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Derivative Nonlinear Schrödinger Equation with General Cubic Nonlinearity

宮崎大学 教育文化学部 北 直泰 (NAOYASU KITA)
Faculty of Education and Culture, Miyazaki University

1 Introduction and Main Theorem

We consider the Cauchy problem for the nonlinear Schrödinger equation which includes the first order derivatives of unknown function in its nonlinearity :

$$\begin{cases} i\partial_t u &= -\frac{1}{2}\partial_x^2 u + \mathcal{N}(u, \partial_x u), \\ u(0,x) &= u_0(x), \end{cases} \tag{1.1}$$

where u is unknown function from $(t,x) \in \mathbf{R}\times\mathbf{R}$ to $\mathbf{C}$. The derivatives ∂_t and ∂_x denote $\partial/\partial t$ and $\partial/\partial x$, respectively. The nonlinearity $\mathcal{N}(u,q)$ consists of the cubic polynomial of u, $\bar{u}$, q and $\bar{q}$, i.e.,

$$\mathcal{N}(u,q) = \sum_{j_1+j_2+j_3+j_4=3} C_{j_1j_2j_3j_4} u^{j_1}\bar{u}^{j_2}q^{j_3}\bar{q}^{j_4},$$

where $C_{j_1j_2j_3j_4} \in \mathbf{C}$ and $j_1, \cdots, j_4$ are nonnegative integers.

When the nonlinear term contains the derivatives, it causes the regularity loss unless the special structure is imposed in the nonlinearity. Since the Schrödinger group $U_0(t) = \exp(it\partial_x^2/2)$ does not absove the derivatives in $L_T^\infty(L_x^2)$, we could not make use of contraction mapping principle simply in $L_T^\infty(L_x^2)$ framework, where $L_T^p(L_x^q)$ denotes the function space endowed with the norm $\|f\|_{L_T^p(L_x^q)} = \left(\int_0^T \|f(t,\cdot)\|_{L_x^q}^p\, dt\right)^{1/p}$. Of course, if we impose the special structure on $\mathcal{N}(u,q)$, it is possible to derive a priori estimate so that the energy method works. For the general nonlinearity as in the present case, we refer to Kenig-Ponce-Vega's work [2]. In [2], they derived the crucial smoothing property of $U(t)$ in the new function space $L_x^\infty(L_T^2)$:

$$\|\partial_x \int_0^t U(t-t')F(t')\, dt'\|_{L_x^\infty L_T^2} \le C\|F\|_{L_x^1(L_T^2)},$$

where $\|u\|_{L_x^\infty(L_T^q)} = \sup_x(\int_0^T |u(t,x)|^q\, dt)^{1/q}$ and $\|u\|_{L_x^p(L_T^q)} = \|(\|u(\cdot,x)\|_{L_T^q})\|_{L_x^p}$. This linear estimate recovers the regularity loss in the nonlinearity and the contraction mapping priciple is applicable via the integral equation and obtain the local well-posedness of the solution. In their work, however, one requires the size restriction on the initial data. This is because the estimate $L_x^2(L_T^\infty)\cdot L_x^2(L_T^\infty)\cdot L_x^\infty(L_T^2)\subset L_x^1(L_T^2)$ is applied to the nonlinear term and the quantity $\|u\|_{L_x^2(L_T^\infty)}$ does not expect to be small even when $T\downarrow 0$.

To remove this size restriction, Hayashi-Ozawa [1] applied a nonlinear transformation of unknown function so that the nonlinear component causing the regularity loss is eliminated. They showed that the energy method is still applicable to the general nonlinear case. In [1], they obtained the existence and uniqueness of the solution by assuming that $u_0\in H_x^3$ (the sophisticated estimate likely relaxes this regularity condition into H_x^s with $s>5/2$ since the regularity of u_0 is determined by the estimate of $\|\partial_x^2 u(t)\|_{L_x^\infty}$), where $H_x^s=\{u;\|u\|_{H_x^s}=\|\langle D_x\rangle^s u\|_{L_x^2}<\infty\}$ with $\langle D_x\rangle^\sigma=\mathcal{F}^{-1}\langle\xi\rangle^s\mathcal{F}$ with $\langle\xi\rangle=(1+\xi^2)^{1/2}$.

More recently, Kenig-Ponce-Vega [4] have studied how to remove the size restriction of u_0 and obtained the local well-posedness of the solution. In [4], they write (1.1) as

$$\begin{aligned} i\partial_t u^{(k)} &= -\frac{1}{2}\partial_x^2 u^{(k)}+\mathcal{N}_q(u,\partial_x u)\partial_x u^{(k)}+\mathcal{N}_{\bar q}(u,\partial_x u)\partial_x\bar u^{(k)}+(\text{remainder})\\ &= -\frac{1}{2}\partial_x^2 u^{(k)}+\mathcal{N}_q(u_0,\partial_x u_0)\partial_x u^{(k)}+\mathcal{N}_{\bar q}(u_0,\partial_x u_0)\partial_x\bar u^{(k)}+(\text{remainder}), \quad (1.2)\end{aligned}$$

where $u^{(k)}=\partial_x^k u$, $\mathcal{N}_q(u,q)=\partial_q\mathcal{N}(u,q)$, $\mathcal{N}_{\bar q}(u,q)=\partial_{\bar q}\mathcal{N}(u,q)$ and the remainder consists of at most k-th order derivatives together with $\partial_x(u-u_0)\partial_x u^{(k)}$ etc. They derived the smoothing property of the linear solution to $\mathcal{L}v=F$ in the time-space norm, where

$$\mathcal{L}v=i\partial_t v+\frac{1}{2}\partial_x^2 v-\mathcal{N}_q(u_0,\partial_x u_0)\partial_x v-\mathcal{N}_{\bar q}(u_0,\partial_x u_0)\partial_x\bar v.$$

The merit arising from the representation (1.2) is that $\|\partial_x(u-u_0)\|_{L_x^2(L_T^\infty)}$ or $\|u-u_0\|_{L_x^2(L_T^\infty)}$ included in the remainder is regarded as negligible quantity by taking $T>0$ sufficiently small. Hence, one can apply the contraction mapping principle via the integral equation. In their argument, the theory of pseudo-differential operators is the key to the estimate of v. This suggests that one requires the large regularity of u_0.

Our aim in this work is to minimize the regularity of u_0 without any size restriction and to obtain the local well-posedness of the solution. The idea is based on a gauge transformation different from Hayashi-Ozawa type and a priori estimate in terms of the smoothing properties of $U(t)$ due to Kenig-Ponce-Vega [2]. Concretely speaking, we first modify (1.1) by the following regularization:

$$\begin{cases} i\partial_t u_\nu &= -\frac{1}{2}\partial_x^2 u_\nu+\mathcal{N}(u_\nu,\partial_x\eta_\nu * u_\nu),\\ u_\nu(0,x) &= u_0(x),\end{cases} \quad (1.3)$$

where $\eta_\nu(x)=\nu^{-1}\eta(x/\nu)$ and $\int\eta(x)dx=1$ with $\eta\in C_0^\infty(\mathbf{R})$ and $\nu\in(0,1]$. Since $\eta_\nu *$ provides the regularizing property like $\|\partial_x\eta_\nu * u_\nu\|_{L_x^2}\le C\nu^{-1}\|u_\nu\|_{L_x^2}$, a convenient local solution to (1.3) is constructed via the integral equation. Let $T_\nu\in(0,\infty]$ be the upper time bound for the existence of the solution. To realize the solution to (1.1) by taking $\nu\downarrow 0$, we require the lower uniform bound of T_ν. For this purpose, we derive an a priori estimate in the Banach space Y_T with the norm:

$$|||u|||_{Y_T} = \|u\|_{L_T^\infty(H_x^s)}+\|\langle D_x\rangle^{s-3/2}\partial_x^2 u\|_{L_x^\infty(L_T^2)}+\max_{j=0,1}\|\langle D_x\rangle^\mu\partial_x^j u\|_{L_x^2(L_T^\infty)},$$

where $s>0$ will be specified later and $\mu>0$ is small. This is the remarkably differnt point from the usual energy method. To seek for the a priori estimate, we apply the gauge transformation given by the pseudo-differential operator and, roughly speaking, eliminate the heavy term in the nonlinearity of (1.2) after diagonalizing the system of $\vec{u}_\nu = (u_\nu, \bar{u}_\nu)^t$ (see section 2). This kind of elimination is available especially in one space dimension. In our argument, the regularity condition on u_0 are essentially given by (so-called) the estimate of maximal function, i.e., $\|\partial_x U(t)u_0\|_{L^2_x(L^\infty_T)} \le C\|u_0\|_{H^\sigma_x}$, where $\sigma > 3/2$. Our main theorem in this article is

Theorem 1.1 *Let $u_0 \in H^s_x$ with $s > 3/2$. Then, we have the following assertions.*

(1) *For some $T > 0$, there exists a unique solution u to (1.1) such that $u \in C([0,T]; H^s_x) \cap Y_T$.*

(2) *Let u' be the solution to (1.1) with initial data $u_0' \in B_\rho(u_0) \equiv \{v_0; \|v_0 - u_0\|_{H^s_x} < \rho\}$ where $\rho > 0$ is sufficiently small. Then, for some $T' \in (0,T)$, we have*

$$\begin{aligned} \|u' - u\|_{L^\infty_{T'}(H^s_x)} &\le C\|u_0' - u_0\|_{H^s_x}, \\ \|\langle D_x\rangle^{s-3/2}\partial_x^2(u' - u)\|_{L^\infty_x(L^2_{T'})} &\le C\|u_0' - u_0\|_{H^s_x}. \end{aligned}$$

We now close this section by introducing several notations. The quantity $\|\cdot\|_X$ denotes the norm of a Banach space X. Let $\mathcal{B}(X;Y)$ be the set of bounded operators from X to Y. When $X = Y$, we simply write $\mathcal{B}(X;X)$ as $\mathcal{B}(X)$. The summation space is defined by $X + Y = \{x + y; x \in X \text{ and } y \in Y\}$ with the norm $\|f\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y; f = x + y, x \in X \text{ and } y \in Y\}$. Let $L^p_x(L^r_T)$ and $L^r_T(L^p_x(\mathbf{R}))$ be the function spaces $L^p(\mathbf{R}; L^r[0,T])$ and $L^r([0,T]; L^p_x)$, respectively. The fractional order differentiaion D^s_x stands for $\mathcal{F}^{-1}|\xi|^s\mathcal{F}$. We sometimes use $\hat{f}$ or $\mathcal{F}f$ for the Fourier transform. Throughout this paper, C denotes a positive constant which is independent of $\nu \in (0,1]$ and does not diverge as $\varphi \to u_0$ in H^s_x. Also, C_φ denotes a positive constant which is independent of $\nu \in (0,1]$ but may possibly diverge as $\varphi \to u_0$ in H^s_x.

