数理解析研究所講究録
第 1561 巻 2007 年 71-78

Extensions of (weakly) null-additive, monotone set functions from rings to generated algebras *

東京工業大学・知能システム科学専攻 室伏 俊明 (Toshiaki MUROFUSHI)
Department of Computational Intelligence and Systems Science,
Tokyo Institute of Technology

Abstract. This paper shows that the greatest and least monotone extensions of a null-additive [resp. weakly null-additive], monotone set function from a ring of subsets to the algebra generated by the ring are null-additive [resp. weakly null-additive]. In addition, the paper characterizes all the (weakly) null-additive, monotone extensions.

1 Introduction

The existence of a null-additive, monotone extension of a null-additive, monotone set function from a ring of subsets to the algebra generated by the ring has been shown by Pap [3] and Wu and Sun [6]; Pap considered extensions in two cases, but Wu and Sun pointed out that there was an error in the second case and Pap's extension in the first case essentially applies to the second one, and showed that their extension of a weakly null-additive, monotone set function is weakly null-additive. This paper points out that their extension is the greatest monotone extension, and shows that the least monotone extension of a null-additive [resp. weakly null-additive], monotone set function also is null-additive [resp. weakly null-additive]. Furthermore, the paper characterizes all the (weakly) null-additive, monotone extensions.

The paper is organized as follows. Section 2 provides definitions and properties of basic concepts. Section 3 shows the above-mentioned results. We omit the proofs of the results; for the proofs, see [2]. Section 4 gives several examples of (weakly) null-additive, monotone extensions.

Throughout the paper, $\top$ is a positive extended real number, i.e., $0 < \top \leq \infty$, and the closed interval $[0, \top]$ of the real line is considered as the codomain of functions. In addition, we assume $\sup \emptyset = 0$ and $\inf \emptyset = \top$. The difference and symmetric difference of sets A and B are denoted by $A \setminus B$ and $A \triangle B$, respectively.

*This work is partially supported by a grant from the Ministry of Education, Culture, Sports, Science and Technology, the 21st Century COE Program "Creation of Agent-Based Social Systems Sciences."

2 Preliminaries

Definition 1. *Let* $(\boldsymbol{U}, \curlyvee)$ *be an upper semilattice with ordering* $\preceq$, *and* $\mu : \boldsymbol{U} \to [0, \top]$.

(i) μ *is said to be* monotone *if* $\mu(R) \le \mu(S)$ *whenever* R, $S \in \boldsymbol{U}$ *and* $R \preceq S$.

(ii) [5] μ *is said to be* null-additive *if* $\mu(R \curlyvee N) = \mu(R)$ *whenever* R, $N \in \boldsymbol{U}$ *and* $\mu(N) = 0$.

(iii) [5] μ *is said to be* weakly null-additive *if* $\mu(N_1 \curlyvee N_2) = 0$ *whenever* N_1, $N_2 \in \boldsymbol{U}$ *and* $\mu(N_1) = \mu(N_2) = 0$.

As is well-known, null-additivity implies weak null-additivity.

Throughout the paper, X is a nonempty set and $\mathcal{R}$ is a ring of subsets of X. A *set function* is a function $\mu : \mathcal{R} \to [0, \top]$ such that $\mu(\emptyset) = 0$, where $\top$ is a standard upper bound of the possible values of μ; for example, if μ is regarded as a generalization of ordinary measures, then $\top = \infty$, and, if μ is regarded as a generalization of probability measures, then $\top = 1$. We denote the family of null sets with respect to μ by $\mathcal{N}_\mu$, that is, $\mathcal{N}_\mu = \{N \mid \mu(N) = 0\}$.

The following lemmas are immediate consequences of Definition 1.

Lemma 1. [5] *Let* μ *be a monotone set function on a ring* $\mathcal{R}$. *The following conditions are equivalent to each other.*

(a) μ *is null-additive.*

(b) $\mu(R \triangle N) = \mu(R)$ *whenever* $R \in \mathcal{R}$ *and* $N \in \mathcal{N}_\mu$.

(c) $\mu(R \setminus N) = \mu(R)$ *whenever* $R \in \mathcal{R}$ *and* $N \in \mathcal{N}_\mu$.

Lemma 2. *Let* μ *be a monotone set function on a ring* $\mathcal{R}$. *Then* μ *is weakly null-additive iff* $\mathcal{N}_\mu$ *is an ideal of* $\mathcal{R}$.

