The $L^p$ boundedness of wave operators for Schrödinger operators

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1 Introduction

Let $H = -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^m$, $m \geq 1$, with real valued potential $V(x)$ such that $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > 2$, where $\langle x \rangle = (1 + x^2)^{1/2}$. Then, it is well known that

1. $H$ is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^m)$ with domain $D(H) = H^2(\mathbb{R}^m)$ and $C_0^\infty(\mathbb{R}^m)$ is a core;

2. the spectrum $\sigma(H)$ of $H$ consists of an absolutely continuous part $[0, \infty)$, and at most a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities;

3. the singular continuous spectrum and positive eigenvalues are absent from $\sigma(H)$.

We denote the point and the absolutely continuous spectral subspaces of $\mathcal{H}$ for $H$ by $\mathcal{H}_p$ and $\mathcal{H}_{ac}$ respectively, and the orthogonal projections in $\mathcal{H}$ onto the respective subspaces by $P_p$ and $P_{ac}$. We write $H_0 = -\Delta$ for the free Schrödinger operator.

4. The wave operators $W_\pm$ defined by the following limits in $\mathcal{H}$:

$$W_\pm = \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$$

exist and are complete in the sense that $\text{Image } W_\pm = \mathcal{H}_{ac}$.

5. $W_\pm$ satisfy the so called intertwining property and the absolutely continuous part of $H$ is unitarily equivalent to $H_0$ via $W_\pm$: For Borel functions $f$ on $\mathbb{R}$, we have

$$f(H)P_{ac}(H) = W_\pm f(H_0)W_\pm^*.$$  (1.1)
It follows from the intertwining property (1.1) that, if $X$ and $Y$ are Banach spaces such that $L^2(\mathbb{R}^m) \cap X$ and $L^2(\mathbb{R}^m) \cap Y$ are dense in $X$ and $Y$ respectively, then,

$$
\|f(H)P_{ac}(H)\|_{B(X,Y)} \leq \|W_\pm\|_{B(Y)} \|f(H_0)\|_{B(X,Y)} \|W_\pm^*\|_{B(X)} = C \|f(H_0)\|_{B(X,Y)} \tag{1.2}
$$

Here it is important that the constant $C = \|W_\pm\|_{B(Y)} \|W_\pm^*\|_{B(X)}$ is independent of the function $f$. Thus, the mapping property of $f(H)P_{ac}(H)$ from $X$ to $Y$ may be deduced from that of $f(H_0)$, once we know that $W_\pm$ are bounded in $X$ and in $Y$. Note that the solutions $u(t)$ of the Cauchy problem for the Schrödinger equation

$$
i \partial_t u = (-\Delta + V)u, \quad u(0) = \varphi$$

and $v(t)$ of the wave equation

$$\partial_t^2 v = (\Delta - V)v, \quad v(0) = \varphi, \quad \partial_t v(0) = \psi$$

are given in terms of the functions of $H$, respectively by

$$u(t) = e^{-itH} \varphi, \quad \text{and} \quad v(t) = \cos(t\sqrt{H}) \varphi + \frac{\sin(t\sqrt{H})}{\sqrt{H}} \psi.$$

It follows that, if $W_\pm$ are bounded in Lebesgue spaces $L^p(\mathbb{R}^m)$ for $1 \leq p \leq \infty$ and if the initial states $\varphi$ and $\psi$ belong to the continuos spectral subspace $\mathcal{H}_c(H)$, then the $L^p$-$L^q$ estimates for the propagators of the respective equations may be deduced from the well known $L^p$-$L^q$ estimates for the free propagators $e^{-itH_0}$ or $\cos(t\sqrt{H_0})$ and $\sin(t\sqrt{H_0})/\sqrt{H_0}$ (if $\varphi$ and $\psi$ are eigenfunctions of $H$, the behavior of $u(t)$ and $v(t)$ are trivial). In particular, we have the so called dispersive estimates for the Schrödinger equation

