数理解析研究所講究録
第 1561 巻 2007 年 7-10

Relation between Equilibrium points for a Differential Inclusion and Solutions of a Variational Inequality

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We introduce a relation between equiliblium points for some differential inclusion and solutions of a variational inequality. At first we show convergence theorems of solutions for some differential inclusions.

1. Convergence Theorem for Differential Inclusions

Let X be a Hilbert space with an inner product $(,)$, let $K \subset X$ be a non-empty closed convex subset of X, let A be a set-valued mapping from K to 2^X with convex compact set-value, and let $x()\ : [0,\infty) \to X$. We consider a differential inclusion $DI(A,K)$ for A and K as follows:

$$DI(A,K)\ :\ \begin{cases} x(t) \in K \text{ for any } t \in [0,\infty), \\ \dot{x}(t) \in -A(x(t)) \text{ for almost any } t \in (0,\infty). \end{cases}$$

Next we give some definitions.

定義 (Def.) (1) $x()$ is called a trajectory of $DI(A,K)$, if a mapping $x()\ : [0,\infty) \to K$ is absolutely continuous and satisfies $DI(A,K)$.

(2) A point $x^* \in K$ is said to be an equilibrium point for $DI(A,K)$ if $0 \in -A(x^*)$.

Let $\varphi : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex function. A subdifferential $\partial\varphi : X \to 2^X$ is defined by

$$\partial\varphi(x) = \{w \in X : \varphi(y) \geq \varphi(x) + (w, y-x) \text{ for any } y \in X\}.$$

Then it is well-known that $x^* \in K$ is an equilibrium point of $DI(\partial\varphi, K)$ if and only if x^* is a minimum point of φ , i.e., $\varphi(x^*) = \min_{x\in X}\varphi(x)$. It is also known that $\partial\varphi$ has a property of ***demipositivity***.

定義 (Def.) A set-valued mapping $A : X \to 2^X$ is said to be ***demi***positive if (1),(2) and (3) hold:

(1) $(v, x-y) \geq 0$ for all $x \in X, y \in A^{-1}(0)$ *and* $v \in A(x)$.

(2) There exists $y_0 \in A^{-1}(0)$ such that $0 \in A(x)$ whenever $(v, x-y_0) = 0$ for all $v \in A(x)$.

(3) For the y_0 in (2), if $x_n \to x$, $v_n \in A(x_n)$, $\{v_n\}$ is bounded and $\lim_{n\to\infty}(v_n, x_n - y_0) = 0$, then $0 \in A(x)$.

(Remark) If A satisfies (1) and (2), A is called *firmly* positive.

Bruck showed the convergence theorems with respect to a *demi*positive mapping.

定理 (Theorem1) ([1] Bruck, 1974)

Suppose $A : X \to 2^X$ is *demi*positive and that $x()\ : [0,\infty) \to X$ is an ***absolutely continuous*** mapping satisfying

$$\begin{cases} x(t) \in D(A) \textit{ for all } t \geq 0, \\ \dot{x}(t) \in -A(x(t)) \textit{ for almost all } t > 0, \\ \|x(t)\| \in L^\infty(0,\infty). \end{cases}$$

Then there exists $x^* = \text{w-}\lim_{t\to\infty} x(t)$ and $x^* \in A^{-1}(0)$.

定理 (Theorem2) ([1] Bruck,1974) Let $\varphi : X \to (-\infty,\infty]$ be a proper lower semi-continuous convex even function with a minimum. Then there exists a unique solution $x()\ : [0,\infty) \to X$, which is absolutely continuous on $[\delta,\infty)$ for all $\delta > 0$, satisfying

$$\begin{cases} x(t) \in D(\partial\varphi) \textit{ for all } t > 0, \\ \dot{x}(t) \in -\partial\varphi(x(t)) \textit{ for almost all } t > 0, \end{cases}$$

and there exists $x^* = s - \lim_{t\to\infty} x(t)$ such that $\varphi(x^*) = \min_{x\in X} \varphi(x)$.

We shall introduce the sufficient conditions of ***demipositivity.***

定理 (Theorem3) ([1] Bruck,1974) A set-valued mapping $A : X \to 2^X$ is *demi*positive when the following one of (a)-(e) holds:

(a) A is a subdifferential $\partial\varphi$ of a proper lower semi-continuous convex function $\varphi : X \to [-\infty,\infty)$ with a minimum in X.

(b) A is $I - T$, where I is an identity function and T is a non-expansive mapping with a fixed point.

(c) A is maximal monotone, odd and firmly positive.

(d) A is maximal monotone and $\mathit{int}A^{-1}(0) \neq \emptyset$.

(e) A is maximal monotone, firmly positive and weakly closed.

(Remark) (1) $A : X \to 2^X$ is called monotone if $(u - v, x - y) \geq 0$ for any $x,y \in D(A)$ and $u \in A(x), v \in A(y)$.

(2) A is called maximal monotone if it is not properly contained in any other monotone subset of X.

(3) A is said to be weakly closed if $x_n \to x$, $v_n \to v$, $v_n \in A(x_n)$ and then $v \in A(x)$.

There are many results of approximations of equilibrium points for Maximal operators ([2], [3], etc.)

2. Solutions of Variational Inequality and Equilibrium points of Differential

Variational Inequality

Let $F : K \to 2^X$ be a upper semi-continuous set-valued mapping such that $F(x)$ is a non-empty convex compact subset of X for any $x \in X$, where F is said to be upper semi-continuous if for any open set U containing $F(x_0)$ there exists a neighborhood V of x_0 such that $F(V) \subset U$, where $F(V) = \cup_{x \in V} F(x)$. We give some definitions for solutions of variational inequalities for F and K.