If a monotone set function μ on $\mathcal{R}$ is null-additive, then, since $\mathcal{N}_\mu$ is an ideal of $\mathcal{R}$ by Lemma 2, we can consider the quotient ring

$$\mathcal{R}/\mathcal{N}_\mu = \{R \triangle \mathcal{N}_\mu \mid R \in \mathcal{R}\},$$

where $R \triangle \mathcal{N}_\mu = \{R \triangle N \mid N \in \mathcal{N}_\mu\}$, and due to Lemma 1 we can define a monotone, extended-real-valued function M on $\mathcal{R}/\mathcal{N}_\mu$ by

$$\mathrm{M}(R \triangle \mathcal{N}_\mu) = \mu(R);$$

note that $\mathrm{M}(\mathcal{N}_\mu) = \mu(\emptyset) = 0$ and that $\mathrm{M}(R \triangle \mathcal{N}_\mu) > 0$ whenever $R \triangle \mathcal{N}_\mu \neq \mathcal{N}_\mu$. Conversely, if $\mathcal{N}$ is an ideal of $\mathcal{R}$, and if M is a monotone, extended-real-valued function defined on $\mathcal{R}/\mathcal{N}$ such that $\mathrm{M}(\mathcal{N}) = 0$, then we can define a null-additive, monotone set function μ on $\mathcal{R}$ by

$$\mu(R) = \mathrm{M}(R \triangle \mathcal{N}).$$

Moreover, if $\mathrm{M}(R \triangle \mathcal{N}) > 0$ whenever $R \triangle \mathcal{N} \neq \mathcal{N}$, then it holds that $\mathcal{N}_\mu = \mathcal{N}$.

For any ring $\mathcal{R}$ of subsets of X, let $\mathcal{A}(\mathcal{R})$ be the algebra on a set X generated by $\mathcal{R}$, that is, $\mathcal{A}(\mathcal{R}) = \bigcap\{\mathcal{A} \mid \mathcal{A} \text{ is an algebra on } X \text{ containing } \mathcal{R}\}$.

Proposition 1. [4] *The algebra $\mathcal{A}(\mathcal{R})$ generated by a ring $\mathcal{R}$ on a set X is given by $\mathcal{A}(\mathcal{R}) = \mathcal{R} \cup \complement\mathcal{R}$, where $\complement\mathcal{R} := \{X \setminus R \mid R \in \mathcal{R}\}$.*

As is well known, the following lemmas hold.

Lemma 3. *The following five conditions are equivalent to each other:*

(a) $X \in \mathcal{R}$,

(b) $\mathcal{R}$ *is an algebra,*

(c) $\mathcal{A}(\mathcal{R}) = \mathcal{R}$,

(d) $\complement\mathcal{R} = \mathcal{R}$,

(e) $\mathcal{R} \cap \complement\mathcal{R} \neq \emptyset$.

Lemma 4. (i) *If $R \in \mathcal{R}$ and $A \in \mathcal{A}(\mathcal{R})$, then $R \cap A \in \mathcal{R}$ and $R \setminus A \in \mathcal{R}$.*

(ii) *If $C \in \complement\mathcal{R}$ and $A \in \mathcal{A}(\mathcal{R})$, then $C \cup A \in \complement\mathcal{R}$.*

(iii) *If C, $D \in \complement\mathcal{R}$, then $C \cap D \in \complement\mathcal{R}$.*

(iv) *If $C \in \complement\mathcal{R}$ and $R \in \mathcal{R}$, then $C \setminus R \in \complement\mathcal{R}$.*

(v) *If $A \in \mathcal{A}(\mathcal{R})$ and $C \in \complement\mathcal{R}$, then $A \setminus C \in \mathcal{R}$.*

It follows from (i) that $\mathcal{R}$ is an ideal in $\mathcal{A}(\mathcal{R})$, and from (ii) and (iii) that $\complement\mathcal{R}$ is a filter in $\mathcal{A}(\mathcal{R})$.

3 (Weakly) null-additive extensions

In this section, μ is assumed to be a monotone set function from a ring $\mathcal{R}$ of subsets of a set X into $[0, \top]$.

Definition 2. (i) *The set function μ^* on $\mathcal{A}(\mathcal{R})$ is defined by*

$$\mu^*(A) = \inf\{\mu(R) \mid A \subset R \in \mathcal{R}\} \tag{1}$$

for $A \in \mathcal{A}(\mathcal{R})$.

