partial P(t, a)}{\partial a} &= -\mu(a)P(t, a), \\ P(t, 0) &= \int_0^\omega f(a)P(t, a)da, \\ P(0, a) &= P_0(a), \end{aligned} \tag{1.11}$$

where $P_0(a) = \sum_{j=1}^n U_j(0, a)$ is a given initial data. Moreover, by adding term to term in the eigenvalue problem (1.10), we can obtain the following linear eigenvalue problem:

$$\begin{aligned} \left(-\frac{\partial}{\partial a} - \mu(a) \right) P^*(a) &= r^* P^*(a), \\ P^*(0) &= \int_0^\omega f(a)P^*(a)da, \end{aligned} \tag{1.12}$$

where $P^*(a) := \sum_{i=1}^n U_i^*(a)$. From (1.12), we know that $P^*(a) = P^*(0)e^{-r^*a}\ell(a)$, where $\ell(a) := \exp(-\int_0^a \mu(\sigma)d\sigma)$ is the survival function. Inserting the expression of P^* into the boundary condition, we know that the eigenvalue r^* must be a root of the Euler-Lotka characteristic equation:

$$\int_0^\omega e^{-za} f(a)\ell(a)da = 1, \quad z \in \mathbf{C}. \tag{1.13}$$

Then it is well known that (1.13) has a unique dominant real root r_0 , which is called the *intrinsic rate of natural increase*.

Therefore the host population system (1.11) has a unique persistent age profile (normalized age distribution) given by

$$\psi(a) := \frac{e^{-r_0 a} \ell(a)}{\int_0^\omega e^{-r_0 \sigma} \ell(\sigma) d\sigma} = \psi(0) e^{-r_0 a} \ell(a), \tag{1.14}$$

and ψ is called the *stable age profile*. Therefore under the Assumption 1.1, persistent solutions of the homogeneous epidemic system (1.1) have a unique growth

rate r_0 . Then the remaining problem is to look for the persistent distributions U^* and to examine their stability.

Intuitively speaking, if the host population age distribution approaches a stable age distribution independently from the epidemic, we could expect that the long-run behavior (stability and instability of the persistent solutions) of the basic system would be determined by the limiting system whose host population has already attained the stable age distribution. In order to show this intuition to hold, first we have to consider the dynamics of age profile more precisely. In the following we call the homogeneous epidemic system under the Assumption 1.1 the *homogeneous epidemic system in the stable population*.

2 Age profile dynamics of the stable population

For the stable population model, its initial data are called *trivial* if $f(a+t)P_0(a) = 0$ for almost all $a \in [0, \omega]$ and all $t \geq 0$, which means that the support of P_0 starts beyond the maximum reproductive age ([9], p.22) (that is, the initial individuals are too old to be fertile), otherwise they are called *non-trivial*. It follows from the stable population theory ([11], [9]) that if a given initial data $P_0(a)$ is non-trivial, the age distribution converges to the stable age profile:

$$\lim_{t \rightarrow \infty} \frac{P(t, a)}{\int_0^\omega P(t, a) da} = \psi(a). \quad (2.1)$$

That is, ψ is a relatively stable age distribution. Once the stable age profile ψ is attained, it is persistent and the population grows exponentially with the Malthusian parameter r_0 .

In order to show the stability principle for the epidemic system in the next section, here we introduce some properties of the *ratio age distribution* w defined by:

$$w(t, a) = \frac{P(t, a)}{N(t)\psi(a)}, \quad (2.2)$$

where $N(t) = \int_0^\omega P(t, a) da$ is the total size of the population at time t and $P(t, a)/N(t)$ is the age profile (normalized age distribution) of the host population at time t , so w is the ratio of the age profile at time t to the stable age profile. The ratio distribution is well defined as long as the initial data P_0 is nontrivial.