$$\|e^{-itH} P_c(H) \varphi\|_\infty \leq C|t|^{-\frac{m}{2}} \|\varphi\|_1.$$

In this lecture we would like to briefly survey the current status of the study of the mapping property of $W_\pm$ in Lebesgue spaces $L^p(\mathbb{R}^m)$. We say that 0 is a resonance of $H$, if there is a solution $\varphi$ of $(-\Delta + V(x))\varphi(x) = 0$ such that $|\varphi(x)| \leq C|x|^{2-m}$ but $\varphi \notin \mathcal{H}$ and call such a solution $\varphi(x)$ a resonance function of $H$; $H$ is of generic type, if 0 is neither an eigenvalue nor a resonance of $H$, otherwise of exceptional type. Note that there is no zero resonance if $m \geq 5$. We shall see that the mapping property of $W_\pm$ in $L^p(\mathbb{R}^m)$ spaces is fairly well understood when $H$ is of generic type although the conditions on potentials for the $L^p$-boundedness of $W_\pm$ are far
from optimal and the end point problem, viz. the problem for the case $p = 1$ and $p = \infty$ is not settled completely in the cases $m = 1$ and $m = 2$. On the other hand, if $H$ is of exceptional type, the situation is much less satisfactory: We have essentially no results when $m = 2$ and only a partial result for $m = 4$; when dimensions $m = 3$ or $m \geq 5$, we know that $W_{\pm}$ are bounded in $L^{p}(\mathbb{R}^{m})$ for $p$ between $m/m - 2$ and $m/2$, however, we have only partial answers for what happens for $p$ outside this interval. We should also emphasize that these results are obtained only for operators $-\Delta + V(x)$ and, the problem is completely open when magnetic fields are present or when the metric of the space is not flat.

The general reference are as follows: For one dimension $m = 1$ see [3]; [17] and [8] for $m = 2$, [16] and [9] for $m = 4$, [15] and [19] for odd $m \geq 3$, and [16] and [5] for even $m \geq 6$.

2 One dimensional case

In one dimension we have the fairly satisfactory result. The following result is due to D'Ancona and Fanelli ([3], see [14, 1] for earlier results).

**Theorem 2.1.** (1) Suppose $\langle x \rangle^{2}V(x) \in L^{1}(\mathbb{R}^{1})$. Then, $W_{\pm}$ are bounded in $L^{p}$ for all $1 < p < \infty$.

(2) Suppose $\langle x \rangle V(x) \in L^{1}(\mathbb{R}^{1})$ and $H$ is of generic type, then $W_{\pm}$ are bounded in $L^{p}$ for all $1 < p < \infty$.

**Remark 2.2.** We believe that $W_{\pm}$ are not bounded in $L^{1}$ nor in $L^{\infty}$ and that $W_{\pm}$ are bounded from Hardy space $H^{1}$ into $L^{1}$ and $L^{\infty}$ into BMO. However, we do not know the definite answer yet.

The proof of Theorem 2.1 employs the expression of $W_{\pm}$ in terms of the scattering eigenfunctions $\varphi_{\pm}(x, \xi)$ of $H$:

$$W_{\pm}u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi_{\pm}(x, \xi) \hat{u}(\xi) d\xi$$

as in earlier works [14, 1]) and uses some detailed properties of $\varphi_{\pm}(x, \xi)$. The functions $\varphi_{\pm}(x, \xi)$ are obtained by solving the Lippmann-Schwinger equation

$$\varphi_{\pm}(x, \xi) = e^{ix\xi} + \frac{1}{2i\xi} \int_{-\infty}^{\infty} e^{\pm i|\xi-\nu|V(y)} \varphi_{\pm}(y, \xi) dy$$

and it can be expressed in terms of Jost functions. We refer [3] for the details.
3 Higher dimensional case $m \geq 2$

In higher dimensions $m \geq 2$, the situation is not as satisfactory as in the one dimensional case: We believe that the conditions on the potentials in the following theorems are far from optimal.

When $m \geq 2$, the problem has been studied by using the stationary representation formula of wave operators which expresses $W_\pm$ in terms of the boundary values of the resolvent. We write

$$G(\lambda) = (H - \lambda^2)^{-1}, \quad G_0(\lambda) = (H_0 - \lambda^2)^{-1}.$$  \quad \lambda \in \mathbb{C}^+$$where $\mathbb{C}^+ = \{z \in \mathbb{C}: \Im z > 0\}$ is the upper half plane. We write

$$\mathcal{H}_s = L^2_s(\mathbb{R}^m) = L^2(\mathbb{R}^m, \langle x \rangle^{2s}dx)$$

for the weighted $L^2$ spaces. We recall the well known limiting absorption principle (LAP) for $G_0(\lambda)$ and $G(\lambda)$ due to Agmon and Kuroda (see [11]). For Banach spaces $X, Y$, $B_{\infty}(X, Y)$ is the space of compact operators from $X$ to $Y$; $a_-$ for $a \in \mathbb{R}$ stands for an arbitrary number smaller than $a$.

Lemma 3.1. (1) Let $1/2 < \sigma$. Then, $G_0(\lambda)$ is a $B_{\infty}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\sigma})$ valued function of $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ of class $C^{(\sigma-\frac{1}{2})_-}$. For non-negative integers $j < \sigma - \frac{1}{2}$,

$$\|G_0^{(j)}(\lambda)\|_{B(\mathcal{H}_{\sigma},\mathcal{H}_{-\sigma})} \leq C_{j\sigma} |\lambda|^{-1}, \quad |\lambda| \geq 1.$$ (3.1)

(2) Let $\frac{1}{2} < \sigma, \tau < m - \frac{3}{2}$ satisfy $\sigma + \tau > 2$. Then, $G_0(\lambda)$ is a $B_{\infty}(\mathcal{H}_{\sigma}, \mathcal{H}_{-\tau})$-valued function of $\lambda \in \overline{\mathbb{C}^+}$ of class $C^{\rho_*}$, $\rho_* = \min(\tau + \sigma - 2, \tau - 1/2, \sigma - 1/2)$.