定義 (Def.) [4] (1) $x^* \in K$ is called a solution of Stampacchia variational inequality $SVI(F,K)$ if there exists $\xi^* \in F(x^*)$ such that $(\xi^*, y - x^*) \geq 0$ *for all* $y \in K$.
定義 (Def.) (2) $x^* \in K$ is called a solution of Strong Minty variational inequality $SMVI(F,K)$ if *for all* $y \in K$, $(\eta, y - x^*) \geq 0$ *for all* $\eta \in F(y)$.

定義 (Def.) (3) $x^* \in K$ is called a solution of Weak Minty variational inequality $WMVI(F,K)$ if for any $y \in K$ there exists $\eta_0 \in F(y)$ such that $(\eta_0, y - x^*) \geq 0$.

F is said to be pseudomonotone if for all $x,y \in K$ there exists $u \in F(x)$ such that $(u, y - x) \geq 0$ then $(v, y - x) \geq 0$ for all $v \in F(y)$. If F is pseudomonotone, the set of solutions of $SVI(F,K)$ coincides with the set of solutions of $SMVI(F,K)$, $WMVI(F,K)$. Results with respect to the convergence theorems of variational inequalities are shown in many approaches ([5], [6], [7]).

Let $T_K(x) = \{v \in X : x + a_n v_n \in K, a_n > 0, a_n \to 0, v_n \to v (n \to \infty)\}$ and let $N_K(x) = \{y \in X : (y, v) \leq 0 \textit{ for any } v \in T_K(x)\}$. $T_K(x)$ is called a tangent cone and $N_K(x)$ is called a normal cone. The following differential inclusion is said to be a differential variational inequality $DVI(F,K)$.

$$DVI(F,K) : \begin{cases} x(t) \in K \textit{ for all } t \in [0,\infty), \\ \dot{x}(t) \in -(F + N_K)(x(t)) \textit{ for a.e. } t \in [0,\infty). \end{cases}$$

And we call the following differential inclusion a projected differential inclusion $PDI(F,K)$.

$$PDI(F,K) : \begin{cases} x(t) \in K \textit{ for all } t \in [0,\infty), \\ \dot{x}(t) \in P_{T_{K(x)}}(-F)(x(t)) \textit{ for a.e. } t \in [0,\infty), \end{cases}$$

where $P_{T_{K(x)}}$ is a projection onto $T_K(x(t))$. It is shown that $x(t)$ is a solution of $DVI(F,K)$ if and only if $x(t)$ is a solution of $PDI(F,K)$.

G.P.Crespi and M.Rocca showed the following theorems ([8]).

定理 (Theorem (G.P.Crespi and M.Rocca,2004)) Let $x^* \in K$ be an equilibrium point of $DVI(F,K)$ and assume that F is pseudomonotone. Then every solution $x(t)$ of $DVI(F,K)$ satisfies that

$$\|x(t) - x^*\| \leq \|x(s) - x^*\| \quad \textit{for} \quad t \geq s.$$

We shall introduce the relation between equilibrium points of $DVI(F,K)$ and solutions of $SVI(F,K)$.

定理 (Theorem) Let $K \subset X$ be a closed convex subset, and let $F : X \to 2^X$ be an upper semi-continuous mapping with non-empty convex and compact values. Assume F is pseudomonotone. Then, the following (a) and (b) are equivalent:
(a) $x^* \in K$ is an equilibrium point of $DVI(F,K)$. (b) $x^* \in K$ is a solution of $SVI(F,K)$.

There are many results of convergence theorems to solutions of $SVI(F,K)$ by using iterative schemes and also given many results of approximating solutions. We try to study the approximation theory and the iterative methods in order to find an equilibrium point of $DI(F,K)$ with respect to the fixed point theory with good compositions of operator F of a large class of set-valued mappings.([9])

[Refferences]

[1] Ronald E. Bruck,Jr. "Asymptotic Convergence of Nonlinear Contraction Semigroups in Hilbert Space," Bull.Ams.,1974
[2] S.Kamimura and W. Takahashi, "Approximation Solutions of Maximal Monotone Operators in Hilbert Spaces," J. Approx. Theory 106-2(2000), 226-240
[3] W. Takahashi, "Nonlinear Functional Analysis," Yokohama Publishers, 2002
[4] D. Kinderlehrer and G. Stampacchia, "An Introduction to Variational Inqualities and Their Applications," Academic Press, 1980
[5] U. Mosco, "Convergence of Convex Sets and Solutions of Variational Inequalities," Adv. Math. 3(1969), 510-585
[6] H. Manaka Tamura, "Regularization of Nonlinear Variational Inequalities with pseudomonotone operators and Mosco-perturbed Domains," Proceedings of NACA, 2001
[7] J. C. Yao, "Variational Inqualities with generalized monotone operators," Mathematics of Operators Research 19(1994), 691-705
[8] G. Crespi and M. Rocca, "Minty Variational Inequalities and Monotone Trajectories of Differential Inclusions," Journal inequalities in Pure and Applied Mathmatics, vol.5, Issue 2, Article 48, 2004
[9] H. Manaka Tamura, "A Note on Stević's Iteration Method," J. Math. Anal. Appl. 314(2006), 382-389