2 Deformation of (1.3)

In this section, we deform (1.1) by using a gauge transformation defined by a pseudo-differential operator so that the uniform bound of $\|u_\nu\|_{Y_T}$ $(0 < \nu \le 1)$ is derived. Let $u^{(1)}_\nu = \partial_x u_\nu$. Then, $u^{(1)}_\nu$ satisfies

$$\begin{aligned} i\partial_t u^{(1)}_\nu &= -\frac{1}{2}\partial_x^2 u^{(1)}_\nu + \mathcal{N}_q(u_\nu, \eta_\nu * u^{(1)}_\nu)\partial_x\eta_\nu * u^{(1)}_\nu + \mathcal{N}_{\bar{q}}(u_\nu, \eta_\nu * u^{(1)}_\nu)\partial_x\eta_\nu * \bar{u}^{(1)}_\nu \\ &\quad + \mathcal{N}_u(u_\nu, \eta_\nu * u^{(1)}_\nu)\eta_\nu * u^{(1)}_\nu + \mathcal{N}_{\bar{u}}(u_\nu, \eta_\nu * u^{(1)}_\nu)\eta_\nu * \bar{u}^{(1)}_\nu, \end{aligned}$$

where $\mathcal{N}_u$ and $\mathcal{N}_{\bar{u}}$ stand for the partial derivatives of $\mathcal{N}(u,q)$ with respective to u and $\bar{u}$. Since $\partial_x \bar{u}^{(1)}_\nu$ does not vanish by the gauge transformation, we first eliminate it by the diagonalization. To this end, we employ the systemized representation of the above equation. Namely, let $\vec{u}^{(1)}_\nu = (u^{(1)}_\nu, \bar{u}^{(1)}_\nu)^t$ and write

$$i\partial_t \vec{u}^{(1)}_\nu = -\frac{1}{2}\mathbf{A}\partial_x^2 \vec{u}^{(1)}_\nu + \mathbf{B}_\nu(u_\nu)\partial_x\eta_\nu * \vec{u}^{(1)}_\nu + \vec{P}_\nu(u_\nu), \tag{2.1}$$

where $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{B}_\nu(u) = \begin{pmatrix} \mathcal{N}_q(u, \partial_x\eta_\nu * u) & \mathcal{N}_{\bar{q}}(u, \partial_x\eta_\nu * u) \\ -\overline{\mathcal{N}_{\bar{q}}(u, \partial_x\eta_\nu * u)} & -\overline{\mathcal{N}_q(u, \partial_x\eta_\nu * u)} \end{pmatrix}$ and $\vec{P}_\nu(u)$ is

$$\vec{P}_\nu(u) = \begin{pmatrix} \mathcal{N}_u(u, \partial_x\eta_\nu * u)\partial_x\eta_\nu * u + \mathcal{N}_{\bar{u}}(u, \partial_x\eta_\nu * u)\partial_x\eta_\nu * \bar{u} \\ -\overline{\mathcal{N}_{\bar{u}}(u, \partial_x\eta_\nu * u)}\partial_x\eta_\nu * u - \overline{\mathcal{N}_u(u, \partial_x\eta_\nu * u)}\partial_x\eta_\nu * \bar{u} \end{pmatrix}.$$

(Step 1) Diagonalization. Let $\varphi(x) \in C_0^\infty(\mathbf{R})$ (which will be taken sufficiently close to u_0 in X^s so that $u_\nu(t) - \varphi$ is small when $t \downarrow 0$). We write (2.1) as

$$\begin{aligned} i\partial_t\vec{u}_\nu^{(1)} &= -\frac{1}{2}\mathbf{A}\partial_x^2\vec{u}_\nu^{(1)} + \mathbf{B}_\nu(\varphi)\partial_x\eta_\nu * \vec{u}_\nu^{(1)} \\ &\quad +(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\partial_x\eta_\nu * \vec{u}_\nu^{(1)} + \vec{P}_\nu(u_\nu), \end{aligned} \tag{2.2}$$

Some readers might wander why we do not take $\varphi = u_0$. The answer to this question will be shown at the end of this section. Let

$$\vec{v}_\nu = (\mathbf{I} - \mathbf{J}_\nu\langle D_x\rangle^{-2}\partial_x\eta_\nu *)\vec{u}_\nu^{(1)}, \tag{2.3}$$

where $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{J}_\nu = \begin{pmatrix} 0 & -\mathcal{N}_{\bar{q}}(\varphi, \partial_x\eta_\nu * \varphi) \\ -\overline{\mathcal{N}_{\bar{q}}(\varphi, \partial_x\eta_\nu * \varphi)} & 0 \end{pmatrix}$. By the commutator relation like

$$\begin{aligned} [(\mathbf{I} - \mathbf{J}_\nu\langle D_x\rangle^{-2}\partial_x\eta_\nu *), -\frac{1}{2}\mathbf{A}\partial_x^2] &= \begin{pmatrix} 0 & -\mathcal{N}_{\bar{q}}(\varphi, \partial_x\eta_\nu * \varphi) \\ -\overline{\mathcal{N}_{\bar{q}}(\varphi, \partial_x\eta_\nu * \varphi)} & 0 \end{pmatrix} \langle D_x\rangle^{-2}\partial_x^3\eta_\nu * \\ &\quad -\mathbf{A}((\partial_x\mathbf{J}_\nu)\langle D_x\rangle^{-2}\partial_x^2 + \frac{1}{2}(\partial_x^2\mathbf{J}_\nu)\langle D_x\rangle^{-2}\partial_x)\eta_\nu *, \end{aligned}$$

we see that

$$\begin{aligned} i\partial_t\vec{v}_\nu &= -\frac{1}{2}\mathbf{A}\partial_x^2\vec{v}_\nu + \mathbf{B}_{\nu,diag}(\varphi)\partial_x\eta_\nu * \vec{v}_\nu \\ &\quad +(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\partial_x^2\eta_\nu * \vec{u}_\nu + \vec{Q}_\nu(\varphi, u_\nu), \end{aligned} \tag{2.4}$$

where $\vec{u}_\nu = (u_\nu, \bar{u}_\nu)^t$ and $\mathbf{B}_{\nu,diag}(\varphi)$ denotes the diagonal part of $\mathbf{B}_\nu(\varphi)$ and

$$\begin{aligned} \vec{Q}_\nu(\varphi, u) &= -\mathbf{J}_\nu\langle D_x\rangle^{-2}\partial_x\eta_\nu * \mathbf{B}_\nu(u)\partial_x^2\eta_\nu * \vec{u} + (\mathbf{I} - \mathbf{J}_\nu\langle D_x\rangle^{-2}\partial_x\eta_\nu *)\vec{P}_\nu(u) \\ &\quad +\mathbf{B}_{\nu,diag}(\varphi)\partial_x\eta_\nu * (\mathbf{J}_\nu\langle D_x\rangle^{-2}\partial_x^2\eta_\nu * \vec{u}) - \mathbf{B}_{\nu,off}(\varphi)(\mathbf{I} + \langle D_x\rangle^{-2}\partial_x^2)\partial_x^2\eta_\nu * \vec{u} \\ &\quad -\mathbf{A}((\partial_x\mathbf{J}_\nu)\langle D_x\rangle^{-2}\partial_x^3\eta_\nu * \vec{u} + \frac{1}{2}(\partial_x^2\mathbf{J}_\nu)\langle D_x\rangle^{-2}\partial_x^2\eta_\nu * \vec{u}), \end{aligned}$$

with $\vec{u} = (u, \bar{u})^t$ and $\mathbf{B}_{\nu,off}(\varphi) = \mathbf{B}_\nu(\varphi) - \mathbf{B}_{\nu,diag}$.