(ii) *The set function μ_* on $\mathcal{A}(\mathcal{R})$ is defined by*

$$\mu_*(A) = \sup\{\mu(R) \mid R \in \mathcal{R},\ R \subset A\} \tag{2}$$

for $A \in \mathcal{A}(\mathcal{R})$.

Since $\inf \emptyset = \top$, it follows that

$$\mu^*(A) = \begin{cases} \mu(A) & \text{if } A \in \mathcal{R}, \\ \top & \text{if } A \in \mathcal{A}(\mathcal{R}) \setminus \mathcal{R}. \end{cases} \tag{3}$$

The set function μ^* is a monotone extension of μ, i.e., it is an extension of μ and is monotone [3], [6], and obviously so is μ_*; hence, if $\mathcal{R}$ is an algebra, $\mu^* = \mu_* = \mu$. In addition, for any monotone extension $\overline{\mu}$ on $\mathcal{A}(\mathcal{R})$ of μ, it follows that $\mu_* \le \overline{\mu} \le \mu^*$. Therefore, μ^* and μ_* are respectively the greatest and least monotone extensions of μ, and obviously, if $\mu_* = \mu^*$, then the monotone extension of μ is unique.

Pap [3], Wu and Sun [6] have shown that the greatest monotone extension μ^* preserves the null-additivity and weak null-additivity of μ.

Theorem 1. *For every monotone set function μ on $\mathcal{R}$, the following hold:*

(i) [3], [6] *If μ is null-additive, then so is μ^*.*

(ii) [6] *If μ is weakly null-additive, then so is μ^*.*

The following is one of our main theorems of this paper, which shows that the least monotone extension μ_* also preserves the null-additivity and weak null-additivity of μ.

Theorem 2. *For every monotone set function μ on $\mathcal{R}$, the following hold:*

(i) *If μ is null-additive, then so is μ_*.*

(ii) *If μ is weakly null-additive, then so is μ_*.*

Proof. See [2]. □

Remark 1. The outer and inner set functions [1] induced by μ are the set functions μ^* and μ_* on 2^X defined by Eqs. (1) and (2) for $A \in 2^X$, respectively. If μ is a null-additive [resp. weakly null-additive] monotone set function on a ring $\mathcal{R}$, then, while the outer and inner set functions μ^* and μ_* induced by μ are null-additive [resp. weakly null-additive] on $\mathcal{A}(\mathcal{R})$ by Theorems 1 and 2, they are not necessarily null-additive or weakly null-additive on 2^X. This fact is shown by the following example.

Consider the real line $\mathbb{R}$ as the whole set X. Let $\mathcal{R}$ be the ring generated by the family $\mathcal{I} = \{(a, b] \mid -\infty < a < b < \infty\}$ of all bounded left half-open intervals, i.e., $\mathcal{R} = \bigcap\{\mathcal{R}_0 \mid \mathcal{R}_0 \text{ is a ring on } \mathbb{R} \text{ containing } \mathcal{I}\}$. Define $\mu : \mathcal{R} \to [0, \infty]$ by

$$\mu(R) = \begin{cases} \infty & \text{if } \{0, 1\} \subset R, \\ \lambda(R) & \text{otherwise}, \end{cases} \qquad (R \in \mathcal{R}),$$

where λ is the Lebesgue measure on $\mathbb{R}$. Obviously μ is monotone, and, since μ vanishes only at the empty set, μ is null-additive. However, neither the outer set function μ^* nor the inner set function μ_* is weakly null-additive. Indeed, $\mu^*(\{0\}) = \mu^*(\{1\}) = 0$ and $\mu^*(\{0, 1\}) = \infty$, and besides, $\mu_*(\mathbb{Q}) = \mu_*(\mathbb{R} \setminus \mathbb{Q}) = 0$ and $\mu_*(\mathbb{R}) = \infty$, where $\mathbb{Q}$ is the set of rational numbers. Note that this implies that the restrictions of μ^* and μ_* to the σ-ring generated by $\mathcal{R}$ are not weakly null-additive.

