ON THE CROSSED BURNSIDE RINGS

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1. Bouc functors

1.1. Notation and Definition. Let \( G \) and \( H \) be finite groups. An \((H, G)\)-biset, or a biset shortly, is a set with a left \((H \times G^{op})\)-action, i.e., a set \( U \) with a left \( H \)-action and a right \( G \)-action which commute.

If \( K \) is another group, and \( V \) is a \((K, H)\)-biset, then the product \( V \times U \) by the right action of \( H \) given by \((v, u)h = (vh, h^{-1}u)\) for \( v \in V, u \in U, \) and \( h \in H \). The class of \((v, u)\) in \( V \times H U \) is denoted by \((v, H \ u)\). The set \( V \times H U \) is a \((K, G)\)-biset for the action given by

\[ k(v, H \ u)g = (kv, H \ ug) \]

for \( k \in K, g \in G, u \in U, \) and \( v \in V \).

Denote by \( \mathcal{C}_p \) the following category:

- The objects of \( \mathcal{C}_p \) are the finite \( p \)-groups.
- If \( P \) and \( Q \) are finite \( p \)-groups, then \( \text{Home}_p(P, Q) = B(Q \times P^{op}) \) is the Burnside group of finite \((Q, P)\)-bisets. An element of this group is called a virtual \((Q, P)\)-biset.
- The composition of morphisms is \( \mathbb{Z} \)-bilinear, and if \( P, Q, R \) are finite \( p \)-groups, if \( U \) is a finite \((Q, P)\)-biset, and \( V \) is a finite \((R, Q)\)-biset, then the composition of \((v, u)\) and \((w, t)\) is the \((v, w)\) in \( V \times Q U \). The identity morphism \( 1_P \) of the \( p \)-group \( P \) is the class of the set \( P \), with left and right action by multiplication.

Let \( \mathcal{F}_p \) denote the category of additive functors from \( \mathcal{C}_p \) to the category \( \mathbb{Z} \)-Mod of abelian groups. An object of \( \mathcal{F}_p \) is called a Bouc functor (defined over finite \( p \)-groups, with values in \( \mathbb{Z} \)-Mod) (see [Th06], [Bo06]).

1.2. Notation. The Bouc functor of Burnside group will denote by \( B \). The Bouc functor of rational representations will denote by \( R \). The \( G \)-poset of the family of all subgroups of a finite group \( G \) will denote by \( S(G) \). If \( X \) is a \( G \)-set, denote by \( G \setminus X \) a family of \( G \)-orbits, and by \( [G \setminus X] \) a set of representatives of \( G \setminus X \).

2. The Dade group

2.1. Some known Dade groups: The structure of \( D(P) \) is known for any 2-group \( P \) of normal 2-rank 1: when \( P \) is generalized quaternion, the result is due to Dade, and the other cases have been solved by Carlson and Thévenaz:

Theorem 2.2. (Dade [Da78a], Carlson-Thévenaz [CT00])

1. \( D(C_{2^n}) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \), and \( D(C_{2^n}) \cong (\mathbb{Z}/2\mathbb{Z})^{n} \), if \( n \geq 3 \).
2. \( D(D_{2^n}) \cong \mathbb{Z}2^{n-3} \).
3. \( D(SD_{2^n}) \cong \mathbb{Z}2^{n-4} \oplus \mathbb{Z}/2\mathbb{Z} \).
4. \( D(Q_{2^n}) \cong \mathbb{Z}2^{n-5} \oplus \mathbb{Z}/4\mathbb{Z} \), for \( n \geq 4 \).
5. \( D(Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), if the ground field contains all cubic roots of unity, and \( D(Q_8) \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \) otherwise.

The ingredient of the present note is

Theorem 2.3. (Bouc-Thévenaz [BT00] Theorem 10.4) There is an exact sequence of functors

\[ 0 \rightarrow QD \xrightarrow{\alpha} QB \xrightarrow{\epsilon} QR \rightarrow 0 \]

where \( \epsilon(P) : QB(P) \rightarrow QR \) is the morphism mapping a \( P \)-set to the corresponding permutation module over \( Q \).
We could determine the difference \( \text{rank} B^c(P) - \text{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) \) by using a result as follows:

**Theorem 2.4.** (Bouc-Thévenaz [BT00] Theorem A) The torsion-free rank of the Dade group \( \mathcal{D}(P) \) is equal to the number of conjugacy classes of non-cyclic subgroups of \( P \).

3. **The Crossed Burnside Ring and the Rational Representations of Drinfel'd Double**

3.1. **Definition.** Let \( M \) be one of the Bouc functors \( \mathbb{Q}D, \mathbb{Q}B \) and \( \mathbb{Q}R_{\mathbb{Q}} \). We use a construction of Dress for Mackey functors for obtaining a module from \( M \). Let \( P \) be a \( p \)-group. Now we set

\[
M(X) = \left( \bigoplus_{x \in X} M(P_x) \right)^P = \left\{ (m(x)) \in \bigoplus_{x \in X} M(P_x) \mid g(m(x)) = m(gx) \forall g \in P \right\},
\]

where \( P_x \) is the stabilizer of \( x \) in \( P \).