(Step2) Gauge Transformation. To eliminate $\mathbf{B}_{\nu,diag}(\eta_\nu * \varphi)\eta_\nu * \vec{v}_\nu$ on the right hand side of (2.4), we set $\vec{w}_\nu \equiv \mathbf{K}_\nu(x, i^{-1}\partial_x)\vec{v}_\nu = \mathbf{K}_\nu\vec{v}_\nu$ where $\mathbf{K}_\nu(x, i^{-1}\partial_x)$ is the pseudo-differential operator with the symbol:

$$\mathbf{K}_\nu(x, \xi) = \begin{pmatrix} \exp(-\hat{\eta}(\nu\xi)\partial_x^{-1}\mathcal{N}_q(\varphi, \partial_x\eta_\nu * \varphi)) & 0 \\ 0 & \exp(-\hat{\eta}(\nu\xi)\partial_x^{-1}\overline{\mathcal{N}_q(\varphi, \partial_x\eta_\nu * \varphi)}) \end{pmatrix},$$

where $\partial_x^{-1} f$ denotes $\int_{-\infty}^{x} f(y)\, dy$. This transformation yields

$$i\partial_t \vec{w}_\nu = -\frac{1}{2}\mathbf{A}\partial_x^2 \vec{w}_\nu + \mathbf{K}_\nu(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\eta_\nu * \partial_x^2 \vec{u}_\nu + \vec{R}_\nu(\varphi, u_\nu), \tag{2.5}$$

where $\vec{R}_\nu(\varphi, u_\nu) = (1/2)\mathbf{A}(\partial_x^2 \mathbf{K}_\nu)\vec{v}_\nu + \mathbf{K}_\nu \vec{Q}_\nu(\varphi, u_\nu)$ and the symbol of $(\partial_x^2 \mathbf{K}_\nu)$ is defined by $\partial_x^2 \mathbf{K}_\nu(x, \xi)$. Since the remainder $\vec{R}_\nu(\varphi, u_\nu)$ contains the large order derivatives of φ, we could not replace φ by u_0.

3 Preliminaries

In this section, we introduce several key estimates frequently used in our argument. In what follows, we employ the brief notation GF for $\int_0^t U(t-t')F(t')\, dt'$. The smoothing property of $U(t)$ and G plays an important role to recover the regularity loss arising from the nonlinearity. Hereafter, we assume that $0 < T < 1$.

Lemma 3.1 *Let $p \in [2, \infty]$ and $q \in [2, \infty)$. Then, we have*

$$\|D_x^{1/2} U(t)\phi\|_{L_x^\infty(L_T^2)} \le C\|\phi\|_{L_x^2}, \tag{3.1}$$
$$\|\partial_x GF\|_{L_x^\infty(L_T^2)} \le C\|F\|_{L_x^1(L_T^2)}, \tag{3.2}$$
$$\|D_x^{1/2} GF\|_{L_T^\infty(L_x^2)} \le C\|F\|_{L_x^1(L_T^2)}. \tag{3.3}$$

Proof of Lemma 3.1. All the estimates in Lemma 3.1 are given in [3; Theorem 2.3, Corollary 2.3]. □

Let us call $\|f(\cdot, x)\|_{L_T^\infty}$ "the maximal function of $f(t,x)$". We next give the estimates for the maximal function. Remark that the estimate (3.5) essenntially determines the regularity constraint of the initial data.

Lemma 3.2 *Let $\sigma > 1/2$. Then, we have*

$$\|U(t)\phi\|_{L_x^2(L_T^\infty)} \le C\|\phi\|_{H_x^\sigma}, \tag{3.4}$$
$$\|GF\|_{L_x^2(L_T^\infty)} \le CT^{1/4}(1+T)^{\sigma/2-1/4}\|\langle D_x\rangle^{\sigma-1/2} F\|_{L_x^1(L_T^2)}. \tag{3.5}$$

Proof of Lemma 3.2. For the estimate (3.4), see [5]. The estimate (3.5) is proved in [6], where the estimate of maximal function is derived for the linearized Benjamin-Ono equation but the derivation in [6] is similarly applied to the Schrödinger equation. In (3.5), the power of T is extracted by the normal scaling argument. □

When we apply the fractional order derivative to the nonlinear term, we often use Leibniz' type rule described in the following.

Lemma 3.3 *(1) Let* $\sigma \in (0,1)$, $\sigma_1, \sigma_2 \in [0,\sigma]$ *with* $\sigma = \sigma_1 + \sigma_2$. *Also, let* $p, r \in (1,\infty)$ *and* $p_1, p_2, r_1, r_2 \in (1,\infty)$ *with* $1/p = 1/p_1 + 1/p_2$ *and* $1/r = 1/r_1 + 1/r_2$. *Then, we have*

$$\|D_x^\sigma(fg) - (D_x^\sigma f)g - f(D_x^\sigma g)\|_{L_x^p(L_T^r)} \leq C\|D_x^{\sigma_1} f\|_{L_x^{p_1}(L_T^{r_1})}\|D_x^{\sigma_2} g\|_{L_x^{p_2}(L_T^{r_2})}. \quad (3.6)$$

Moreover, for $\sigma_1 = 0$, *the value* $r_1 = \infty$ *is allowed.*

(2) Let $\sigma, \sigma_1, \sigma_2$ *as in (1). Also,* $p_1, p_2, r_1, r_2 \in (1,\infty)$ *satisfy* $1 = 1/p_1 + 1/p_2$ *and* $1/2 = 1/r_1 + 1/r_2$. *Then, we have*

$$\|D_x^\sigma(fg) - (D_x^\sigma f)g - f(D_x^\sigma g)\|_{L_x^1(L_T^2)} \leq C\|D_x^{\sigma_1} f\|_{L_x^{p_1}(L_T^{r_1})}\|D_x^{\sigma_2} g\|_{L_x^{p_2}(L_T^{r_2})}. \quad (3.7)$$

Proof of Lemma 3.3. See [4; Appendix]. □

In the nonlinear estimate, we often encounter the lower order derivatives like $D_x^{s-3/2}\partial_x u$ and $\partial_x^2 u$ etc. The following interpolation helps us estimate these quantities. In particular, we require the end point case, i.e., $p_0 = 1, p_1 = \infty, r_0 = \infty$ and $r_1 = 2$.

Lemma 3.4 *Let* $\sigma = (1-\theta)\sigma_0 + \theta\sigma_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$ *and* $1/r = (1-\theta)/r_0 + \theta/r_1$ *with* $\theta \in [0,1]$ *and* $p_0, p_1, r_0, r_1 \in [1,\infty]$. *Then, for* $f \in \mathcal{S}(\mathbf{R}; C^\infty[0,T])$, *we have*

$$\begin{aligned}\|D_x^\sigma f\|_{L_x^p(L_T^r)} \leq\ & \sup_{\lambda\in\mathbf{R}}(e^{-\lambda^2}\|D_x^{\sigma_0+i\lambda(\sigma_1-\sigma_0)} f\|_{L_x^{p_0}(L_T^{r_0})})^{1-\theta} \\ & \times \sup_{\lambda\in\mathbf{R}}(e^{1-\lambda^2}\|D_x^{\sigma_1+i\lambda(\sigma_1-\sigma_0)} f\|_{L_x^{p_1}(L_T^{r_1})})^{\theta}. \quad (3.8)\end{aligned}$$

Proof of Lemma 3.4. Let $f, g \in C_0^\infty(\mathbf{R}; C^\infty[0,T])$ and

$$g_z(t,x) = \|g(\cdot,x)\|_{L_T^{r'}}^{(1-z)(p'/p_0'-r'/r_0')+z(p'/p_1'-r'/r_1')}|g(t,x)|^{(1-z)r'/r_0'+zr'/r_1'}\mathrm{sgn} g(t,x)$$

with $z \in \mathbf{C}$ and $1/p + 1/p' = 1/r + 1/r' = 1$. By the three line theorem on the strip $\{z; 0 \leq \mathrm{Re} z \leq 1\}$, we see that

$$\begin{aligned}|e^{z^2}((g_z, D_x^{(1-z)\sigma_0+z\sigma_1} f))| \leq\ & \sup_\lambda |e^{-\lambda^2}((g_{i\lambda}, D_x^{\sigma_0+i\lambda(\sigma_1-\sigma_0)} f))|^{1-\mathrm{Re} z} \\ & \times \sup_\lambda |e^{(1+i\lambda)^2}((g_{1+i\lambda}, D_x^{\sigma_1+i\lambda(\sigma_1-\sigma_0)} f))|^{\mathrm{Re} z}, \quad (3.9)\end{aligned}$$

where $((\cdot,\cdot))$ denotes the integration of time-space variables. Take $z = \theta$. Then, Hölder's inequality gives the bound of the right hand side of (3.9) like

$$\|g\|_{L_x^{p'}(L_T^{r'})}\sup_\lambda(e^{-\lambda^2}\|D_x^{\sigma_0+i\lambda(\sigma_1-\sigma_0)} f\|_{L_x^{p_0}(L_T^{r_0})})^{1-\theta}\sup_\lambda(e^{1-\lambda^2}\|D_x^{\sigma_1+i\lambda(\sigma_1-\sigma_0)} f\|_{L_x^{p_1}(L_T^{r_1})})^{\theta}.$$

Then, the duality argument yields Lemma 3.4. □

We next show the estimate of the gauge transform $\mathbf{K}_\nu(x, i^{-1}\partial_x)$.

Lemma 3.5 *Let $p, r \in [1,\infty]$ and $\sigma \in [0,1)$. Then, we have*

$$
\begin{aligned}
&\|D_x^\sigma \mathbf{K}_\nu(x, i^{-1}\partial_x)\vec{f}\|_{L_x^p(L_T^r)} \\
&\leq \; C\exp(C\|\varphi\|_{H_x^s})\|\langle D_x\rangle^\sigma \vec{f}\|_{L_x^p(L_T^r)}.
\end{aligned} \tag{3.10}
$$

In the above inequality, we may replace $\|\cdot\|_{L_x^p(L_T^r)}$ by $\|\cdot\|_{L_x^p}$.