If $\overline{\mu}$ is a monotone extension of a monotone set function μ on a ring $\mathcal{R}$ to the algebra $\mathcal{A}(\mathcal{R})$, then obviously $\mathcal{N}_\mu = \mathcal{N}_{\mu^*} \subset \mathcal{N}_{\overline{\mu}} \subset \mathcal{N}_{\mu_*}$; note that $\mathcal{N}_{\mu_*} \cap \mathcal{R} = \mathcal{N}_{\overline{\mu}} \cap \mathcal{R} = \mathcal{N}_{\mu^*} \cap \mathcal{R} = \mathcal{N}_\mu$. Now, we show our second main theorem.

Theorem 3. *Let $\overline{\mu}$ be a monotone extension of a monotone set function μ on $\mathcal{R}$ to $\mathcal{A}(\mathcal{R})$, and $\mathcal{N}_\mu \subsetneqq \mathcal{N}_{\overline{\mu}}$.*

(i) *If $\overline{\mu}$ is null-additive, then $\overline{\mu} = \mu_*$.*

(ii) *If $\overline{\mu}$ is weakly null-additive, then $\mathcal{N}_{\overline{\mu}} = \mathcal{N}_{\mu_*}$.*

Proof. See [2]. □

As a direct consequence of the above theorem, we can obtain the following theorem, which characterizes the (weakly) null-additive monotone extensions. Note that condition (b) of (i) follows from the remark just below Lemma 2. In either case (i) or (ii), μ_* satisfies condition (a) and μ^* satisfies condition (b).

Theorem 4. *Assume $X \notin \mathcal{R}$.*

(i) *Let μ be a null-additive, monotone set function on $\mathcal{R}$. Then $\overline{\mu}$ is a null-additive, monotone extension of μ on $\mathcal{A}(\mathcal{R})$ if and only if* (a) *or* (b) *below holds:*

(a) $\overline{\mu} = \mu_*$.

(b) *There exists a monotone function $\mathrm{M}_{\complement}$ defined on $(\complement\mathcal{R})/\mathcal{N}_\mu = \{C \triangle \mathcal{N}_\mu \mid C \in \complement\mathcal{R}\}$ such that*

$$\overline{\mu}(A) = \begin{cases} \mu(A) & \text{if } A \in \mathcal{R}, \\ \mathrm{M}_{\complement}(A \triangle \mathcal{N}_\mu) & \text{if } A \in \complement\mathcal{R}, \end{cases}$$

and that $\mathrm{M}_{\complement}$ satisfies both of the following conditions:

(b-1) $\mathrm{M}_{\complement}(C \triangle \mathcal{N}_\mu) \geq \mu_*(C)$ *for all* $C \in \complement\mathcal{R}$,

(b-2) $\mathrm{M}_{\complement}(C \triangle \mathcal{N}_\mu) > 0$ *for all* $C \in \complement\mathcal{R}$.

(ii) *Let μ be a weakly null-additive, monotone set function on $\mathcal{R}$. Then $\overline{\mu}$ is a weakly null-additive, monotone extension of μ on $\mathcal{A}(\mathcal{R})$ if and only if there exists a monotone function $\mu_{\complement}$ defined on $\complement\mathcal{R}$ such that*

$$\overline{\mu}(A) = \begin{cases} \mu(A) & \text{if } A \in \mathcal{R}, \\ \mu_{\complement}(A) & \text{if } A \in \complement\mathcal{R}, \end{cases}$$

and that $\mu_{\complement}$ satisfies

$$\mu_{\complement}(C) \geq \mu_*(C) \quad \text{for all } C \in \complement\mathcal{R}$$

and one of the following two conditions:

(a) $\{N \in \complement\mathcal{R} \mid \mu_{\complement}(N) = 0\} = \mathcal{N}_{\mu_*} \setminus \mathcal{N}_\mu$,

(b) $\{N \in \complement\mathcal{R} \mid \mu_{\complement}(N) = 0\} = \emptyset$.