**Corollary 3.2.** Let \( P \) be a \( p \)-group and \( X \) a \( P \)-set. Then there is an exact sequence of \( \mathbb{Q} \)-vector spaces

\[
0 \longrightarrow \mathbb{Q}D(X) \xrightarrow{\alpha} \mathbb{Q}B(X) \xrightarrow{\epsilon} \mathbb{Q}R_{\mathbb{Q}}(\mathcal{D}(P)) \longrightarrow 0.
\]

3.3. **Notation.** We denote by \( B^c(P) \) the crossed Burnside ring of \( P \), i.e. the Grothendieck ring of the category of finite crossed \( P \)-sets over \( P^c \), for relations given by decomposition into disjoint union of crossed \( P \)-sets, the ring structure being induced by the product of crossed \( P \)-sets. Also we denote by \( R_{\mathbb{Q}}(\mathcal{D}(P)) \) the rational representation ring of the Drinfel'd double \( \mathcal{D}(P) = (\mathbb{Q}P)^* \otimes \mathbb{Q}P \) for the group algebra \( \mathbb{Q}P \).

**Corollary 3.4.** Let \( P \) be a \( p \)-group. Then there is an exact sequence of \( \mathbb{Q} \)-vector spaces

\[
0 \longrightarrow \mathbb{Q}D(P^c) \xrightarrow{\alpha} \mathbb{Q}B^c(P) \xrightarrow{\epsilon} \mathbb{Q}R_{\mathbb{Q}}(\mathcal{D}(P)) \longrightarrow 0.
\]

In particular, we have

\[
\text{rank} B^c(P) = \text{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) + \dim_{\mathbb{Q}} \mathbb{Q}D(P^c).
\]

**Corollary 3.5.** Let \( P \) be a \( p \)-group. Then the following numbers are equal:

1. \( \text{rank} B^c(P) \).
2. \( \text{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) + \sum_{g \in [P \setminus P^c]} \dim_{\mathbb{Q}} \mathbb{Q}D(C_P(g)) \).
3. \( \sum_{Q \in [P \setminus S(P)]} \frac{|C_P(Q)|}{|N_P(Q)|} \cdot |Q| \left( \sum_{x \in Q \setminus Q'} \frac{1}{|\prime x|} \right) \).
4. \( \sum_{Q \in [P \setminus S(P)]} |N_P(Q) \setminus C_P(Q)| \).
5. \( \sum_{g \in [P \setminus P^c]} |C_P(g) \setminus S(C_P(g))| \).

**Corollary 3.6.** Let \( P \) be a \( p \)-group. Then

\[
\text{rank} B^c(P) = \text{rank} R_{\mathbb{Q}}(\mathcal{D}(P)) + \sum_{g \in [P \setminus P^c]} |C_P(g) \setminus \mathcal{S}_{\text{non}}(C_P(g))|,
\]

where \( \mathcal{S}_{\text{non}}(C_P(g)) \) is the \( C_P(g) \)-poset of non-cyclic subgroups of \( C_P(g) \) with \( C_P(g) \)-action defined by conjugation.

**Corollary 3.7.** Let \( P \) be a cyclic \( p \)-group. Then

\[
\text{rank} B^c(P) = \text{rank} R_{\mathbb{Q}}(\mathcal{D}(P)).
\]
3.8. Some small 2-groups. We summarize basic facts on the structure of the centralizers of the representative of a conjugacy class of dihedral, semi-dihedral and generalized quaternion 2-groups (see, for instance, III.17 of [Er90]). In the rest of the paper, we always denote by $z$ the central elements of order 2 of the group considered. Suppose that

$$D_{2^n} = \langle x, y| x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1} \rangle$$

is a dihedral group of order $2^n$ ($n \geq 2$). Then the centralizers of 1 and $z$ are $D_{2^n}$. The centralizers of $y$ and $xy$ are Klein four groups. The centralizers of the representative of the other $2^{n-2} - 1$ conjugacy classes are cyclic subgroups ($n \geq 3$). Suppose that

$$SD_{2^n} = \langle x, y| x^{2^{n-1}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \rangle$$

is a semi-dihedral group of order $2^n$ ($n \geq 4$). Then the centralizers of 1 and $z$ are $SD_{2^n}$. The centralizer of $y$ is a Klein four group. The centralizers of the representative of the other $2^{n-2}$ conjugacy classes are cyclic subgroups. Suppose that

$$Q_{2^n} = \langle x, y| x^{2^{n-2}} = y^2 = 1, y^{-1}xy = x^{-1+2^{n-2}} \rangle$$

is a generalized quaternion group of order $2^n$ ($n \geq 3$). Then the centralizers of 1 and $z$ are $Q_{2^n}$. The centralizers of the representative of the other $2^{n-2} + 1$ conjugacy classes are cyclic subgroups.

Corollary 3.9. Let $P$ be a dihedral group $D_{2^n}$ of order $2^n$ ($n \geq 2$). Then

$$\text{rank} B^c(P) - \text{rank} R_Q(D(P)) = 4n - 4.$$  

Corollary 3.10. Let $P$ be a semi-dihedral group $SD_{2^n}$ of order $2^n$ ($n \geq 4$). Then

$$\text{rank} B^c(P) - \text{rank} R_Q(D(P)) = 4n - 7.$$  

Corollary 3.11. Let $P$ be a generalized quaternion group $Q_{2^n}$ of order $2^n$ ($n \geq 3$). Then

$$\text{rank} B^c(P) - \text{rank} R_Q(D(P)) = 4n - 10.$$  

REFERENCES