Lemma 3.2. (1) Assume $|V(x)| \leq C(x)^{-\delta}$ for some $\delta > 1$. Let $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$. Then, $G(\lambda)$ is a $B_{\infty}(\mathcal{H}_{\gamma}, \mathcal{H}_{-\gamma})$ valued function of $\lambda \in \overline{\mathbb{C}^+} \setminus \{0\}$ of class $C^{(\gamma-\frac{1}{2})_-}$. For $0 \leq j < \gamma - \frac{1}{2}$,

$$\|G^{(j)}(\lambda)\|_{B(\mathcal{H}_{\gamma},\mathcal{H}_{-\gamma})} \leq C_{j\gamma} |\lambda|^{-1}, \quad |\lambda| \geq 1.$$ (3.2)

(2) Assume $|V(x)| \leq C(x)^{-\delta}$ for some $\delta > 2$ and that $H$ is of generic type. Let $1 < \gamma < \delta - 1$. Then $G(\lambda)$ is a $B_{\infty}(\mathcal{H}_{\gamma}, \mathcal{H}_{-\gamma})$ valued function of $\lambda \in \overline{\mathbb{C}^+}$ of class $C^{(\gamma-1)_-}$.

Using the boundary values of the resolvents on the real line, wave operators may be written in the following form (see [10]):

$$W_\pm u = u - \frac{1}{\pi i} \int_0^\infty G(\mp \lambda)V(G_0(\lambda) - G_0(-\lambda)) \lambda ud\lambda.$$ (3.3)

In what follows, we shall deal with $W_-$ only and we denote it by $W$ for brevity.
3.1 Born terms

If we formally expand the second resolvent equation into the series

\[ G(\lambda)V = (1 + G_0(\lambda)V)^{-1}G_0(\lambda)V = \sum_{n=1}^{\infty}(-1)^{n-1}(G_0(\lambda)V)^n \]

and substitute the right side for \( G(\lambda)V \) in the stationary formula (3.3), then we have the formal expansion of \( W \):

\[ W = 1 - \Omega_1 + \Omega_2 - \cdots \]  (3.4)

where for \( n = 1, 2, \ldots \),

\[ \Omega_n u = \frac{1}{\pi i} \int_0^\infty (G_0(\lambda)V)^n(G_0(\lambda) - G_0(-\lambda))u\lambda d\lambda. \]

This is called the Born expansion of the wave operator, the sum

\[ I - \Omega_1 + \cdots + (-1)^n\Omega_n \]

the \( n \)-th Born approximation of \( W_- \) and the individual \( \Omega_n \) the \( n \)-th Born term. The Born terms \( \Omega_n \) may be computed more or less explicitly and they can be expressed as superpositions of one dimensional convolution operators: We write \( \Sigma \) for the \( m - 1 \) dimensional unit sphere. Define the function \( K_{n}(t, \ldots, t_{n}, \omega, \ldots, \omega_{n}) \) of \( t_{1}, \ldots, t_{n} \in R \) and \( \omega_1, \ldots, \omega_n \in \Sigma \) by

\[ K_n(t, \ldots, t_n, \omega, \ldots, \omega_n) = C^n \int_{R_1^+} e^{i(t_1s_1+\cdots+t_n s_n)/2}(s_1 \cdots s_n)^{m-2} \prod_{j=1}^{n} \hat{V}(s_j \omega_j - s_{j-1} \omega_{j-1})ds_1 \cdots ds_n \]  (3.5)

where \( s_0 = 0, R_+ = (0, \infty) \) and \( C \) is an absolute constant. Then \( \Omega_n u(x) \) may be written in the form

\[ \int_{R_1^+ \times I} \left( \int_{\Sigma \times \Sigma} K_n(t, \ldots, t_n, \omega, \ldots, \omega_n)f(\overline{x} + \rho)d\omega_1 \ldots d\omega_n \right) dt_1 \cdots dt_n \]  (3.6)

where \( I = (2x \cdot \omega_n, \infty) \) is the range of integration with respect to \( t_n, \overline{x} = x - 2(\omega_n, x)\omega_n \) is the reflection of \( x \) along the \( \omega_n \) axis and \( \rho = t_1 \omega_1 + \cdots + t_n \omega_n \).