Inserting $P = N\psi w$ into (1.11), we can derive that w satisfies the following system:

$$\begin{aligned} \frac{\partial w(t, a)}{\partial t} + \frac{\partial w(t, a)}{\partial a} &= (r_0 - h(w(t, \cdot)))w(t, a), \\ w(t, 0) &= \int_0^\omega \pi(\sigma)w(t, \sigma) d\sigma, \\ w(0, a) &= w_0(a) \in \Sigma, \end{aligned} \quad (2.3)$$

where $\pi(a) := e^{-r_0 a} f(a) \ell(a)$, Σ is the state space of w given by

$$\Sigma := \left\{ \phi \in L^1_+(0, \omega) : \int_0^\omega \psi(a) \phi(a) da = 1 \right\},$$

and $h : \Sigma \rightarrow \mathbf{R}$ is a functional defined by

$$h(\phi) := \int_0^\omega (f(a) - \mu(a)) \psi(a) \phi(a) da.$$

It is easy to see that $h(1) = r_0$ and $h(w(t, \cdot))$ gives the growth rate of the host population at time t , that is, $N'(t)/N(t) = h(w(t, \cdot))$.

Once the solution w of (2.3) is given, the host population P is recovered as

$$P(t, a) = N(0) e^{\int_0^t h(w(\sigma, \cdot)) d\sigma} \psi(a) w(t, a). \quad (2.4)$$

It follows that $P(t, a)$ becomes the persistent solution $N(0) e^{r_0 t} \psi(a)$ if and only if $w(t, a) = w^* \equiv 1$, which is the unique nontrivial equilibrium solution of (2.3).

From the well-known strong ergodicity theorem for the stable population, it is easy to see that the ratio distribution w converges to $w^* \equiv 1$ exponentially in L^1 norm as $t \rightarrow \infty$ (see [9] Chapter II). However, more precisely, let us show that $\|w - w^*\|/\|w_0 - w^*\|$ is decreasing exponentially, which result is needed in the next section.

Let us consider the system (2.3) as a Cauchy problem in $L^1(0, \omega)$:

$$\frac{dw(t)}{dt} = A_0 w(t) + G(w(t)), \quad w(0) = w_0 \in \Sigma, \quad (2.5)$$

where A_0 is a population operator defined by $(A_0 \phi)(a) = -d\phi(a)/da$ with the domain given by

$$D(A_0) = \left\{ \phi \in AC[0, \omega] : \phi(0) = \int_0^\omega \pi(a) \phi(a) da \right\}.$$

and the nonlinear term G is given by

$$G(\phi) := (r_0 - h(\phi)) \phi.$$

Since G is a continuously Fréchet differentiable nonlinear perturbation, for each $w_0 \in L^1$ there exists a maximal interval of existence $[0, T_{w_0})$ and a unique continuous function $t \rightarrow w(t; w_0)$ from $[0, T_{w_0})$ to L^1 such that

$$w(t; w_0) = e^{tA_0} w_0 + \int_0^t e^{A_0(t-\sigma)} G(w(\sigma; w_0)) d\sigma, \quad t \in [0, T_{w_0}), \quad (2.6)$$

and $T_{w_0} = \infty$ or $\limsup_{t \rightarrow T_{w_0}} \|w(t; w_0)\| = \infty$ and if $w_0 \in D(A_0)$, then $w(t; w_0) \in D(A_0)$ and $w(t; w_0)$ is continuously differentiable and satisfies (2.6) on $[0, T_{w_0})$ ([16]; [19], Prop. 4.16). Since (2.3) is induced from the stable population model (1.11) by the transformation (2.4), we know that $T_{w_0} = \infty$ if w_0 is a nontrivial data, and the following holds:

Lemma 2.1 *Let Σ_0 be the set of nontrivial initial data. Then $\Sigma \cap \Sigma_0$ is positively invariant with respect to the solution semiflow of the system (2.3).*

Next consider the linearized equation of (2.5) at the equilibrium point $w^* \equiv 1$:

$$\frac{d\zeta(t)}{dt} = (A_0 + G'[w^*])\zeta(t), \quad (2.7)$$

where $\zeta(t, a) := w(t, a) - w^*$ and $G'[w^*] : L^1 \rightarrow L^1$ is a linear operator with one-dimensional range defined as

$$G'[w^*]\zeta := -h(\zeta)w^*,$$

Then $A_0 + G'[w^*]$ generates a strongly continuous semigroup $T(t) = \exp((A_0 + G'[w^*])t)$, because $A_0 + G'[w^*]$ is a bounded linear perturbation of the population semigroup generator A_0 .