Proof of Lemma 3.5. It sufficies to consider the pseudo-differential operator with the symbol like $k_\nu(x,\xi) = \exp(\hat{\eta}(\nu\xi)\psi(x))$, where $\psi = \partial_x^{-1}\mathcal{N}_q(\varphi, \eta_\nu * \partial_x\varphi)$ or $\partial_x^{-1}\overline{\mathcal{N}_q(\varphi, \eta_\nu * \partial_x\varphi)}$. We first show that $k_\nu(x, i^{-1}\partial_x) \in \mathcal{B}(L_x^p L_T^r)$. Note that $k_\nu(x, i^{-1}\partial_x) - I$ has the integral kernel given by

$$
\begin{aligned}
|[k_\nu(x, i^{-1}\partial_x) - I](x,y)| &= \left|(2\pi\nu)^{-1}\int \{\exp(\hat{\eta}(\xi)\psi(x)) - 1\}e^{i\xi(x-y)/\nu}\, d\xi\right| \\
&\leq C_N \exp(C\|\psi\|_{L_x^\infty})\nu^{-1}\langle (x-y)/\nu\rangle^{-N},
\end{aligned}
$$

where the last inequality in the above follows from the integration by parts. Therefore, Young's inequality yields $k_\nu(x, i^{-1}\partial_x) = I + (k_\nu(x, i^{-1}\partial_x) - I) \in \mathcal{B}(L_x^p(L_T^r))$.

We next show that $[\langle D_x\rangle^\sigma, k_\nu(x, i^{-1}\partial_x)] \in \mathcal{B}(L_x^p(L_T^r))$ and its operator norm is bounded by $C\|\partial_x\psi\|_{L_x^\infty}\exp(C\|\psi\|_{L_x^\infty})$. Note that the integral kernel of $[\langle D_x\rangle, k_\nu(x, i^{-1}\partial_x)]$ is given by the oscillatory integral like

$$
\begin{aligned}
L(x,y) &\equiv (2\pi)^{-2}\iiint e^{i(x-z)\xi}\langle\xi\rangle^\sigma \times e^{i(z-y)\zeta}(k_\nu(z,\zeta) - k_\nu(x,\zeta))\, d\xi d\zeta dz \\
&= (2\pi)^{-2}\iiint e^{i(x-z)\xi} i^{-1}\partial_\xi\langle\xi\rangle^\sigma \times e^{i(z-y)\zeta}\int_0^1 \partial_x k_\nu(\theta z + (1-\theta)x, \zeta)\, d\theta d\xi d\zeta dz.
\end{aligned}
$$

Since

$$
\left|\int e^{i(x-z)\xi}\partial_\xi\langle\xi\rangle^\sigma\, d\xi\right| \leq C_N|x-z|^{-\sigma}\langle x-z\rangle^{-N}
$$

and

$$
\begin{aligned}
&\left|\int e^{i(z-y)\zeta}\int_0^1 \partial_x k_\nu(\theta z + (1-\theta)x, \zeta)\, d\theta d\zeta\right| \\
&\leq C_N\|\partial_x\psi\|_{L_x^\infty}\exp(C\|\psi\|_{L_x^\infty})\nu^{-1}\langle (z-y)/\nu\rangle^{-N},
\end{aligned}
$$

we see that

$$
\begin{aligned}
|L(x,y)| &\leq C_N\|\partial_x\psi\|_{L_x^\infty}\exp(C\|\psi\|_{L_x^\infty})\int |x-z|^{-\sigma}\langle x-z\rangle^{-N}\nu^{-1}\langle (z-y)/\nu\rangle^{-N}\, dz \\
&\leq C_N\|\partial_x\psi\|_{L_x^\infty}\exp(C\|\psi\|_{L_x^\infty})|x-y|^\sigma\langle x-y\rangle^{-N}.
\end{aligned}
$$

Thus, Young's inequality yields $[\langle D_x\rangle^\sigma, k_\nu(x, i^{-1}\partial_x)] \in \mathcal{B}(L_x^p(L_T^r))$. Since $\langle D_x\rangle^\sigma - D_x^\sigma \in \mathcal{B}(L_x^p(L_T^r))$, we obtain Lemma 3.5. □

4 Nonlinear Estimates

When we apply Lemma 3.1 to the nonlinearity, we require the nonlinear estimates given in the following two lemmas. In what follows, we only consider the case $s \in (3/2, 2)$.

Lemma 4.1 *Let s as in Theorem 1.1 and $\mu \in (0,1)$. Then, there exist $C > 0$ and $\theta \in (0,1)$ such that*

$$\begin{aligned}
&\|\langle D_x\rangle^{s-3/2}(fg\partial_x h)\|_{L_x^1(L_T^2)} \\
&\leq C\|f\|_{L_x^2(L_T^\infty)}\|g\|_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x h\|_{L_x^\infty(L_T^2)} \\
&\quad +C\|\langle D_x\rangle^\mu f\|^\theta_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x f\|^{1-\theta}_{L_x^\infty(L_T^2)}\|g\|_{L_x^2(L_T^\infty)} \\
&\quad \times\|\langle D_x\rangle^\mu h\|^{1-\theta}_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x h\|^\theta_{L_x^\infty(L_T^2)} \\
&\quad +C\|f\|_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^\mu g\|^\theta_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x g\|^{1-\theta}_{L_x^\infty(L_T^2)} \\
&\quad \times\|\langle D_x\rangle^\mu h\|^{1-\theta}_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x h\|^\theta_{L_x^\infty(L_T^2)} \qquad (4.1)\\
&\|\langle D_x\rangle^{s-3/2}(fg\partial_x h)\|_{L_T^1(L_x^2)} \\
&\leq CT^{1/2}\|f\|_{L_T^\infty(H_x^{s-1})}(\|g\|_{L_x^2(L_T^\infty)}+\|g\|_{L_T^\infty(H_x^{s-1})}) \\
&\quad \times(\|\langle D_x\rangle^{s-3/2}\partial_x h\|_{L_x^\infty(L_T^2)}+\|h\|_{L_T^\infty(H_x^{s-1})}). \qquad (4.2)
\end{aligned}$$

Lemma 4.2 *Let $\vec{R}_\nu(\varphi, u_\nu)$ defined in section 2 and $s' < s$. Then, we have*

$$\begin{aligned}
\|\vec{R}_\nu(\varphi,u_\nu)\|_{L_T^1(H_x^{s-1})} &\leq C_\varphi T(|||u_\nu|||_{Y_T}+|||u_\nu|||^3_{Y_T}), \qquad (4.3)\\
\|\vec{R}_\nu(\varphi,u_\nu)-\vec{R}_{\nu'}(\varphi,u_{\nu'})\|_{L_T^1(H_x^{s'-1})} &\leq C_\varphi T(1+|||u_\nu|||^2_{Y_T}+|||u_{\nu'}|||^2_{Y_T})|||u_\nu-u_{\nu'}|||_{Y_T} \\
&\quad +C_\varphi(\nu^\beta+\nu'^\beta)(1+|||u_\nu|||_{Y_T}+|||u_{\nu'}|||_{Y_T})^3. \qquad (4.4)
\end{aligned}$$

Proof of Lemma 4.1. Applying $\langle D_x\rangle^{s-3/2} - D_x^{s-3/2} \in \mathcal{B}(L_x^1(L_T^2))$ and Lemma 3.3, we see that

$$\begin{aligned}
\|\langle D_x\rangle^{s-3/2}(fg\partial_x h)\|_{L_x^1(L_T^2)} &\leq \|f\|_{L_x^2(L_T^\infty)}\|g\|_{L_x^2(L_T^\infty)}\|D_x^{s-3/2}\partial_x h\|_{L_x^\infty(L_T^2)} \\
&\quad +C\|D_x^{s-3/2}(fg)\|_{L_x^{\tilde{p}}(L_T^{\tilde{r}})}\|\partial_x h\|_{L_x^p(L_T^r)} \\
&\quad +C\|fg\partial_x h\|_{L_x^1(L_T^2)},
\end{aligned}$$

where $1/p = (1-\theta)/2+\theta/\infty$, $1/r = (1-\theta)/\infty+\theta/2$, $1/p+1/\tilde{p} = 1$ and $1/r+1/\tilde{r} = 1/2$ together with $1 = (1-\theta)\mu/2+\theta(s-1/2-\mu/2)$. Using Lemma 3.4, we have

$$\begin{aligned}
\|\partial_x h\|_{L_x^p(L_T^r)} &\leq (\sup_\lambda e^{-\lambda^2}\|D_x^{\mu/2+i\lambda(s-1/2-\mu)}\mathcal{F}^{-1}\mathrm{sgn}\ \xi\mathcal{F}h\|_{L_x^2(L_T^\infty)})^{1-\theta} \\
&\quad \times(\sup_\lambda e^{1-\lambda^2}\|D_x^{s-1/2-\mu/2+i\lambda(s-1/2-\mu)}\mathcal{F}^{-1}\mathrm{sgn}\ \xi\mathcal{F}h\|_{L_x^\infty(L_T^2)})^\theta \\
&\leq C\|\langle D_x\rangle^\mu h\|^{1-\theta}_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x h\|^\theta_{L_x^\infty(L_T^2)}, \qquad (4.5)
\end{aligned}$$

where we made use of

$$\|D_x^{\mu/2+i\lambda(s-1/2-\mu)}\langle D_x\rangle^{-\mu}(\mathcal{F}^{-1}\text{sgn }\xi\mathcal{F})\|_{\mathcal{B}(L_x^2(L_T^\infty))}\le C\langle\lambda\rangle^N,$$
$$\|D_x^{s-3/2-\mu/2}\langle D_x\rangle^{-(s-3/2)}\|_{\mathcal{B}(L_x^\infty(L_T^2))}\le C\langle\lambda\rangle^N$$

with N sufficiently large. By the similar argument to derive (4.5), we have

$$\begin{aligned}
&\|D_x^{s-3/2}(fg)\|_{L_x^{\tilde p}(L_T^{\tilde r})}\\
&\le C(\|D_x^{s-3/2}f\|_{L_x^{2\tilde p/(\tilde p-2)}(L_T^{\tilde r})}\|g\|_{L_x^2(L_T^\infty)}+\|f\|_{L_x^2(L_T^\infty)}\|D_x^{s-3/2}g\|_{L_x^{2\tilde p/(\tilde p-2)}L_T^{\tilde r}})\\
&\le C\|\langle D_x\rangle^\mu f\|_{L_x^2(L_T^\infty)}^\theta\|\langle D_x\rangle^{s-3/2}\partial_x f\|_{L_x^\infty(L_T^2)}^{1-\theta}\|g\|_{L_x^2(L_T^\infty)}\\
&\quad +C\|f\|_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^\mu g\|_{L_x^2(L_T^\infty)}^\theta\|\langle D_x\rangle^{s-3/2}\partial_x g\|_{L_x^\infty(L_T^2)}^{1-\theta}. \qquad (4.6)
\end{aligned}$$