We define \( m_* = (m-1)/(m-2) \) for \( m \geq 3 \). If \( m \geq 3 \), we have with \( \sigma > 1/m_* \) that

\[ \|K_1\|_{L^1(R \times \Sigma)} \leq C\|\mathcal{F}(\langle x \rangle^{\sigma}V)\|_{L^{m_*}(R^m)}^{\frac{1}{m_*}}, \]  (3.7)

\[ \|K_n\|_{L^1(R^n \times \Sigma^n)} \leq C^n\|\mathcal{F}(\langle x \rangle^{2\sigma}V)\|_{L^{m_*}(R^m)}^{\frac{1}{m_*}}, \quad n \geq 2, \]  (3.8)

(see [15], page 569) and we obtain the following lemma.
Lemma 3.3. Let $m \geq 3$ and $\sigma > 1/m_\ast$. Then, there exists a constant $C > 0$ such that for any $1 \leq p \leq \infty$
\[
\|\Omega_1 u\|_p \leq C \|\mathcal{F}(\langle x \rangle^\sigma V)\|_{L^{m_\ast}(\mathbb{R}^m)} \| u \|_p,
\]
(3.9)
\[
\|\Omega_n u\|_p \leq C^n \|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m}(\mathbb{R}^m)}^n \| u \|_p, \quad n = 2, \ldots
\]
(3.10)

It follows that the series (3.4) converges in the operator norm of $B(L^p)$ for any $1 \leq p \leq \infty$ if $\|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m_\ast}(\mathbb{R}^m)}$ is sufficiently small and we obtain the following theorem.

Theorem 3.4. Suppose $m \geq 3$ and $V$ satisfies $\mathcal{F}(\langle x \rangle^{2\sigma} V) \in L^{m_\ast}(\mathbb{R}^m)$ for some $\sigma > 1/m_\ast$. Then, there exists a constant $C > 0$ such that $W_\pm$ are bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ provided that $\|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m}(\mathbb{R}^m)} < C$.

Note that that $H$ is of generic type if $\|\mathcal{F}(\langle x \rangle^{2\sigma} V)\|_{L^{m_\ast}(\mathbb{R}^m)}$ is sufficiently small. We remark that the condition $\mathcal{F}(\langle x \rangle^\sigma V) \in L^{m_\ast}(\mathbb{R}^m)$ requires some smoothness of $V$ if the dimension $m$ becomes larger. Recall that a certain smoothness condition on $V$ is necessary for $W_\pm$ to be bounded in $L^p$ for all $1 \leq p \leq \infty$ by virtue of the counter-example of Golberg-Vissan ([6]) for the dispersive estimates for dimensions $m \geq 4$.

In dimension $m = 2$, the factor $(s_1 \ldots s_n)^{m-2}$ is missing from (3.5) and it is evident that estimates (3.7) nor (3.8) do not hold. Nonetheless, we have the following result.

Lemma 3.5. Let $m = 2$. Then, for any $s > 1$ and $1 < p < \infty$, we have
\[
\|\Omega_1 u\|_p \leq C_{ps} \|\langle x \rangle^s V\|_2 \| u \|_p.
\]
If $\tilde{\chi}(\lambda) \in C^\infty(\mathbb{R})$ vanishes near $\lambda = 0$, then for any $s > 2$ and $1 < p < \infty$, we have
\[
\|\Omega_2 \tilde{\chi}(H_0) u\|_p \leq C_{ps} \|\langle x \rangle^s V\|_2^2 \| u \|_p.
\]

3.2 High energy estimate

We let $\chi \in C_0^\infty(\mathbb{R})$ and $\tilde{\chi} \in C^\infty(\mathbb{R})$ be such that
\[
\chi(\lambda) = 1 \text{ for } |\lambda| < \epsilon, \chi(\lambda) = 0 \text{ for } |\lambda| > 2\epsilon \text{ for some } \epsilon > 0
\]
and $\chi(\lambda^2) + \tilde{\chi}(\lambda)^2 = 1$ for all $\lambda \in \mathbb{R}$.

Then, the high energy part of the wave operator $W\tilde{\chi}(H_0)$ may be studied by a unified method for all $m \geq 2$ and we may show that $W$ is bounded in $B(L^p(\mathbb{R}^m))$ for all $1 \leq p \leq \infty$ when $m \geq 3$ and for $1 < p < \infty$ for $m = 2$:
Theorem 3.6. Let $V$ satisfy $|V(x)| \leq C\langle x \rangle^{-\delta}$ for some $\delta > m + 2$. Suppose, in addition, that $\mathcal{F}(x^\sigma V) \in L^{m*}(\mathbb{R}^m)$ if $m \geq 4$. Then $W_{\pm} \tilde{\chi}(H_0)$ is bounded in $B(L^p(\mathbb{R}^m))$ for all $1 \leq p \leq \infty$ when $m \geq 3$ and for $1 < p < \infty$ for $m = 2$.