From the biological meaning, the state space of the perturbation ζ is

$$\Sigma^* := \left\{ \phi \in L^1_+(0, \omega) : \int_0^\omega \psi(a)\phi(a)da = 0 \right\}.$$

Then it is easy to see that Σ^* is positively invariant with respect to the semiflow $T(t)$. So we write the restriction of $T(t)$ to the subspace Σ^* as $T_{\Sigma^*}(t)$.

Lemma 2.2 *For the linearized semigroup $T_{\Sigma^*}(t)$, there exist numbers $\epsilon > 0$ and $M(\epsilon) \geq 1$ such that*

$$\|T_{\Sigma^*}(t)\| \leq M(\epsilon)e^{-\epsilon t}. \quad (2.8)$$

Proof: Since the semigroup generated by the population operator with a compact perturbation is eventually compact ([11], [19] Prop. 4.14), it follows that $\omega_0(A_0 + G'[w^*]) = \sup\{\Re z : z \in P_\sigma(A_0 + G'[w^*])\}^2$. Hence the restriction $T_{\Sigma^*}(t)$ is also an eventually compact semigroup. Then if there exists a negative dominant eigenvalue for $A_0 + G'[w^*]$ on the subspace Σ^* , then there exist $M(\epsilon) \geq 1$ and $\epsilon > 0$ such that $\|T_{\Sigma^*}(t)\| \leq M(\epsilon)e^{-\epsilon t}$. So let us consider the eigenvalue equation for $A_0 + G'[w^*]$ on Σ^* :

$$(\lambda - (A_0 + G'[w^*]))\phi = 0, \quad \phi \in D(A_0) \cap \Sigma^*.$$

Then we have

$$\lambda\phi + \phi' + h(\phi) = 0, \quad \phi(0) = \int_0^\omega \pi(a)\phi(a)da,$$

which is solved formally as follows:

$$\phi(a) = \phi(0)e^{-\lambda a} - h(\phi) \int_0^a e^{-\lambda(a-s)} ds.$$

² $P_\sigma(A)$ denotes the point spectrum of the operator A and $\omega_0(A)$ denotes the spectral bound of the semigroup e^{tA} (see [15])

Inserting the above formal solution into the boundary condition, we have

$$\phi(0) = \phi(0) \int_0^\omega \pi(a)e^{-\lambda a} da - h(\phi) \int_0^\omega \pi(a) \int_0^a e^{-\lambda(a-s)} ds da.$$

On the other hand, since $\phi \in \Sigma^*$, we obtain a condition

$$\phi(0) \int_0^\omega \psi(a)e^{-\lambda a} da - h(\phi) \int_0^\omega \psi(a) \int_0^a e^{-\lambda(a-s)} ds da = 0.$$

Then we have a simultaneous equation system for the unknown numbers $\phi(0)$ and $h(\phi)$ as

$$\begin{pmatrix} \Psi_{11}(\lambda) & \Psi_{12}(\lambda) \\ \Psi_{21}(\lambda) & \Psi_{22}(\lambda) \end{pmatrix} \begin{pmatrix} \phi(0) \\ h(\phi) \end{pmatrix} = 0,$$

where

$$\begin{pmatrix} \Psi_{11}(\lambda) & \Psi_{12}(\lambda) \\ \Psi_{21}(\lambda) & \Psi_{22}(\lambda) \end{pmatrix} := \begin{pmatrix} \int_0^\omega \pi(a)e^{-\lambda a} da - 1 & - \int_0^\omega \pi(a) \int_0^a e^{-\lambda(a-s)} ds da, \\ \int_0^\omega \psi(a)e^{-\lambda a} da & - \int_0^\omega \psi(a) \int_0^a e^{-\lambda(a-s)} ds da, \end{pmatrix}$$

Let $\Psi(\lambda) = (\Psi_{ij}(\lambda))_{1 \leq i, j \leq 2}$. Then λ is an eigenvalue of our interest if and only if $\lambda \in \Lambda := \{\lambda \in \mathbf{C} : \det(I - \Psi(\lambda)) = 0\}$. It is easy to see that $\Lambda = \{\lambda \in \mathbf{C} \setminus \{0\} : \Psi_{11}(\lambda) = 0\}$. By using the standard argument for the Euler–Lotka characteristic equation, we know that all eigenvalues associated with eigenvectors in Σ^* have negative real part. \square

Proposition 2.3 *Let $w(t)$ be the solution of (2.5). Then there exists a number $\eta > 0$ such that if $\|w_0 - w^*\| < \eta$, it follows that*

$$\|w(t, \cdot) - w^*\| \leq M(\epsilon)e^{-\epsilon t} \|w_0 - w^*\|, \quad (2.9)$$

for some numbers $M(\epsilon) \geq 1$ and $\epsilon > 0$, where $M(\epsilon)$ is independent from the initial data w_0 .