Also, we can show that

$$\begin{aligned}
\|fg\partial_x h\|_{L_x^1(L_T^2)} &\le \|f\|_{L_x^2(L_T^\infty)}\|g\|_{L_x^2(L_T^\infty)}\|\partial_x h\|_{L_x^\infty(L_T^2)}\\
&\le C\|\langle D_x\rangle^\mu f\|_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^\mu g\|_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x h\|_{L_x^\infty(L_T^2)}. \qquad (4.7)
\end{aligned}$$

Combining (4.5)–(4.7), we obtain (4.1). To prove (4.2), we apply the Leibniz' rule for fractional order derivatives. We have

$$\begin{aligned}
\|D_x^{s-3/2}(fg\partial_x h)\|_{L_T^1(L_x^2)} &\le \|fgD_x^{s-3/2}\partial_x h\|_{L_T^1(L_x^2)}+T\|D_x^{s-3/2}(fg)\|_{L_T^\infty(L_x^{2+4/\varepsilon})}\|\partial_x h\|_{L_T^\infty(L_x^{2+\varepsilon})}\\
&\equiv I_1+I_2.
\end{aligned}$$

By Hölder's inequality and Sobolev's embedding, I_1 is estimated as

$$\begin{aligned}
I_1 &\le T^{1/2}\|f\|_{L_x^\infty(L_T^\infty)}\|g\|_{L_x^2(L_T^\infty)}\|D_x^{s-3/2}\partial_x h\|_{L_x^\infty(L_T^2)}\\
&\le CT^{1/2}\|f\|_{L_T^\infty(H_x^{s-1})}\|g\|_{L_x^2(L_T^\infty)}\|\langle D_x\rangle^{s-3/2}\partial_x h\|_{L_x^\infty(L_T^2)}.
\end{aligned}$$

As for I_2, Leibniz' rule and Sobolev's embedding yield

$$I_2\le CT\|f\|_{L_T^\infty(H_x^{s-1})}\|g\|_{L_T^\infty(H_x^{s-1})}\|h\|_{L_T^\infty(H_x^{s-1})}.$$

Hence, we obtain (4.2). □

Proof of Lemma 4.2. By the H_x^{s-1}-boundedness of $\mathbf{K}_\nu$, we see that

$$\|\vec R_\nu(\varphi,u_\nu)\|_{L_T^1(H_x^{s-1})}\le C_\varphi T\|u_\nu\|_{L_T^\infty(H_x^s)}+\|\vec Q_\nu(\varphi,u_\nu)\|_{L_T^1(H_x^{s-1})}.$$

To estimate $\|\vec Q_\nu(\varphi,u_\nu)\|_{L_T^1(H_x^{s-1})}$, it suffices to consider

$$\begin{aligned}
&\|\langle D_x\rangle^{s-1}\mathbf{J}_\nu\langle D_x\rangle^{-2}\partial_x\eta_\nu*\mathbf{B}_\nu(u_\nu)\partial_x^2\eta_\nu*\vec u_\nu\|_{L_T^1(L_x^2)}\\
&\le C\|\langle D_x\rangle^{s-3}\partial_x\mathbf{B}_\nu(u_\nu)\partial_x^2\eta_\nu*\vec u_\nu\|_{L_T^1(L_x^2)}\\
&\le C\|\mathbf{B}_\nu(u_\nu)\partial_x^2\eta_\nu*\vec u_\nu\|_{L_T^1(L_x^2)}\\
&\le CT^{1/2}\|\mathbf{B}_\nu(u_\nu)\eta_\nu*\partial_x^2\vec u_\nu\|_{L_x^2(L_T^2)}\\
&\le CT^{1/2}\|u_\nu\|_{L_x^\infty(L_T^\infty)}\|u_\nu\|_{L_x^2(L_T^\infty)}\|u_\nu\|_{L_x^\infty(L_T^2)}\\
&\le CT^{1/2}\|u_\nu\|_{Y_T}^3.
\end{aligned}$$

The proof of (4.4) likewise follows. We note that $\nu^\beta+\nu'^\beta$ arises from the estimates of $\mathbf{K}_\nu-\mathbf{K}_{\nu'}$, $\mathbf{J}_\nu-\mathbf{J}_{\nu'}$ and $(\eta_\nu-\eta_{\nu'})*$ which cause the slight loss of regularity. □

5 A priori estimate in Y_T and convergence of u_ν

To obtain the a priori estimate of u_ν for $\nu \in (0,1]$, we use the following integral representations:

$$\begin{aligned}
\overrightarrow{w}_\nu &= \mathbf{U}(t)\overrightarrow{w}_{\nu,0} - i\mathbf{G}\mathbf{K}_\nu(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\eta_\nu * \partial_x^2 \overrightarrow{u}_\nu \\
&\quad -i\mathbf{G}\overrightarrow{R}_\nu(\varphi, u_\nu), && (5.1)\\
u_\nu &= U(t)u_0 - iG\mathcal{N}(u_\nu, \partial_x u_\nu), && (5.2)
\end{aligned}$$

where $\mathbf{U}(t) = \exp(it\mathbf{A}\partial_x^2/2)$, $\mathbf{G}\vec{F} = \int_0^t \mathbf{U}(t-\tau)\vec{F}(\tau)d\tau$ and $\overrightarrow{w}_{\nu,0} = \mathbf{K}_\nu(\partial_x\vec{u}_0 + \mathbf{J}_\nu\eta_\nu * \vec{u}_0)$ with $\vec{u}_0 = (u_0, \bar{u}_0)^t$. The construction of the approximating solution u_ν in Y_T is simple. In fact, by applying Lemma 3.1, 3.2 to (5.2) and in virtue of the regularization due to $\eta_\nu *$ together with Lemma 3.3, the nonlinear term is, for instance, estimated as

$$\begin{aligned}
&\|D_x^{s-3/2}\partial_x\mathcal{N}(u_\nu, \partial_x\eta_\nu * u_\nu)\|_{L_x^1(L_T^2)} \\
&\le C\nu^{-N}T^{1/2}(\max_{j=0,1}\|\langle D_x\rangle^\mu \partial_x^j u_\nu\|_{L_x^2(L_T^\infty)})\|u_\nu\|_{L_T^\infty(H_x^s)}^2.
\end{aligned}$$

Thus, by taking $T > 0$ sufficiently small, the contraction mapping priciple successfully works in Y_T. The local solution u_ν is continuated as long as $\|u_\nu(t)\|_{H_x^s}$ is finite. Note that $|\!|\!| u_\nu |\!|\!|_{Y_T}$ is continuous with respect to T.

For brief description, we define several norms as follows

$$\begin{aligned}
|\!|\!| u |\!|\!|_{Y_T} &= \|u\|_{L_T^\infty(H_x^s)} + \|\langle D_x\rangle^{s-3/2}\partial_x^2 u\|_{L_x^\infty(L_T^2)} + \max_{j=0,1}\|\langle D_x\rangle^\mu \partial_x^j u\|_{L_x^2(L_T^\infty)} \\
&\equiv |\!|\!| u |\!|\!|_{initial} + |\!|\!| u |\!|\!|_{smooth} + |\!|\!| u |\!|\!|_{maxim}.
\end{aligned}$$

To ensure the convergence of the nonlinearity as $\nu \downarrow 0$, we require the Cauchy property of $\{u_\nu\}_{\nu\in(0,1]}$. Note that the proof fails when we consider $|\!|\!| u_\nu - u_{\nu'} |\!|\!|_{Y_T}$, since the estimate $\|(\eta_\nu - \eta_{\nu'})u_\nu\|_{H_x^s} \le C(\nu^\beta + \nu'^\beta)\|u_\nu\|_{H_x^{s+\beta}}$ indicates the regularity loss. Therefore, we employ the function space slightly weaker than Y_T, i.e.,

$$|\!|\!| u |\!|\!|_{Z_T} = \|u\|_{L_T^\infty(H_x^{s'})} + \|\langle D_x\rangle^{s'-3/2}\partial_x^2 u\|_{L_x^\infty(L_T^2)} + \max_{j=0,1}\|\langle D_x\rangle^{\mu'} \partial_x^j u\|_{L_x^2(L_T^\infty)},$$

where $s' < s$ and $\mu' < \mu$. The key proposition to obtain our main theorem is

Proposition 5.1 (a priori estimate) *The following assertions hold.*

(1) Let $T_\nu = \sup\{T'; |\!|\!| u_\nu |\!|\!|_{Y_\tau} < 2C_0\delta_0$ for $0 < \tau < T'\}$. Then, $\liminf_{\nu\downarrow 0} T_\nu = T_0 > 0$,

(2) Let $\|u_0\|_{H_x^s} \le \delta_0$ and $T \in (0, T_0]$ sufficiently small. Then, we have

$$\begin{aligned}
&|\!|\!| u_\nu |\!|\!|_{Y_T} \le 2C_0\delta_0, && (5.3)\\
&|\!|\!| u_\nu - u_{\nu'} |\!|\!|_{Z_T} \le C_\varphi(\nu^\beta + \nu'^\beta)(1 + 4C_0\delta_0)^3, && (5.4)
\end{aligned}$$

where C_0 and C_φ do not depend on $\nu \in (0,1]$ but C_φ may diverge as $\varphi \to u_0$ in H_x^s.