We outline the proof. We write $\nu = (m - 2)/2$. Iterating the resolvent equation, we have

$$G(\lambda)V = \sum_{1}^{2n}(-1)^{j-1}(G_{0}(\lambda)V)^{j} + G_{0}(\lambda)N_{n}(\lambda)$$

where $N_{n}(\lambda) = (VG_{0}(\lambda))^{n-1}VG(\lambda)V(G_{0}(\lambda)V)^{n}$. If we substitute this for $G(\lambda)V$ in the stationary formula (3.3), we obtain

$$W\tilde{\chi}(H_0)^2 = \tilde{\chi}(H_0)^2 + \sum_{j=1}^{2n}(-1)^{j}\Omega_{j}\tilde{\chi}(H_0)^2 - \tilde{\Omega}_{2n+1},$$

(3.11)

$$\tilde{\Omega}_{2n+1} = \frac{1}{i\pi} \int_{0}^{\infty} G_{0}(\lambda)N_{n}(G_{0}(\lambda) - G_{0}(-\lambda))\tilde{\Psi}(\lambda)d\lambda,$$

(3.12)

where $\tilde{\Psi}(\lambda) = \lambda\overline{\chi}(\lambda^{2})^{2}$. The operators $\tilde{\chi}(H_0)$ and $\Omega_{1}\tilde{\chi}(H_0)^2, \ldots, \Omega_{2n}\tilde{\chi}(H_0)^2$ are bounded in $L^p(\mathbb{R}^m)$ for any $1 \leq p \leq \infty$ if $m \geq 3$ and for $1 < p < \infty$ if $m = 2$ by virtue of Lemma 3.3 and Lemma 3.5, since $\tilde{\chi}(H_0)$ is clearly bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ and $m \geq 2$. We then show that, for sufficiently large $n$, $\tilde{\Omega}_{2n+1}$ is also bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ and $m \geq 2$ by showing that its integral kernel

$$\tilde{\Omega}_{2n+1}(x, y) = \frac{1}{\pi i} \int_{0}^{\infty} \langle N_{n}(\lambda)(G_{0}(\lambda) - G_{0}(-\lambda))\delta_{y}, G_{0}(-\lambda)\delta_{x}\rangle\lambda\Psi^{2}(\lambda^{2})d\lambda,$$

where $\delta_{a} = \delta(x-a)$ is the unit mass at the point $x = a$, satisfies the estimate that

$$\sup_{x \in \mathbb{R}^m} \int |\tilde{\Omega}_{2n+1}(x, y)|dy < \infty \quad \text{and} \quad \sup_{y \in \mathbb{R}^m} \int |\tilde{\Omega}_{2n+1}(x, y)|dx < \infty.$$

(3.13)

It is a result of Schur's lemma that estimates (3.13) imply that $\tilde{\Omega}_{2n+1}$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$. Note that $[G_{0}(\lambda)\delta_{y}](x) = G_{0}(\lambda, x-y)$ is the integral kernel of $G_{0}(\lambda)$ and $G_{0}(\lambda, x)$ is given by

$$G_{0}(\lambda, x) = \frac{e^{i\lambda|x|}}{2(2\pi)^{\nu+\frac{1}{2}}\Gamma(\nu + \frac{1}{2})|x|^{m-2}} \int_{0}^{\infty} e^{-t}t^{\nu-\frac{1}{2}}\left(\frac{t}{2} - i\lambda|x|\right)^{\nu-\frac{1}{2}}dt.$$
As a slight modification of the argument is necessary for the case \( m = 2 \), we restrict ourselves to the case \( m \geq 3 \) and, for definiteness, we assume \( m \) is even in what follows in this subsection. We define
\[
\tilde{G}_0(\lambda, z, x) = e^{-i\lambda|x|}G_0(\lambda, x - z)
\]
and
\[
T_{\pm}(\lambda, x, y) = \langle N_n(\lambda)\tilde{G}_0(\pm\lambda, \cdot, y), \tilde{G}_0(-\lambda, \cdot, x)\rangle \tag{3.15}
\]
so that
\[
\tilde{\Omega}_{2n+1}(x, y) = \frac{1}{\pi i} \int_0^\infty (e^{i\lambda(|x|+|y|)}T_+(\lambda, x, y) - e^{i\lambda(|x|-|y|)}T_-(\lambda, x, y)) \tilde{\Psi}(\lambda) d\lambda. \tag{3.16}
\]
We may compute derivatives \( \tilde{G}_0^{(j)}(\lambda, z, x) \) with respect to \( \lambda \) using Leibniz's formula. If we set \( \psi(z, x) = |x-z|-|x| \), they are linear combinations over \((\alpha, \beta)\) such that \( \alpha + \beta = j \) of
\[
\frac{e^{i\lambda\psi(z, x)}\psi(z, x)^\alpha}{|x-z|^{m-2-\beta}} \int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}\left(\frac{t}{2} - i\lambda|x-z|\right)^{\nu-\frac{1}{2}-\beta} dt.
\]
Since \( |\psi(z, x)|^\alpha \leq \langle x \rangle^j \) for \( 0 \leq \alpha \leq j \) and
\[
|z-x| \leq C_\epsilon \left|\frac{t}{2} - i\lambda|x-z|\right| \leq C_\epsilon(t + \lambda|x-z|)
\]
when \( |\lambda| \geq 1 \), we have for \( |\lambda| \geq \epsilon \)
\[
\left|\left(\frac{\partial}{\partial \lambda}\right)^j \tilde{G}_0(\lambda, z, x)\right| \leq C_j \left(\frac{\langle x \rangle^j}{|x-z|^{m-2}} + \frac{\lambda^{\frac{m-3}{2}}\langle x \rangle^j}{|x-z|^{\frac{m-1}{2}}}\right). \tag{3.17}
\]
for \( j = 0, 1, 2, \ldots \).