Proof: Inserting $w^* + \zeta(t)$, $\zeta \in \Sigma^*$ into (2.5), we have

$$\frac{d\zeta(t)}{dt} = (A_0 + G'[w^*])\zeta(t) + R(\zeta(t)), \quad (2.10)$$

where

$$R(\zeta(t)) := G(w^* + \zeta(t)) - G'[w^*]\zeta(t) \in \Sigma^*.$$

From the Fréchet differentiability, there exists a continuous increasing function $k(r)$, $r \geq 0$ with $k(0) = 0$ such that

$$\|G(w^* + \zeta) - G(w^*) - G'[w^*]\zeta\| = \|R(\zeta)\| \leq k(r)\|\zeta\|,$$

for all $\zeta \in L^1$ such that $\|\zeta\| \leq r$. Applying the variation of constants formula to (2.10), we have

$$\zeta(t) = T_{\Sigma^*}(t)\zeta(0) + \int_0^t T_{\Sigma^*}(t-s)R(\zeta(s))ds, \quad (2.11)$$

From Lemma 2.2, if we can choose $\gamma < 0$ such that $\|T_{\Sigma^*}(t)\| \leq M(\gamma)e^{\gamma t}$ with some $M(\gamma) \geq 1$. Choose $r > 0$ such that $k(r) \leq -\gamma/2M(\gamma)$ and let $\eta \leq r/M(\gamma)$. Suppose that $\|\zeta(0)\| < \eta$ and let $0 < t_1 \leq \infty$ be the largest extended real number such that $\|\zeta(t)\| \leq r$ for $0 \leq t < t_1$. Then for $0 \leq t < t_1$, we have

$$\begin{aligned} \|\zeta(t)\| &\leq M(\gamma)e^{\gamma t}\|\zeta(0)\| + M(\gamma) \int_0^t e^{\gamma(t-s)} \|R(\zeta(s))\| ds, \\ &\leq M(\gamma)e^{\gamma t}\|\zeta(0)\| + M(\gamma)e^{\gamma t} \int_0^t e^{-\gamma s} k(r) \|\zeta(s)\| ds. \end{aligned}$$

Therefore we obtain

$$e^{-\gamma t}\|\zeta(t)\| \leq M(\gamma)\|\zeta(0)\| + \frac{-\gamma}{2} \int_0^t e^{-\gamma s} \|\zeta(s)\| ds.$$

From the Gronwall inequality, we have

$$e^{-\gamma t}\|\zeta(t)\| \leq M(\gamma)\|\zeta(0)\|e^{-\frac{\gamma}{2}t}.$$

Thus we arrive at the estimate as $\|\zeta(t)\| \leq M(\gamma)e^{\frac{\gamma}{2}t}\|\zeta(0)\| \leq r$, so we know that $t_1 = \infty$ and (2.9) holds for $-\epsilon = \gamma/2$. \square

3 Linearized stability for the homogeneous epidemic system

For the homogeneous epidemic system, once the stable age distribution is attained, we can rewrite the basic system into the normalized system in which the nonlinear term becomes bilinear (the mass action type). In the following, we prove that the local stability [instability] of steady states of the normalized system implies the stability [instability] of persistent solutions of the original homogeneous system.

Let us introduce the *ratio distributions* for each of the epidemiological classes as follows:

$$u_j(t, a) := \frac{U_j(t, a)}{N(t)\psi(a)}. \quad (3.1)$$

In the following we assume that $N(t) > 0$ for all $t \geq 0$, which holds if the initial age profile is non-trivial.