To prove Proposition 5.1, we need two lemmas. The first one indicates that the estimates of u_ν is replaced by those of $\overrightarrow{w}_\nu$.

Lemma 5.2 *Let $s > s' > 3/2$ and $\nu, \nu' > 0$ sufficiently small. Then, we have*

$$\|u_\nu\|_{L^\infty_T(H^s_x)} \le C(\|\vec{w}_\nu\|_{L^\infty_T(H^{s-1}_x)} + \|u_\nu\|_{L^\infty_T(L^2_x)}), \tag{5.5}$$

$$\|\langle D_x\rangle^{s-3/2}\partial_x^2 u_\nu\|_{L^\infty_x(L^2_T)} \le C\|\langle D_x\rangle^{s-3/2}\partial_x \vec{w}_\nu\|_{L^\infty_x(L^2_T)} + C_\varphi T^{1/2}\|u_\nu\|_{L^\infty_T(H^s_x)}, \tag{5.6}$$

$$\begin{aligned}&\|u_\nu - u_{\nu'}\|_{L^\infty_T(H^{s'}_x)}\\ &\quad\le C(\|\vec{w}_\nu - \vec{w}_{\nu'}\|_{L^\infty_T(H^{s'}_x)} + \|u_\nu - u_{\nu'}\|_{L^\infty_T(L^2_x)}) + C_\varphi(\nu^\beta + \nu'^\beta)|||u_\nu|||_{Y_T},\end{aligned} \tag{5.7}$$

$$\begin{aligned}&\|\langle D_x\rangle^{s'-3/2}\partial_x^2(u_\nu - u_{\nu'})\|_{L^\infty_x(L^2_T)}\\ &\quad\le C\|\langle D_x\rangle^{s'-3/2}\partial_x(\vec{w}_\nu - \vec{w}_{\nu'})\|_{L^\infty_x(L^2_T)} + C_\varphi T^{1/2}\|u_\nu - u_{\nu'}\|_{L^\infty_T(H^{s'}_x)}\\ &\qquad + C_\varphi(\nu^\beta + \nu'^\beta)|||u_\nu|||_{Y_T},\end{aligned} \tag{5.8}$$

where β is a small positive constant.

Proof of Lemma 5.2. Since $\vec{w}_\nu = \mathbf{K}_\nu(\partial_x \vec{u}_\nu + \mathbf{J}_\nu\eta_\nu * \vec{u}_\nu)$, we see that

$$\langle D_x\rangle^\sigma\partial_x^{j-1}\vec{w}_\nu = \mathbf{K}_\nu\langle D_x\rangle^\sigma\partial_x^j\vec{u}_\nu + [\langle D_x\rangle^\sigma\partial_x^{j-1}, \mathbf{K}_\nu]\partial_x\vec{u}_\nu + \langle D_x\rangle^\sigma\partial_x^{j-1}\mathbf{K}_\nu\mathbf{J}_\nu\eta_\nu * \vec{u}_\nu. \tag{5.9}$$

Let $\tilde{\mathbf{K}}_\nu = \tilde{\mathbf{K}}_\nu(x, i^{-1}\partial_x)$ be the pseudo-differential operator of the symbol:

$$\tilde{\mathbf{K}}_\nu(x,\xi) = \begin{pmatrix} \exp(\hat{\eta}(\nu\xi)\partial_x^{-1}\mathcal{N}_q(\varphi, \partial_x\eta_\nu * \varphi)) & 0 \\ 0 & \exp(\hat{\eta}(\nu\xi)\partial_x^{-1}\overline{\mathcal{N}_q(\varphi, \partial_x\eta_\nu * \varphi)}) \end{pmatrix}.$$

Note that $\tilde{\mathbf{K}}_\nu$ plays a rôle like the inverse of $\mathbf{K}_\nu$. Then, from (5.9), it follows that

$$\begin{aligned}\langle D_x\rangle^\sigma\partial_x^j\vec{u}_\nu &= \tilde{\mathbf{K}}_\nu\langle D_x\rangle^\sigma\partial_x^{j-1}\vec{w}_\nu - (\tilde{\mathbf{K}}_\nu\mathbf{K}_\nu - I)\langle D_x\rangle^\sigma\partial_x^j\vec{u}_\nu\\ &\quad -\tilde{\mathbf{K}}_\nu([\langle D_x\rangle^\sigma\partial_x^{j-1}, \mathbf{K}_\nu]\partial_x\vec{u}_\nu + \langle D_x\rangle^\sigma\partial_x^{j-1}\mathbf{K}_\nu\mathbf{J}_\nu\eta_\nu * \vec{u}_\nu).\end{aligned} \tag{5.10}$$

Taking $\sigma = s-1$ and $j = 1$ in (5.10) and applying Lemma 3.5–3.3 together with $[\langle D_x\rangle^\sigma, \mathbf{K}_\nu] \in \mathcal{B}(L^2_x; H_x^{-(1-\sigma)})$ uniformly in $\nu \in (0,1]$, we have

$$\|u_\nu\|_{L^\infty_T(H^s_x)} \le C\|\vec{w}_\nu\|_{L^\infty_T(H^{s-1}_x)} + C_\varphi\nu^\beta\|u_\nu\|_{L^\infty_T(H^s_x)} + C\|u_\nu\|_{L^\infty_T(H^{s-1}_x)}.$$

Taking $\nu > 0$ so small that $C_\varphi\nu^\beta < 1/4$ and applying $\|u_\nu\|_{L^\infty_T(H^{s-1}_x)} \le \varepsilon\|u_\nu\|_{L^\infty_T(H^s_x)} + C_\varepsilon\|u_\nu\|_{L^\infty_T(L^2_x)}$, we obtain (5.5). To prove (5.6), we let $\sigma = s - 3/2$ and $j = 2$ in (5.10). Then, it follows that

$$\begin{aligned}\|\langle D_x\rangle^{s-3/2}\partial_x^2 u_\nu\|_{L^\infty_x(L^2_T)} &\le C\|\langle D_x\rangle^{s-3/2}\partial_x\vec{w}_\nu\|_{L^\infty_x(L^2_T)} + C_\varphi\nu^\beta\|\langle D_x\rangle^{s-3/2}\partial_x^2 u_\nu\|_{L^\infty_x(L^2_T)}\\ &\quad + C_\varphi(\varepsilon\|\langle D_x\rangle^{s-3/2}\partial_x^2 u_\nu\|_{L^\infty_x(L^2_T)} + C_\varepsilon\|u_\nu\|_{L^\infty_x(L^2_T)}).\end{aligned}$$

Taking $\nu, \varepsilon > 0$ small and applying $\|u_\nu\|_{L^\infty_x(L^2_T)} \le T^{1/2}\|u_\nu\|_{L^\infty_T(L^\infty_x)} \le CT^{1/2}\|u_\nu\|_{L^\infty_T(H^s_x)}$, we obtain (5.6). The estimates (5.7) and (5.8) follow from the description:

$$\begin{aligned}\langle D_x\rangle^\sigma\partial_x^j(\vec{u}_\nu - \vec{u}_{\nu'}) &= \tilde{\mathbf{K}}_\nu\langle D_x\rangle^\sigma\partial_x^{j-1}(\vec{w}_\nu - \vec{w}_{\nu'}) - (\tilde{\mathbf{K}}_\nu\mathbf{K}_\nu - I)\langle D_x\rangle^\sigma\partial_x^j(\vec{u}_\nu - \vec{u}_{\nu'})\\ &\quad -\tilde{\mathbf{K}}_\nu[\langle D_x\rangle^\sigma\partial_x^j, \mathbf{K}_{\nu'}]\partial_x(\vec{u}_\nu - \vec{u}_{\nu'})\\ &\quad -\tilde{\mathbf{K}}_\nu\langle D_x\rangle^\sigma\partial_x^{j-1}(\mathbf{K}_\nu - \mathbf{K}_{\nu'})(\partial_x\vec{u}_\nu + \mathbf{J}_\nu\eta_\nu * \vec{u}_\nu)\\ &\quad -\tilde{\mathbf{K}}_\nu\langle D_x\rangle^\sigma\partial_x^{j-1}\mathbf{K}_{\nu'}(\mathbf{J}_\nu\eta_\nu * \vec{u}_\nu - \mathbf{J}_{\nu'}\eta_{\nu'}\vec{u}_{\nu'}).\end{aligned}$$

Note that the coefficient $\nu^\beta + \nu'^\beta$ appears in the estimates of $\mathbf{K}_\nu - \mathbf{K}_{\nu'}$, $\mathbf{J}_\nu - \mathbf{J}_{\nu'}$ and $(\eta_\nu - \eta_{\nu'})*$. $\square$

The second lemma shows that one can make $|||u_\nu - \varphi|||_{maxim}$ and $|||u_\nu|||_{smooth}$ (appearing in the nonlinear estimates) small enough by taking φ close to u_0 and $T > 0$ small.