Note that \( \tilde{G}_0(\lambda, z, x) \sim C|x-z|^{2-m} \) near \( z = x \) and \( \tilde{G}_0(\lambda, z, x) \notin L^2_{\text{loc}}(\mathbb{R}^m_z) \) for a fixed \( x \) if \( m \geq 4 \). However, the LAP (3.1) implies
\[
\|\langle x \rangle^{-\gamma-j}G_0^{(j)}(\lambda)\langle x \rangle^{-\gamma-j}\|_{B(H^s, H^{s+2})} \leq C_{sj\gamma}|\lambda|, \quad |\lambda| \geq \epsilon \tag{3.18}
\]
for any \( \gamma > 1/2 \), \( s \in \mathbb{R} \) and \( j = 0, 1, \ldots \) and \( k \) times application of \( G_0(\lambda)V \) to \( \tilde{G}_0(\lambda, \cdot, x) \), \( k > (m-2)/2 \), makes it into a function in \( L^2_{-\gamma}(\mathbb{R}^m_z) \) for any \( \gamma > 1/2 \). Thus, if we take \( n = k > (m-2)/2 \), \( T_{\pm}(\lambda, x, y) \) are well defined continuous functions of \((x, y)\) which are \((m+2)/2\) times continuously differentiable with respect to \( \lambda \). This, however, produces the increasing factor \( \lambda^k \) by virtue.
of the increase of the norm of (3.18). We, therefore, take \( n \) larger so that \( n > m \) and use the fact (3.1) that \( \| x \rangle^{-\gamma-j}G_0^{(j)}(\lambda)\langle x \rangle^{-\gamma-j}\|_{B(L^2,L^2)} \leq C|\lambda|^{-1} \) decays as \( \lambda \to \pm \infty \). Then, the decay property of extra factors \((G_0^{(j)}(\lambda)V)^{n-k}\) cancels this increasing factor and makes \( T_{\pm}(\lambda, x, y) \) integrable with respect to \( \lambda \). Using also the fact that \( \tilde{G}_0(\lambda, \cdot, x) \sim |x|^{-\frac{m-1}{2}} \) as \( |x| \to \infty \), we in this way obtain the following estimate:

**Lemma 3.7.** Let \( 0 \leq s \leq \frac{m+2}{2} \). We have

\[
\left| \left( \frac{\partial}{\partial \lambda} \right)^s T_{\pm}(\lambda, x, y) \right| \leq C_{ns} \lambda^{-3} \langle x \rangle^{-\frac{m-1}{2}} \langle y \rangle^{-\frac{m-1}{2}} \quad (3.19)
\]

To obtain the desired estimate for \( \tilde{\Omega}_{2n+1}(x, y) \), we apply integration by parts \( 0 \leq s \leq (m+2)/2 \) times with respect to the variable \( \lambda \) in (3.16):

\[
\int_0^\infty e^{i\lambda(|x|\pm|y|)}T_{\pm}(\lambda, x, y)\tilde{\Psi}(\lambda)d\lambda = \frac{1}{(|x|\pm|y|)^s} \int_0^\infty e^{i\lambda(|x|\pm|y|)} \left( \frac{\partial}{\partial \lambda} \right)^s \left( T_{\pm}(\lambda, x, y)\tilde{\Psi}(\lambda) \right) d\lambda
\]

and estimate the right hand side by using (3.19). We obtain

\[
|\tilde{\Omega}_{n+1}(x, y)| \leq C \sum_{\pm} \langle |x|\pm|y| \rangle^{-\frac{m+2}{2}} \langle x \rangle^{-\frac{m-1}{2}} \langle y \rangle^{-\frac{m-1}{2}}.
\]

It is then an easy exercise to show that \( \tilde{\Omega}_{n+1}(x, y) \) satisfies the estimate (3.13).