Note that under the Assumption 1.1, we have

$$w(t, a) = \frac{P(t, a)}{N(t)\psi(a)} = \sum_{j=1}^n u_j(t, a) = \langle e, u(t, a) \rangle, \quad (3.2)$$

where we define $e := (1, 1, \dots, 1)^T$ so that for $u \in Y := (L^1(0, \omega))^n$, we have

$$\langle e, u(a) \rangle := \sum_{i=1}^n u_i(a).$$

Now in order to make a concrete normalization argument possible, we introduce a special homogeneity condition:

Assumption 3.1 *The force of infection is given by an expression of the form*

$$\int_0^\omega \beta(a, \sigma) \frac{U_j(t, \sigma)}{N(t)} d\sigma, \quad (3.3)$$

where β is the transmission probability function (which is assumed to be essentially bounded and nonnegative) and U_j is an infectious class. Other transition entries in the matrix $G(a; U)$ are given bounded nonnegative functions independent of the population density U .

Inserting $U_j(t, a) = N(t)\psi(a)u_j(t, a)$ into (1.1), the homogeneous epidemic system in the stable population is reduced to a Cauchy problem on the Banach space $Y_+ = (L_+^1(0, \omega))^n$:

$$\frac{du(t)}{dt} = \tilde{A}u(t) + \tilde{F}(u(t)) + B(u(t)), \quad u(0) = u_0. \quad (3.4)$$

where the operators \tilde{A} , \tilde{F} and B acting on Y are defined as follows:

$$\tilde{A}\phi := -d\phi/da,$$

$$D(\tilde{A}) := \left\{ \phi \in Y : \phi_i \in AC[0, \omega], \phi(0) = \int_0^\omega \tilde{M}(a)\phi(a)da, \right\},$$

where $\tilde{M}(a) := (k_{ij}\pi(a))_{1 \leq i, j \leq n}$ and $\tilde{F}(\phi)(a) = \tilde{G}(a, \phi)\phi$ where the force of infection term in \tilde{G} is given by the formula as

$$\lambda[a|\phi_j] := \int_0^\omega \beta(a, \sigma)\psi(\sigma)\phi_j(\sigma)d\sigma\phi_k. \quad (3.5)$$

Other elements of the matrix $\tilde{G}(a; U)$ are the same as the corresponding elements of G . The operator \tilde{A} is a generator of the multistate population semigroup $e^{t\tilde{A}}$ ([11]). The nonlinear term B is given by

$$B(\phi) = z(\phi)\phi, \quad (3.6)$$

where $z : Y_+ \rightarrow \mathbf{R}_+$ is a functional given by

$$z(\phi) := r_0 - h(\langle e, \phi \rangle). \quad (3.7)$$

Conversely if the system (3.4) is given, by using its solution u we can recover the total size $N(t)$ from the equation:

$$\frac{1}{N(t)} \frac{dN(t)}{dt} = r_0 - z(u(t)), \quad N(0) = \int_0^\omega P(0, a)da,$$

Once we can determine $N(t)$ from the above equation, U_j and P can be recovered by (3.1) and (3.2).

Let K_1 be a closed convex set given by

$$K_1 := \{\phi \in Y_+ : \langle e, \phi(a) \rangle = 1\}.$$

Moreover, from the definition (3.2), biologically meaningful solutions $u(t)$ of (3.4) should take a value in the closed convex set:

$$K_2 := \left\{ \phi \in Y_+ : \int_0^\omega \psi(a) \langle e, \phi(a) \rangle da = 1 \right\}.$$

Then it follows from $\int_0^\omega \psi(a) da = 1$ that $K_1 \subset K_2$ and $z(\phi) = 0$ if $\phi \in K_1$.

Lemma 3.2 *The sets K_1 and $K_2 \cap \Sigma_0$ are positively invariant with respect to the solution semiflow of (3.4).*

Proof: In the system (2.5), if $w(0, a) = w^*$, we have $w(t, a) = w^*$ for all $t > 0$, which means that the positive orbit u is included in K_1 if $u(0) \in K_1$. Next suppose that $u(0) \in K_2 \cap \Sigma_0$. Then it follows that

$$\frac{d \langle e, u(t) \rangle}{dt} = A_0 \langle e, u(t) \rangle + z(u) \langle e, u(t) \rangle. \quad (3.8)$$

The equation (3.8) is equivalent to (2.5) with $w = \langle e, u \rangle$, hence it follows from Lemma 2.2 that $\langle e, u(t) \rangle$ stays in the state space $\Sigma_0 \cap \Sigma$, which implies that $u(t) \in K_2 \cap \Sigma_0$. \square

Now we adopt a technical assumption, which would be satisfied for many cases:

Assumption 3.3 *The operator $\tilde{F} : Y \rightarrow Y$ is locally Lipschitz continuous and continuously Fréchet differentiable.*

Lemma 3.4 *Let $u(t)$ be a solution of (3.4). Then there exists $B_0 > 0$ and $\epsilon_1 > 0$ such that*

$$\|B(u(t))\| \leq B_0 e^{-\epsilon_1 t} \|u(t)\| \|w_0 - w^*\|, \quad (3.9)$$

where $w_0(a) := \langle e, u(0, a) \rangle$.