4 Examples

Let μ be a monotone set function on a ring $\mathcal{R}$. Then for any nondecreasing function $\varphi : [0, \top] \to [0, \top]$ satisfying $\varphi(r) \geq r$ for all $r \in [0, \top]$, the set function $\overline{\mu}_\varphi$ defined below is a monotone extension of μ on $\mathcal{A}(\mathcal{R})$:

$$\overline{\mu}_\varphi(A) = \begin{cases} \mu(A) & \text{if } A \in \mathcal{R}, \\ \varphi(\mu_*(A)) & \text{if } A \in \mathcal{A}(\mathcal{R}) \setminus \mathcal{R}. \end{cases} \tag{4}$$

If μ is weakly null-additive, then so is $\overline{\mu}_\varphi$. If $\mu_* \neq \mu^*$, then there is an monotone extension $\overline{\mu}_\varphi$ different from μ_* and μ^*; there is $C \in \complement\mathcal{R}$ such that $\mu_*(C) < \mu^*(C)$, and there is $\varphi : [0, \top] \to [0, \top]$ such that $\varphi(r) \geq r$ for all $r \in [0, \top]$ and $\mu_*(C) < \varphi(\mu_*(C)) < \mu^*(C)$.

As mentioned before, if $\overline{\mu}$ is a monotone extension of a monotone set function μ on $\mathcal{R}$ to $\mathcal{A}(\mathcal{R})$, then $\mathcal{N}_\mu = \mathcal{N}_{\mu^*} \subset \mathcal{N}_{\overline{\mu}} \subset \mathcal{N}_{\mu_*}$. Regardless whether μ is (weakly) null-additive or not, the following cases can occur:

Case I. $\mathcal{N}_{\mu^*} = \mathcal{N}_{\overline{\mu}} = \mathcal{N}_{\mu_*}$,

Case II. $\mathcal{N}_{\mu^*} = \mathcal{N}_{\overline{\mu}} \subsetneqq \mathcal{N}_{\mu_*}$,

Case III. $\mathcal{N}_{\mu^*} \subsetneqq \mathcal{N}_{\overline{\mu}} = \mathcal{N}_{\mu_*}$,

Case IV. $\mathcal{N}_{\mu^*} \subsetneqq \mathcal{N}_{\overline{\mu}} \subsetneqq \mathcal{N}_{\mu_*}$.

By Theorem 3, if $\overline{\mu}$ is weakly null-additive, then Case IV does not hold.

In what follows, consider the set $\mathbb{N}$ of positive integers as the whole set X, and let $\mathcal{R}$ be the ring of finite subsets of $\mathbb{N}$. Then $\complement\mathcal{R}$ is the family of cofinite subsets of $\mathbb{N}$, and $\mathcal{A}(\mathcal{R}) = \mathcal{R} \cup \complement\mathcal{R}$. For a function $f : \mathbb{N} \to [0, \top]$, we write $N_f = \{n \in \mathbb{N} \mid f(n) = 0\}$ and $T_f = \{n \in \mathbb{N} \mid f(n) = \top\}$.

Example 1. Consider a function $f : \mathbb{N} \to [0, \top]$, and let μ be the set function on $\mathcal{R}$ defined by

$$\mu(R) = \bigvee_{n \in R} f(n) \qquad (R \in \mathcal{R}),$$

where $\bigvee$ stands for supremum. Then, by definition, μ is monotone and null-additive, and $\mathcal{N}_\mu = \{N \in \mathcal{R} \mid N \subset N_f\} = 2^{N_f} \cap \mathcal{R}$. The least monotone extension μ_* is given as

$$\mu_*(A) = \bigvee_{n \in A} f(n)$$

for $A \in \mathcal{A}(\mathcal{R})$, and it follows that

$$\mathcal{N}_{\mu_*} = \{N \in \mathcal{A}(\mathcal{R}) \mid N \subset N_f\} = \mathcal{N}_\mu \cup \{N \in \complement\mathcal{R} \mid N \subset N_f\}. \tag{5}$$

(The greatest monotone extension μ^* is given by Eq. (3).)

We consider the following three cases:

1. Let T_f be infinite or $\bigvee_{n\in\mathbb{N}\setminus T_f} f(n) = \top$. Then $\mu_* = \mu^*$ and hence the monotone extension of μ is unique. Hence Case I holds, and all the conditions in Theorem 4, (a), (b) of (i) and (a), (b) of (ii), are satisfied.

2. Let T_f be finite and $\bigvee_{n\in\mathbb{N}\setminus T_f} f(n) < \top$. Then, since $\mathbb{N}\setminus T_f \in \mathcal{CR}$ and

$$\mu_*(\mathbb{N}\setminus T_f) = \bigvee_{n\in\mathbb{N}\setminus T_f} f(n) < \top = \mu^*(\mathbb{N}\setminus T_f),$$

it follows that $\mu_* \neq \mu^*$. Now, let $\mathbb{N}\setminus N_f$ be infinite, i.e., N_f be not cofinite. Then it follows from Eq. (5) that $\mathcal{N}_\mu = \mathcal{N}_{\mu^*} = \mathcal{N}_{\mu_*}$; Case I holds again.