### 3.3 Low energy estimate, generic case

By virtue of the intertwining property we have \( W_{\pm}\chi(H_0)^2 = \chi(H)W_{\pm}\chi(H_0) \) and, from (3.3), we may write the low energy part \( W_{\pm}\chi(H_0)^2 \) as the sum of \( \chi(H)\chi(H_0) \) and

\[
\Omega = \frac{i}{\pi} \int_0^\infty \chi(H)G_0(\lambda)V(1+G_0(\lambda)V)^{-1}(G_0(\lambda) - G_0(-\lambda))\chi(H_0)\lambda d\lambda.
\]

(3.20)

Here \( \chi(H_0) \) and \( \chi(H) \) both are integral operators of which the integral kernels satisfy for any \( N > 0 \)

\[
|\chi(H_0)(x, y)| \leq C_N \langle x - y \rangle^{-N}, \quad |\chi(H)(x, y)| \leq C_N \langle x - y \rangle^{-N} \quad (3.21)
\]
and are, a fortiori, bounded in $L^p(\mathbb{R}^m)$ (see [16]). If $H$ is of generic type and $m \geq 3$ is odd, then $(1 + G_0(\lambda)V)^{-1}$ has no singularities at $\lambda = 0$ and we may prove that $\Omega$ is bounded in $L^p(\mathbb{R}^m)$ for all $1 \leq p \leq \infty$ by proving that its integral kernel $\Omega(x, y)$ satisfies the estimate (3.13) by a method similar to the one used for the high energy part. The argument is simpler in the point that we do not have to expand $(1 + G_0(\lambda)V)^{-1}$ since the range of the integration with respect to $\lambda$ in (3.20) is compact and since the integral kernels of $G_0(\lambda)\chi(H_0)$ and $G_0(\lambda)\chi(H)$ have no singularities at the diagonal set by virtue of (3.21). It is, however, more complicated than in the high energy case in that the integral kernels of

$$
\frac{i}{\pi} \int_0^\infty \chi(H)G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}G_0(\pm \lambda)\chi(H_0) \lambda d\lambda,
$$
do not separately satisfy the estimate (3.13) but only their difference does.

If $H$ is of generic type and $m$ is even, then $(1 + G_0(\lambda)V)^{-1}$ or its derivatives contain logarithmic singularities at $\lambda = 0$ which are stronger when the dimensions are lower. Thus, the analysis becomes more involved than the odd case particularly when $m = 2$ and $m = 4$. However, basically the idea as in the odd dimensional case works and we obtain the following theorem. We write $B(x, 1) = \{y \in \mathbb{R}^m : |y - x| < 1\}$.

**Theorem 3.8.** Suppose that $H$ is of generic type:

1. Let $m = 2$. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-6-\varepsilon}$ for some $\varepsilon > 0$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 < p < \infty$.

2. Let $m = 3$. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-5-\varepsilon}$ for some $\varepsilon > 0$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 \leq p \leq \infty$.

3. Let $m = 4$. Suppose that $V$ satisfies for some $q > 2$

$$
\|V\|_{L^q(B(x, 1))} + \|\nabla V\|_{L^q(B(x, 1))} \leq C\langle x \rangle^{-7-\varepsilon}
$$
for some $\varepsilon > 0$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 \leq p \leq \infty$.

4. Let $m \geq 5$. Suppose that $V$ satisfies $|V(x)| \leq C\langle x \rangle^{-m-2-\varepsilon}$ for some $\varepsilon > 0$ in addition to $\mathcal{F}(\langle x \rangle^{2\sigma}V) \in L^{m_*}(\mathbb{R}^m)$ for some $\sigma > 1/m_*$. Then, $W_{\pm}$ are bounded in $L^p$ for all $1 \leq p \leq \infty$.

**Remark 3.9.** When $m = 2$, at the end point, the same remark as in the one dimension applies: We believe $W_{\pm}$ are not bounded in $L^1$ nor in $L^\infty$ at the end point and they are bounded from Hardy space $H^1$ into $L^1$ and $L^\infty$ to BMO. However, we have no proofs.
3.4 Low energy estimate, exceptional case