Proof: It is sufficient to show that there exists $B_0 > 0$ and $\epsilon_1 > 0$ such that $|z(u(t))| \leq B_0 e^{-\epsilon_1 t} \|w_0 - w^*\|$. It follows from $r_0 = \int_0^\omega (f(a) - \mu(a)) \psi(a) da$ that

$$|z(u(t))| \leq \int_0^\omega |f(a) - \mu(a)| \psi(a) |w^* - w(t, a)| da.$$

From the estimate (2.9), there exist numbers $M > 0$ and $\epsilon > 0$ such that $\|w(t, \cdot) - w^*\| \leq M e^{-\epsilon t} \|w_0 - w^*\|$. Hence if we choose B_0 as $B_0 = M \sup_{a \in [0, \omega]} |f(a) - \mu(a)| \psi(a)$, then (3.9) follows immediately. \square

From Lemma 3.4 we can expect that the asymptotic behavior of (3.4) is determined by the system restricted to K_1 :

$$\frac{dv(t)}{dt} = \tilde{A}v(t) + \tilde{F}(v(t)), \quad v \in K_1. \quad (3.10)$$

In the following, we call (3.10) the *normalized system*. From Lemma 3.4, the normalized system can be seen as a limiting equation with respect to (3.4). The normalized system is no other than the ratio age distribution system in case that the host population attains the stable age distribution, that is, we can prove the following:

Lemma 3.5 *Let $u(t)$ be the solution of (3.4). Then $u(t) \in K_1$ if and only if $P(t, a)$ is the persistent solution of (1.11).*

Proof: If $u(t) \in K_1$, we have $\langle e, u(t, a) \rangle = 1$ which implies that

$$\sum_{j=1}^n u_j(t, a) = \frac{1}{N(t)\psi(a)} \sum_{j=1}^n U_j(t, a) = \frac{P(t, a)}{N(t)\psi(a)}.$$

Therefore $P(t, a) = N(t)\psi(a)$. Then $P(t, a)$ is a separation of variable type solution of (1.11), we know that $N(t) = \|P_0\|e^{r_0 t}$ and P becomes the persistent solution. Conversely if we assume that $P(t, a) = Ce^{r_0 t}\psi(a)$ (C is a positive constant), then $N(t) = Ce^{r_0 t}$, hence $\sum_{j=1}^n u_j(t, a) = 1$. \square

Since K_1 is positively invariant with respect to the solution semiflow of (3.4) and (3.4) is reduced to (3.10) on K_1 , it follows that

Lemma 3.6 *Let $V(t)$ be a solution semiflow such that $V(t)v_0 = v(t; v_0)$ where $v(t; v_0)$ denotes the solution of the normalized system (3.10) with the initial data $v(0) = v_0 \in Y$. Then $V(t)(K_1) \subset K_1$.*

Lemma 3.7 *$u^* \in K_2$ is an equilibrium point of (3.4) if and only if $u^* \in K_1$ and u^* is an equilibrium point of the normalized system (3.10), that is,*

$$\tilde{A}u^* + \tilde{F}(u^*) = 0. \quad (3.11)$$

Let $u^* \in K_1$ be an equilibrium point of (3.4). Define a perturbation $\zeta(t, a)$ by $\zeta = u - u^*$, $u \in K_2$. Inserting $u = u^* + \zeta$ into (3.4), we have an equation for the perturbation:

$$\frac{d\zeta(t)}{dt} = \tilde{A}\zeta(t) + \tilde{F}'[u^*]\zeta + R(u^*, \zeta(t)), \quad (3.12)$$

where $\tilde{F}'[u^*]$ is a Fréchet derivative of \tilde{F} at u^* and the residual term R is given by

$$R(u^*, \zeta(t)) := \tilde{F}(u^* + \zeta(t)) - \tilde{F}(u^*) - \tilde{F}'[u^*]\zeta(t) + B(u^* + \zeta(t)). \quad (3.13)$$

