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Finitely generated semigroups with regular congruence classes *

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In this paper, we give characterizations of finitely generated semigroups with regular congruence classes and finitely generated semigroups with finite congruence classes.

1 Presentations of semigroups

Definition. X is a finite alphabet, X^* is the set of all words over X. X^+ is the set of all non-empty words over X, that is, $X^+ = X^* - \{1\}$. Under juxtaposition, X^* is the free monoid with a set X of free generators and X^+ is the free semigroup with a set X of free generators.
A monoid M is *finitely generated* if there exists a finite set of X and there exists a surjective homomorphism of X^* to M which maps an empty word onto the identity element of M.
A semigroup S is *finitely generated* if there exists a finite set of X and there exists a surjective homomorphism of X^+ to S.

Definition (1) Let X be a finite alphabet and R a subset of $X^* \times X^*$. Then R is *string-rewriting system.*

(2) For $u, v \in X^*$, $(w_1, w_2) \in R$, $uw_1v \Rightarrow_R uw_2v$.

The congruence μ_R on X^* (or X^+) generated by $\Rightarrow_R$ is the Thue congruence defined by R.

(3) A monoid S has a *(finite)presentation* if there exists a (finite) set of X, there exists a surjective homomorphism ϕ of X^* to S and there exists a (finie) string-rewriting system R consisting of pairs of words over X such that the Thue congruence μ_R is the congruence $\{(w_1, w_2) \in X^* \times X^* \mid \phi(w_1) = \phi(w_2)\}$.

*This is an absrtact and the paper will appear elsewhere.

2 Semigroups with regular congruence classes

Definition. A semigroup S has *regular congruence classes* if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to S such that for each words $w \in X^+$ $\phi^{-1}(\phi(w))$ is a regular language.

Definition. X is a finite set of alphabet, X^* is the set of words over X, L is a subset of X^*, is called a *language.* The *syntactic congruence* σ_L on X^* is defined by $w\sigma_L w'$ if and only if the sets $\{(x,y) \in X^* \times X^* \mid xwy \in L\}$, $\{(x,y) \in X^* \times X^* \mid xw'y \in L\}$ are equal to each other. The *syntactic monoid* of L is defined to be a monoid X^*/σ_L

Result 1. *Let L be a language over X. Then L is regular if and only if $Syn(L)$ is a finite monoid.*

Result 2. *Let L be a language of X^*. Then the following are equivalent :*

(1) *L is a σ_L-class in X^*.*

(2) $xLy\ \cap L \neq ((x,y \in X^*) \Rightarrow xLy\ \subseteq L.$

(3) *L is an inverse image $\phi^{-1}(m)$ of a homomorphism ϕ of X^* to a monoid M.*

Result 3. *For every finitely generated monoid M, there exist languages $\{L_m\}_{m\in M}$ of X^* such that M is embedded in the direct product of syntactic monoids.*

Definition. A monoid S is called *residually finite* if for each pair of elements $m, m' \in S$, there exists a conguence on S such that the factor monoid S/μ is finite and $(m,m') \notin \mu$.

Result 4. *If a finitely generated semigroup S has regular congruence classes, then M is residually finite.*

Definition. Let S be a finite generated semigroup. Let X be a finite set and there exists a surjective homomorphism ϕ of X^+ to S. Then the word problem of S is *decidable* if there exists an algorithsm to decide whether $\phi(w_1)$ is equal to $\phi(w_2)$ for each pair of words $w_1, w_2 \in X^+$.

Result 5. The word problem is decidable for finitely generated semigroups with regular congruence classes.

Exampe 1. *A finite semigroup S is semigroup with regular congruence classes.*

Definition. Let S be a semigroup. For any $s \in S$, let $\sigma_s = \{(a,b) \in S \times S \mid xay = s$ if and if $xby = s\ (x,y \in S^1)\}$.

Then σ_s is a congruence on S.

Theorem 1. *A finitely generated semigroup S has regular congruence classes if and only if for any $s \in S$, S/σ_s is a finite semigroup.*

Theorem 2. For a finitely generated semigroup S, it does not depend on presentations of S that S has regular congruence classes.

Theorem 3. *Let S be a finitely generated semigroup with regular congruence classes. Then a subgroup of S is finite.*

Example 2. Let X denote a finite alphabet and $w_1, \cdots, w_r \in X^+$ words over X. Let $I = X^*w_1X^* \cup \cdots \cup X^*w_rX^*$ be an ideal of the free semigroup X^+. Then The Rees factor semigroup X^+/I module I is a (unnecessarily finite) semigroup with regular congruence classes.

Result 6 . (1) *For every finite group G, there exists a regular language L of X^* such that G is isomorphic to $Syn(L)$.*

(2) *If a group G has regular congruence classes, then G is a finite.*

Theorem 4. Let S be a semigroup with regular congruence classes. If S is a completely (0-)simplesemigroup, then S is finite.

Exampe 3. *A residually finite semigroup S is not always a semigroup with regular congruence classes.*

3 Semigroups with finite congruence classes.

Definition. A semigroup S has *finite congruence classes* if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to S such that for each words $w \in X^+$ $\phi^{-1}(\phi(w))$ is a finite set.

Theorem 5. *Let S be a semigroup with finite congruence classes. Then S has no idempotents except* 1 (*that is* , *S possibly has an idempotent*).

Theorem 6. *Let S be a semigroup with regular congruence classes. Then S is a semigroup with finite congruence classes if and only if for any $s \in S$, S/σ_s is a finite nilpotent semigroup with zero.*

Theorem 7. For a finitely generated semigroup S, it does not depend on presentations of S that S has finite congruence classes.

Theorem 8. Let S be a finitely presented semigroup with a string-rewriting system consisting of pairs of words of the same length. Then S is a semigroup with finite congruence classes.

Example 4. Let $X = \{x_1, x_2, \cdots, x_r\}$ and $\mathcal{R} = \{(x_i, x_j) \mid 1 \leq i < j \leq r\}$. Then $X^*/\mathcal{R}^*$ is a monoid with finite congruence classes and is the commutative free monoid.

Theorem 9. Let S be a finitely generated semigroup with a non-overlapping string-rewriting system. Then S is a semigroup with finite congruence classes.

Theorem 10. Any finitely generated subsemigroup of the free semigroup is a semigroup with finite congruence classes.

Example 5 There exists a finitely generated subsemigroup S of the free semigroup which is a semigroup with finite congruence classes but does not have a finite presentation.

Actually, let $X = \{A, B, V, W\}$. Then $V(AB)^n W = VA(BA)^{n-1}W$ in X^+. So the finitely generated subsemigroup $< V, VA, AB, BA, W, BW >$ is isomophic to non-finitely presented semigroup $Y =< a, b, c, d, e, f \mid ac^n e = bd^{n-1} f (n = 0, 1, 2, \cdots) >$. By theorem 10, the semigroup is a semigroup with finite congruence classes.

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Theorem 11. *Any semigroup with either $C(3)$ or $C(2) + T(4)$ has finite congruence classese.* (Refer to [1],[3],[4] and [7] for the conditions $C(p)$, $T(q)$.)

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