We assume $H$ is of exceptional type in this subsection. Then, $(1 + G_0(\lambda)V)^{-1}$ of (3.20) is not invertible at $\lambda = 0$ and it has singularities at $\lambda = 0$. As we have no result when $m = 2$ and only a partial result when $m = 4$ which we mention at the end of this subsection, we assume $m = 3$ or $m \geq 5$ before the statement of Theorem 3.12. We study the singularities of $(1 + G_0(\lambda)V)^{-1}$ as $\lambda \to 0$ by expanding $1 + G_0(\lambda)V$ with respect to $\lambda$ around $\lambda = 0$ and examining the structure of $1 + G_0(0)V$. The result is: If $m \geq 3$ is odd, we have

$$(1 + G_0(\lambda)V)^{-1} = \lambda^{-2}P_0V + \lambda^{-1}A_{-1} + 1 + A_0(\lambda)$$

where $A_{-1}$ is a finite rank operator involving 0 eigenfunctions and the resonance function and $A_0(\lambda)$ has no singularities; if $m \geq 6$ is even, then

$$(1 + G_0(\lambda)V)^{-1} = \frac{P_0V}{\lambda^2} + \sum_{j=0}^2 \sum_{k=1}^2 \lambda^j(\log \lambda)^k D_{jk} + I + A_0(\lambda), \quad (3.22)$$

where $D_{jk}$ are finite rank operators involving 0 eigenfunctions and $A_0(\lambda)$ has no singularities. We substitute this expression for $(1 + G_0(\lambda)V)^{-1}$ in (3.20). Then, the operator produced by $I + A_0(\lambda)$ may be treated as in the previous section for the case when $H$ is of generic type. The operators produced by singular terms may be treated by using the machinaries of harmonic analysis, the weighted inequalities for the Hilbert transform and the Hardy-Littlewood maximal functions, which is a little too complicated to explain here. In this way we obtain the following theorem. We refer the readers to [19] and [5] for the proof respectively for odd and even dimensional case.

**Theorem 3.10.** Suppose that $H$ is of exceptional type.

1. Let $m \geq 3$ be odd. Suppose that $V$ satisfies $|V(x)| \leq C\langle x\rangle^{-m-3-\epsilon}$ for some $\epsilon > 0$ and $\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbb{R}^m)$ in addition for some $\sigma > 1/m_*$. Then, $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ between $m/(m-2)$ and $m/2$.

2. Let $m \geq 6$ be even. Suppose that $V$ satisfies $|V(x)| \leq C\langle x\rangle^{-m-3-\epsilon}$ if $m \geq 8$, $|V(x)| \leq C\langle x\rangle^{-m-4-\epsilon}$ if $m = 6$ for some $\epsilon > 0$ and $\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbb{R}^m)$ for some $\sigma > 1/m_*$ in addition. Then, $W_{\pm}$ are bounded in $L^p(\mathbb{R}^m)$ for $m/(m-2) < p < m/2$.

**Remark 3.11.** When $H$ is of exceptional type, the $W_{\pm}$ are not bounded in $L^p(\mathbb{R}^m)$ if $p > m/2$ and $m \geq 5$, or if $p > 3$ and $m = 3$. This can be deduced from the results on the decay in time property of the propagator
$e^{-itH}P_{ac}$ in the weighted $L^2$ spaces [12, 7], or in $L^p$ spaces [4, 18]. We believe
the same is true for $p$'s on the other side of the interval given in (b), viz.
$1 \leq p \leq m/(m - 2)$ if $m \geq 5$ and $1 \leq p \leq 3/2$ if $m = 3$, but we have again no proofs.

In the case when $m = 2$ or $m = 4$, and if 0 is a resonance of $H$, then the
results of [12] and [7] mentioned above imply that the $W_{\pm}$ are not bounded
in $L^p(\mathbb{R}^m)$ for $p > 2$ and, though proof is missing, we believe that this is
the case for all $p$'s except $p = 2$. However, when $m = 4$ and if 0 is a pure
eigenvalue of $H$ and not a resonance, the $W_{\pm}$ are bounded in $L^p(\mathbb{R}^4)$ for
$4/3 < p < 4$:

**Theorem 3.12.** Let $|V(x)| + |\nabla V(x)| \leq C(x)^{-\delta}$ for some $\delta > 7$. Suppose
that 0 is an eigenvalue of $H$, but not a resonance. Then the $W_{\pm}$ extend to
bounded operators in the Sobolev spaces $W^{k,p}(\mathbb{R}^4)$ for any $0 \leq k \leq 2$ and
$4/3 < p < 4$:

$$\|W_{\pm}u\|_{W^{k,p}} \leq C_p\|u\|_{W^{k,p}}, \quad u \in W^{k,p}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4). \quad (3.23)$$

We do not explain the proof of this theorem and refer the reader to
the recent preprint [8]. Again, the results of [12, 7] imply that the $W_{\pm}$ are
unbounded in $L^p(\mathbb{R}^4)$ if $p > 4$ under the assumption of Theorem 3.12. We
believe that this is the case also for $1 \leq p < 4/3$, though we do not have proofs.

**References**

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