From Lemma 3.4, we obtain an estimate:

$$\|B(u^* + \zeta(t))\| \leq B_0 e^{-\epsilon_1 t} (\|u^*\| + \|\zeta(t)\|) \|w_0 - w^*\|. \quad (3.14)$$

From the definition of Fréchet differentiability, there exists a continuous increasing function $k(r)$, $r \geq 0$ such that $k(0) = 0$ and

$$\|\tilde{F}(v^* + \zeta) - \tilde{F}(u^*) - \tilde{F}'[u^*]\zeta\| \leq k(r)\|\zeta\|, \quad (3.15)$$

for all $\zeta \in Y$ such that $\|\zeta\| \leq r$.

Let $T^*(t)$, $t \geq 0$ be a strongly continuous semigroup of bounded linear operators in the Banach space Y with infinitesimal generator $\tilde{A} + \tilde{F}'[u^*]$. Then if $\omega^* > \omega_0(\tilde{A} + \tilde{F}'[u^*])$, there exists a constant $M(\omega^*) \geq 1$ such that $\|T^*(t)\| \leq M(\omega^*)e^{\omega^* t}$. Now we can prove the following:

Proposition 3.8 *Let $u^* \in K_1$ be an equilibrium solution of (3.4). If $\omega_0(\tilde{A} + \tilde{F}'[u^*]) < 0$, there exist $\eta > 0$, $M \geq 1$ and $\gamma < 0$ such that if $u_0 \in K_2$ and $\|u_0 - u^*\| < \eta$, then for all $t > 0$, it follows that*

$$\|u(t; u_0) - u^*\| \leq M e^{\gamma t} \|u_0 - u^*\|. \quad (3.16)$$

Proof: From (3.12), the perturbation $\zeta(t) := u - u^*$ is given as the continuous solution of the variation of constants formula:

$$\zeta(t) = T^*(t)\zeta(0) + \int_0^t T^*(t-s)R(u^*, \zeta(s))ds. \quad (3.17)$$

Let ω^* be a number such that $0 > \omega^* > \max\{\omega_0(\tilde{A} + \tilde{F}'[u^*]), -\epsilon_1\}$ where ϵ_1 is chosen in Lemma 3.4. Choose $r > 0$ such that $k(s) \leq -\omega^*/4M(\omega^*)$ for $s \in [0, r]$. Let choose $\eta < r$ so small such that

$$0 < \eta \leq \min \left\{ \frac{r}{M(\omega^*)(1 + \frac{B_0\|u^*\|}{\omega^* + \epsilon_1})}, -\frac{\omega^*}{4M(\omega^*)B_0} \right\},$$

Suppose that $\|\zeta(0)\| < \eta$ and let $0 < t_1 \leq \infty$ be the largest extended real number such that $\|\zeta(t)\| \leq r$ for $0 \leq t < t_1$. Note that

$$\|w_0 - w^*\| = \int_0^\omega \left| \sum_{i=1}^n (u_i(0, a) - u_i^*(a)) \right| da \leq \|u(0) - u^*\| = \|\zeta(0)\| < \eta.$$

Then for $0 \leq t < t_1$, we have

$$\begin{aligned} \|\zeta(t)\| &\leq M(\omega^*)e^{\omega^* t} \|\zeta(0)\| + M(\omega^*) \int_0^t e^{\omega^*(t-s)} \|R(u^*, \zeta(s))\| ds, \\ &\leq M(\omega^*)e^{\omega^* t} \|\zeta(0)\| + M(\omega^*)e^{\omega^* t} \int_0^t e^{-\omega^* s} [k(s) + B_0 e^{-\epsilon_1 s} \|\zeta(0)\|] \|\zeta(s)\| ds \\ &\quad + M(\omega^*)e^{\omega^* t} \int_0^t e^{-\omega^* s} B_0 e^{-\epsilon_1 s} \|u^*\| \|\zeta(0)\| ds. \end{aligned}$$

Therefore we obtain

$$e^{-\omega^* t} \|\zeta(t)\| \leq M(\omega^*) \|\zeta(0)\| \left(1 + \frac{B_0 \|u^*\|}{\omega^* + \epsilon_1}\right) + \frac{-\omega^*}{2} \int_0^t e^{-\omega^* s} \|\zeta(s)\| ds.$$