   2-1. Every $\overline{\mu}_\varphi$ defined by Eq. (4) is null-additive and satisfies (b) in Theorem 4 (i).

   2-2. Every monotone extension $\overline{\mu}$ of μ is weakly null-additive and satisfies (a) and (b) in Theorem 4 (ii).

3. Let N_f be cofinite; this implies that T_f is finite and $\bigvee_{n\in\mathbb{N}\setminus T_f} f(n) < \top$. Then, since $N_f \in \mathcal{CR}$ and hence $N_f \in \mathcal{N}_{\mu_*}\setminus\mathcal{N}_\mu$, it follows that $\mathcal{N}_\mu = \mathcal{N}_{\mu^*} \subsetneqq \mathcal{N}_{\mu_*}$.

   3-1. If $\varphi(0) > 0$, then $\overline{\mu}_\varphi$ is null-additive and satisfies (i) (b) and (ii) (b) in Theorem 4; in this case, Case II holds.

   3-2. If $\varphi(0) = 0$ and $\overline{\mu}_\varphi \neq \mu_*$, then $\overline{\mu}_\varphi$ is not null-additive but weakly null-additive, and satisfies (ii) (a) in Theorem 4; in this case, Case III holds.

   3-3. Let $v \in N_f$ and

$$\overline{\mu}_v(A) = \begin{cases} \top & \text{if } v \in A \in \mathcal{CR}, \\ \mu_*(A) & \text{otherwise} \end{cases} \tag{6}$$

   for $A \in \mathcal{A}(\mathcal{R})$. Then $\overline{\mu}_v$ is a monotone extension of μ, and it follows that $\mathcal{N}_\mu = \mathcal{N}_{\mu^*} \subsetneqq \mathcal{N}_{\overline{\mu}_v} = \mathcal{N}_\mu \cup \{N \in \mathcal{CR} \mid N \subset N_f\setminus\{v\}\} \subsetneqq \mathcal{N}_{\mu_*}$. Hence Case IV holds. This $\overline{\mu}_v$ is not weakly null-additive; $\overline{\mu}_v(N_f) = \top > 0$ while $\overline{\mu}_v(\{v\}) = \mu_*(\{v\}) = \mu(\{v\}) = 0$ and $\overline{\mu}_v(N_f\setminus\{v\}) = \mu_*(N_f\setminus\{v\}) = 0$.

Remark 2. In a similar way to the above example, we can construct an example of monotone extensions of a weakly null-additive, monotone set function. For instance, in the same setting as Example 1 with the additional condition $\top > 1$, assume there are $n_0, n_1 \in \mathbb{N}$ such that $f(n_0) = 0$ and $f(n_1) > 1$, and define the set function μ on $\mathcal{R}$ by

$$\mu(R) = \begin{cases} 1 & \text{if } R = \{n_1\}, \\ \bigvee_{n\in R} f(n) & \text{otherwise,} \end{cases} \qquad (R \in \mathcal{R}).$$

Then μ is a weakly null-additive, monotone set function. By assumption, μ is not null-additive; $\mu(\{n_1\}) < \mu(\{n_0, n_1\})$ while $\mu(\{n_0\}) = 0$. About the monotone extension of μ to $\mathcal{A}(\mathcal{R})$, we can make the same argument as the weakly null-additive, monotone extensions in Example 1.

References

[1] D. Denneberg, *Non-additive measures and integrals*, Kluwer Academic, Dordrecht, 1994.

[2] T. Murofushi, Extensions of (weakly) null-additive, monotone set functions from rings of subsets to generated algebras, *Fuzzy Sets and Systems*, submitted.

[3] E. Pap, Extension of null-additive set functions on algebra of subsets, *Novi Sad J. Math.*, **31** (2) 2–13 (2001).

[4] S. Rao and M. Rao, *Theory of Charges*, Academic Press, London, 1980.

[5] Z. Wang, G. J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York, 1992.

[6] C. Wu and B. Sun, A note on the extension of null-additive set function on algebra of subsets, *Appl. Math. Lett.*, doi:10.1016/j.aml.2006.07.011, to appear.