Lemma 5.3 *There exist $\beta > 0$ and $\theta \in (0,1)$ such that*

$$
\begin{aligned}
|||u_\nu - \varphi|||_{maxim} &\leq C\|u_0 - \varphi\|_{H_x^s} + C_\varphi T^\beta (1 + |||u_\nu|||_{Y_T})^3, &(5.11)\\
|||u_\nu|||_{smooth} &\leq C\|u_0 - \varphi\|_{H_x^s} \\
&\quad + C(|||u_\nu - \varphi|||_{maxim} + |||u_\nu - \varphi|||_{maxim}^{1-\theta} |||u_\nu|||_{Y_T}^{\theta})(1 + |||u_\nu|||_{Y_T})^2 \\
&\quad + C_\varphi T^\beta (1 + |||u_\nu|||_{Y_T})^3. &(5.12)
\end{aligned}
$$

Proof of Lemma 5.3. From the integral equation (5.2), it follows that

$$
\begin{aligned}
|||u_\nu - \varphi|||_{maxim} &\leq |||U(t)u_0 - \varphi|||_{maxim} + |||G\mathcal{N}(u_\nu, \partial_x u_\nu)|||_{maxim} \\
&\equiv I_1 + I_2. &(5.13)
\end{aligned}
$$

Note that, by Lemma 3.2,

$$
\begin{aligned}
I_1 &\leq |||U(t)(u_0 - \varphi)|||_{maxim} + |||U(t)\varphi - \varphi|||_{maxim} \\
&\leq C\|u_0 - \varphi\|_{H_x^s} + CT\|\varphi\|_{H_x^\sigma}, &(5.14)
\end{aligned}
$$

where $\sigma > 0$ is sufficiently large. As for the estimate of I_2, we only consider the case $\mathcal{N}(u_\nu, \partial_x\eta_\nu * u_\nu) = (\partial_x\eta_\nu * u_\nu)^3$ and $j = 1$ in the definition of $|||\cdot|||_{maxim}$. Lemma 3.2, 3.3 and 4.1 yield

$$
\begin{aligned}
I_2 &\leq CT^{1/4}\|\langle D_x\rangle_x^{s-3/2}(\partial_x\eta_\nu * u_\nu)^2(\partial_x^2\eta_\nu * u_\nu)\|_{L_x^1(L_T^2)} \\
&\leq CT^{1/4}|||u_\nu|||_{Y_T}^3. &(5.15)
\end{aligned}
$$

Combining (5.13)–(5.15), we obtain (5.11). To prove (5.12), we use (5.1). Then, Lemma 3.1 yields

$$
\begin{aligned}
\|\langle D_x\rangle^{s-3/2}\partial_x\vec{w}_\nu\|_{L_x^\infty(L_T^2)} &\leq \|\langle D_x\rangle^{s-3/2}\partial_x\mathbf{U}(t)\vec{w}_{\nu,0}\|_{L_x^\infty(L_T^2)} \\
&\quad + C\|\langle D_x\rangle^{s-3/2}\mathbf{K}_\nu(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\partial_x^2\eta_\nu * \vec{u}_\nu\|_{L_x^1(L_T^2)} \\
&\quad + C\|\vec{R}_\nu(\varphi, u_\nu)\|_{L_T^1(H_x^{s-1})} \\
&\equiv I_1' + I_2' + I_3'. &(5.16)
\end{aligned}
$$

Note that, to get I_3', we apply Lemma 3.1 (3.1) in the following way:

$$
\begin{aligned}
\|\langle D_x\rangle^{s-3/2}\partial_x\mathbf{G}\vec{R}_\nu\|_{L_x^\infty(L_T^2)} &\leq \int_0^T \|\langle D_x\rangle^{s-3/2}\partial_x\mathbf{U}(\cdot)\mathbf{U}(-t')\vec{R}_\nu\|_{L_x^\infty(L_T^2)}\, dt' \\
&\leq C\|\langle D_x\rangle^{s-3/2}D_x^{1/2}\vec{R}_\nu\|_{L_T^1(L_x^2)}.
\end{aligned}
$$

Let $\vec{\varphi}_\nu = \mathbf{K}_\nu(\partial_x\vec{\varphi} + \mathbf{J}_\nu\eta_\nu * \vec{\varphi})$ with $\vec{\varphi} = (\varphi, \overline{\varphi})^t$. Then, Lemma 3.1 (3.1) gives

$$\begin{aligned} I_1' &\le \|\langle D_x\rangle^{s-3/2}\partial_x\mathbf{U}(t)(\vec{w}_{\nu,0} - \vec{\varphi}_\nu)\|_{L_x^\infty(L_T^2)} + \|\langle D_x\rangle^{s-3/2}\partial_x\mathbf{U}(t)\vec{\varphi}_\nu\|_{L_x^\infty(L_T^2)} \\ &\le C\|\vec{w}_{\nu,0} - \vec{\varphi}_\nu\|_{H_x^{s-1,0}} + C_\varphi T^{1/2} \\ &\le C\|u_0 - \varphi\|_{H_x^s} + C_\varphi T^{1/2}. \end{aligned}$$

We next consider the estimate of I_2'. By Lemma 3.5 and 4.1,

$$\begin{aligned} I_2' &\le C\|\langle D_x\rangle^{s-3/2}(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\eta_\nu * \partial_x^2 u_\nu\|_{L_x^1(L_T^2)} \\ &\le C\|D_x^{s-3/2}(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\eta_\nu * \partial_x^2 u_\nu\|_{L_x^1(L_T^2)} \\ &\quad + C\|(\langle D_x\rangle^{s-3/2} - D_x^{s-3/2})(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\eta_\nu * \partial_x^2 u_\nu\|_{L_x^1(L_T^2)} \\ &\le C(|||u_\nu - \varphi|||_{maxim} + |||u_\nu - \varphi|||_{maxim}^\theta |||u_\nu - \varphi|||_{smooth}^{1-\theta})(1 + |||u_\nu|||_{Y_T})^2, \end{aligned}$$

where we used $\langle D_x\rangle^{s-3/2} - D_x^{s-3/2} \in \mathcal{B}(L_x^1(L_T^2))$ and $\|\partial_x^2 u_\nu\|_{L_x^\infty(L_T^2)} \le |||u_\nu|||_{smooth}$. Since $|||u_\nu - \varphi|||_{smooth} \le |||u_\nu|||_{smooth} + C_\varphi T^{1/2}$, we have

$$\begin{aligned} I_2' &\le C(|||u_\nu - \varphi|||_{maxim} + |||u_\nu - \varphi|||_{maxim}^\theta |||u_\nu|||_{smooth}^{1-\theta})(1 + |||u_\nu|||_{Y_T})^2 \\ &\quad + C_\varphi T^{1/2}(1 + |||u_\nu|||_{Y_T})^2. \end{aligned} \tag{5.17}$$

As for I_3', we apply Lemma 4.2 and observe that

$$I_3' \le C_\varphi T(1 + |||u_\nu|||_{Y_T})^3. \tag{5.18}$$

Combining (5.16)–(5.18) and Lemma 5.2(5.6), we obtain (5.12). □

We are ready for the proof of Proposition 5.1.

Proof of Proposition 5.1. Applying Lemma 3.1, 4.1 and 4.2 to (5.1), we see that

$$\begin{aligned} &\|\overrightarrow{w}_\nu\|_{L_T^\infty(H_x^{s-1})} + \|\langle D_x\rangle^{s-3/2}\partial_x\overrightarrow{w}_\nu\|_{L_x^\infty(L_T^2)} \\ &\le C\|u_0\|_{H_x^s} + C(|||u_\nu - \varphi|||_{maxim} + |||u_\nu - \varphi|||_{maxim}^\theta |||u_\nu|||_{Y_T}^{1-\theta})(1 + |||u_\nu|||_{Y_T})^2 \\ &\quad + C_\varphi T^\beta(1 + |||u_\nu|||_{Y_T})^3. \end{aligned}$$

By Lemma 5.2,

$$\begin{aligned} &|||u_\nu|||_{initial} + |||u_\nu|||_{smooth} \\ &\le C\|u_0\|_{H_x^s} + C(|||u_\nu - \varphi|||_{maxim} + |||u_\nu - \varphi|||_{maxim}^\theta |||u_\nu|||_{Y_T}^{1-\theta})(1 + |||u_\nu|||_{Y_T})^2 \\ &\quad + C_\varphi T^\beta(1 + |||u_\nu|||_{Y_T})^3. \end{aligned} \tag{5.19}$$

Also, applying Lemma 3.2 and 4.1 to (5.2), we have

$$|||u_\nu|||_{maxim} \le C\|u_0\|_{H_x^s} + CT^{1/4}|||u_\nu|||_{Y_T}^3 \tag{5.20}$$

From (5.19)–(5.20), it follows that

$$\begin{aligned} |||u_\nu|||_{Y_T} \le\ & C_0\delta_0 \\ & + C(|||u_\nu - \varphi|||_{maxim} + |||u_\nu - \varphi|||_{maxim}^\theta |||u_\nu|||_{Y_T}^{1-\theta})(1 + |||u_\nu|||_{Y_T})^2 \\ & + C_\varphi T^\beta(1 + |||u_\nu|||_{Y_T})^3. \end{aligned} \tag{5.21}$$

Taking $T \uparrow T_\nu$ in (5.21) if $T_\nu < \infty$, we have

$$\begin{aligned} 2C_0\delta_0 \ \le\ & C_0\delta_0 \\ & + C(\|u_\nu - \varphi\|_{maxim} + \|u_\nu - \varphi\|^{\theta}_{maxim}(2C_0\delta_0)^{1-\theta}) \cdot (1 + 2C_0\delta_0)^2 \\ & + C_\varphi T_\nu^\beta (1 + 2C_0\delta_0)^3. \end{aligned} \tag{5.22}$$