From the Gronwall inequality, we have

$$e^{-\omega^* t} \|\zeta(t)\| \leq M(\omega^*) \|\zeta(0)\| \left(1 + \frac{B_0 \|u^*\|}{\omega^* + \epsilon_1}\right) e^{-\frac{\omega^*}{2} t}.$$

Then we arrive at the estimate as

$$\|\zeta(t)\| \leq M(\omega^*) e^{\frac{\omega^*}{2} t} \left(1 + \frac{B_0 \|u^*\|}{\omega^* + \epsilon_1}\right) \|\zeta(0)\| \leq r.$$

Then we know that $t_1 = \infty$ and (3.16) holds for $\gamma = \omega^*/2$. \square

Here we should note that $\tilde{A} + \tilde{F}'[u^*]$ is the linearized generator of the normalized system (3.10) but it is *not* the linearized generator of the system (3.4), since $B(u(t))$ includes a linear term. Nevertheless Proposition 3.8 tells us that the local stability of equilibrium point of the normalized system (3.10) implies that of the ratio age distribution system (3.4).

In many cases, we can show that the linearized semigroup $e^{(\tilde{A} + \tilde{F}'[u^*])t}$ is eventually compact, so the steady state is locally asymptotically stable if all eigenvalues of the linearized generator $\tilde{A} + \tilde{F}'[u^*]$ have negative real part.

Here we clear the meaning of stability of the persistent solution implied in Proposition 3.8. Observe that the initial data of the homogeneous system (1.1) can be decomposed as follows:

$$U_j(0, a) = N(0) \psi(a) (u_j^*(a) + r_j(a)), \quad 1 \leq j \leq n,$$

where the function $r(a) = (r_1(a), \dots, r_n(a))$ satisfies the condition

$$\int_0^\omega \psi(a) \langle e, r(a) \rangle da = 0,$$

and the condition $\|u_0 - u^*\| < \eta$ can be written as $\|r\| < \eta$. Corresponding to the equilibrium point u^* , we can consider the persistent solution as

$$U^*(t, a) = N_\infty e^{r_0 t} \psi(a) u^*(a),$$

where $N_\infty := \lim_{t \rightarrow \infty} e^{-r_0 t} N(t)$. Then we can show that following:

Proposition 3.9 *There exists a number $\eta > 0$ such that if $\|r\| < \eta$, then*

$$\lim_{t \rightarrow \infty} e^{-r_0 t} \|U(t, \cdot) - U^*(t, \cdot)\| = 0. \quad (3.18)$$

Proof: Observe that

$$\begin{aligned} & \int_0^\omega |U_j(t, a) - U_j^*(t, a)| da \leq \int_0^\omega \left| \frac{U_j(t, a)}{N(t)\psi(a)} - \frac{N_\infty e^{r_0 t}}{N(t)} u_j^*(a) \right| N(t)\psi(a) da \\ & \leq N(t) \sup_{a \in [0, \omega]} |\psi(a)| \left[\int_0^\omega \left| \frac{U_j(t, a)}{N(t)\psi(a)} - u_j^*(a) \right| da + \int_0^\omega |u_j^*(a)| da \left| 1 - \frac{N_\infty e^{r_0 t}}{N(t)} \right| \right]. \end{aligned}$$

From (3.16) and $N(t)e^{-r_0 t} \rightarrow N_\infty > 0$, we obtain (3.18). \square

For the system (3.4), if we choose the initial data such that $u(0) \in K_1$, then $u(t) \in K_1$ for all $t > 0$ since K_1 is positively invariant. In such case, the orbit $u(t) \in K_1$ is described by the normalized system (3.10) since $B(u(t)) = 0$. Therefore the instability of u^* as the equilibrium point of (3.10) implies the instability of u^* as the equilibrium point of (3.4):

Proposition 3.10 *Let u^* be an equilibrium solution of (3.4) and (3.10). If u^* is unstable with respect to the normalized system (3.10), it is also unstable for the system (3.4).*

From the above results, we know that to examine asymptotic behavior of the persistent solutions of the homogeneous epidemic system in the stable population, it is sufficient to study equilibrium points of the normalized system by using the principle of linearized stability in the classical sense.

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