Assume here that $\liminf_{\nu \downarrow 0} T_\nu = 0$. Then, this is the contradiction. Indeed, by taking φ sufficiently close to u_0 in H^s_x, Lemma 5.3 and (5.22) yield $2C_0\delta_0 \le 3/2C_0\delta_0$. Hence, $T_\nu \ge T_0 > 0$ and (5.3) follows. We next prove (5.4). By the integral equation (5.1) and Lemma 3.1, we see that

$$\begin{aligned} & \|\langle D_x\rangle^{s'-3/2}\partial_x(\overrightarrow{w}_\nu - \overrightarrow{w}_{\nu'})\|_{L^\infty_x(L^2_T)} \\ \le\ & C\|\langle D_x\rangle^{s'-3/2}(\mathbf{K}_\nu - \mathbf{K}_{\nu'})(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_\nu(\varphi))\eta_\nu * \partial_x^2\overrightarrow{u}_\nu\|_{L^1_x(L^2_T)} \\ & + C\|\langle D_x\rangle^{s'-3/2}\mathbf{K}_{\nu'}(\mathbf{B}_\nu(u_\nu) - \mathbf{B}_{\nu'}(u_{\nu'}))\eta_\nu * \partial_x^2\overrightarrow{u}_\nu\|_{L^1_x(L^2_T)} \\ & + C\|\langle D_x\rangle^{s'-3/2}\mathbf{K}_{\nu'}(\mathbf{B}_\nu(\varphi) - \mathbf{B}_{\nu'}(\varphi))\eta_\nu * \partial_x^2\overrightarrow{u}_\nu\|_{L^1_x(L^2_T)} \\ & + C\|\langle D_x\rangle^{s'-3/2}\mathbf{K}_{\nu'}(\mathbf{B}_{\nu'}(u_{\nu'}) - \mathbf{B}_{\nu'}(\varphi))(\eta_\nu - \eta_{\nu'}) * \partial_x^2\overrightarrow{u}_\nu\|_{L^1_x(L^2_T)} \\ & + C\|\langle D_x\rangle^{s'-3/2}\mathbf{K}_{\nu'}(\mathbf{B}_{\nu'}(u_{\nu'}) - \mathbf{B}_{\nu'}(\varphi))\eta_{\nu'} * \partial_x^2(\overrightarrow{u}_\nu - \overrightarrow{u}_{\nu'})\|_{L^1_x(L^2_T)} \\ & + \|\vec{R}_\nu(\varphi, u_\nu) - \vec{R}_{\nu'}(\varphi, u_{\nu'})\|_{L^1_T(H^{s'-1}_x)}. \end{aligned}$$

Note that the estimates of integral kernels give

$$\begin{aligned} \|\langle D_x\rangle^{s'-3/2}(\mathbf{K}_\nu - \mathbf{K}_{\nu'})\vec{f}\|_{L^p_x(L^r_T)} \ &\le\ C_\varphi(\nu^\beta + \nu'^\beta)\|\langle D_x\rangle^{s-3/2}\vec{f}\|_{L^p_x(L^r_T)}, \\ \|(\eta_\nu - \eta_{\nu'}) * \vec{f}\|_{L^p_x(L^r_T)} \ &\le\ C(\nu^\beta + \nu'^\beta)\|\langle D_x\rangle^\beta \vec{f}\|_{L^p_x(L^r_T)}. \end{aligned}$$

Then, we have

$$\begin{aligned} & \|\langle D_x\rangle^{s'-3/2}\partial_x(\overrightarrow{w}_\nu - \overrightarrow{w}_{\nu'})\|_{L^\infty_x(L^2_T)} \\ \le\ & C(\|u_{\nu'} - \varphi\|_{maxim} + \|u_\nu\|_{smooth} + C_\varphi T^\beta) \\ & \times(\|u_\nu\|_{Y_T} + \|u_{\nu'}\|_{Y_T})\|u_\nu - u_{\nu'}\|_{Z_T} \\ & + C_\varphi(\nu^\beta + \nu'^\beta)(1 + \|u_\nu\|_{Y_T} + \|u_{\nu'}\|_{Y_T})^3 \end{aligned}$$

By Lemma 3.1 (3.3), it is also possible to derive

$$\begin{aligned} \|\overrightarrow{w}_\nu - \overrightarrow{w}_{\nu'}\|_{L^\infty_T(H^{s'-1}_x)} \ \le\ & C(\|u_{\nu'} - \varphi\|_{maxim} + \|u_\nu\|_{smooth} + C_\varphi T^\beta) \\ & \times(\|u_\nu\|_{Y_T} + \|u_{\nu'}\|_{Y_T})\|u_\nu - u_{\nu'}\|_{Z_T} \\ & + C_\varphi(\nu^\beta + \nu'^\beta)(1 + \|u_\nu\|_{Y_T} + \|u_{\nu'}\|_{Y_T})^3. \end{aligned}$$

Thus, Lemma 5.2 gives

$$\begin{aligned} & \|u_\nu - u_{\nu'}\|_{L^\infty_T(H^{s'}_x)} + \|\langle D_x\rangle^{s'-3/2}\partial_x^2(u_\nu - u_{\nu'})\|_{L^\infty_x(L^2_T)} \\ \le\ & C(\|u_{\nu'} - \varphi\|_{maxim} + \|u_\nu\|_{smooth} + C_\varphi T^\beta)C_0\delta_0\|u_\nu - u_{\nu'}\|_{Z_T} \\ & + C_\varphi(\nu^\beta + \nu'^\beta)(1 + 4C_0\delta_0)^3. \end{aligned} \tag{5.23}$$

Applying Lemma 3.2 to the integral equation (5.2), we can show that

$$\begin{aligned}\max_{j=0,1} \|\langle D_x\rangle^{\mu'}\partial_x^j(u_\nu - u_{\nu'})\|_{L_x^2(L_T^\infty)} &\leq CT^\beta(4C_0\delta_0)^2 \|\!| u_\nu - u_{\nu'} |\!\|_{Z_T} \\ &+ C(\nu^\beta + \nu'^\beta)(4C_0\delta_0)^3. \end{aligned} \tag{5.24}$$

Then, (5.23), (5.24) and Lemma 5.3 yield (5.4). □

We now prove our main theorem.

Proof of Theorem 1.1. By Proposition 5.1(5.3), we can take a convergent subsequence of $\{u_\nu\}_{\nu\in(0,1]}$ such that

$$\begin{aligned}&\lim_{\nu'\downarrow 0} u_{\nu'} = u \quad \text{weakly-}* \text{ in } L_T^\infty(H_x^s),\\ &\lim_{\nu'\downarrow 0}\langle D_x\rangle^{s-3/2}\partial_x^2 u_{\nu'} = \langle D_x\rangle^{s-3/2}\partial_x^2 u \quad \text{weakly-}* \text{ in } L_x^\infty(L_T^2),\\ &\lim_{\nu'\downarrow 0}\langle D_x\rangle^{\mu}\partial_x^j u_{\nu'} = \langle D_x\rangle^{\mu}\partial_x^j u \quad \text{weakly-}* \text{ in } L_x^2(L_T^\infty),\end{aligned}$$

where we identify $L_T^\infty(H_x^s)$ (resp. $L_x^\infty(L_T^2)$ and $L_x^2(L_T^\infty)$) with $(L_T^1(H_x^{-s}))^*$ (resp. $(L_x^1(L_T^2))^*$ and $(L_x^2(L_T^1))^*$). From Proposition 5.1(5.4), it follows that $\mathcal{N}(u_{\nu'}, \eta_{\nu'} * \partial_x u_{\nu'})$ tends to $\mathcal{N}(u, \partial_x u)$ in $L_T^\infty(L_x^2)$ and so u satisfies the integral equation:

$$u = U(t)u_0 - iG\mathcal{N}(u, \partial_x u) \quad \text{in } L_T^\infty L_x^2. \tag{5.25}$$

We next show the continuity in time of u as an H_x^s valued function. In (5.25), it is easy to see that $U(t)u_0 \in C([0,T]; H_x^{s,0})$. As for $G\mathcal{N}(u,\partial_x u) \equiv G\mathcal{N}(t)$, we observe that

$$\begin{aligned}G\mathcal{N}(t+h) - G\mathcal{N}(t) &= U(t+h)\int_t^{t+h} U(-\tau)\mathcal{N}(\tau)d\tau\\ &\quad +(U(t+h)-U(t))\int_0^t U(-\tau)\mathcal{N}(\tau)d\tau\\ &\equiv G_1(h) + G_2(h).\end{aligned} \tag{5.26}$$

Let $I = [t, t+h]$ if $h > 0$ and $I = [t+h, t]$ if $h < 0$. Note that, by the dual estimate of $\|D_x^{1/2}U(t)\phi\|_{L_x^\infty L_I^2} \leq C\|\phi\|_{L_x^2}$, we have $\|D_x^{1/2}\int_I U(-\tau)\mathcal{N}(\tau)d\tau\|_{L_x^2} \leq C\|\mathcal{N}\|_{L_x^1(L_I^2)}$. Then, Lebesgue's convergence theorem yields

$$\begin{aligned}\|D_x^{s-1}\partial_x G_1(h)\|_{L_x^2} &\leq C\|D_x^{s-3/2}\partial_x \mathcal{N}\|_{L_x^1(L_I^2)}\\ &\to 0 \quad \text{as } h \to 0.\end{aligned}$$

Since $\|D_x^{s-1}\partial_x\int_0^t U(-\tau)\mathcal{N}(\tau)d\tau\|_{L_x^2} < \infty$ by Lemma 3.1 (3.3), the strong continuity of the Schrödinger group yields $\lim_{h\to 0} D_x^{s-1}\partial_x G_2(h) = 0$ in L_x^2. Hence, $u \in C([0,T]; H_x^s)$. The uniqueness and Lipschitz' dependence on the initial data follow from the routine work. □

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