数理解析研究所講究録
第 1564 巻 2007 年 76-84

Fixed point subalgebras of lattice vertex operator algebras by an automorphism of order three

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1 Introduction

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra. Thus

$$V=\bigoplus_{m\in\mathbb{Z}}V_{(m)}$$

is a $\mathbb{Z}$-graded vector space over $\mathbb{C}$ and

$$\begin{aligned} Y(\,\cdot\,,z):V&\rightarrow(\operatorname{End}V)[[z,z^{-1}]];\\ v&\mapsto Y(v,z)=\sum_{n\in\mathbb{Z}}v_{n}z^{-n-1}\end{aligned}$$

is a linear map which satisfies a set of axioms. Each v_n is a linear endomorphism of V. For $v\in V$, $Y(v,z)=\sum_{n\in\mathbb{Z}}v_{n}z^{-n-1}$ is called the vertex operator associated with v. The subspace $V_{(m)}$ is called a homogeneous subspace of weight m and every element in $V_{(m)}$ is said to be of weight m. For $v\in V_{(m)}$, we denote its weight by $\operatorname{wt}v$. The generating function

$$\operatorname{ch}V=\sum_{m\in\mathbb{Z}}(\dim V_{(m)})q^{m}$$

of the dimension of each homogeneous subspace is called the character of V.

There are two distinguished elements $\mathbf{1}$ and ω. The element $\mathbf{1}$ is of weight 0 and it is called the vacuum vector. It plays like the unity. In fact, $Y(\mathbf{1},z)=1$, that is, $\mathbf{1}_{-1}=\mathrm{id}_{V}$ and $\mathbf{1}_{n}=0$ if $n\neq-1$. Another important property of $\mathbf{1}$ is the creation property: $v_{-1}\mathbf{1}=v$ for any $v\in V$. The element ω is called the Virasoro element. The operators $L(n)=\omega_{n+1}$, $n\in\mathbb{Z}$ satisfy the Virasoro relation

$$[L(m),L(n)]=(m-n)L(m+n)+\frac{1}{12}(m^{3}-m)\delta_{m+n,0}c,$$

where c is a constant called the central charge. The homogeneous subspace $V_{(m)}$ of weight m is the eigenspace for the operator $L(0)$ with eigenvalue m.

An automorphism g of the vertex operator algebra V is a linear automorphism of V such that $g\omega = \omega$ and $g(u_n v) = (gu)_n(gv)$ for $u, v \in V$, $n \in \mathbb{Z}$. The set $\mathrm{Aut}\, V$ of all automorphisms of V becomes a group under composition. Let g be an automorphism of the vertex operator algebra V. Then g leaves the weight m subspace $V_{(m)}$ invariant. Moreover, the space $V^g = \{v \in V \mid gv = v\}$ of fixed points is a subalgebra, which is called an orbifold.

Orbifold is an important theme in the theory of vertex operator algebras. In fact, many interesting examples of orbifolds are known. On the other hand, it is difficult to study them. There are several reasons. One is that V^g is in general more complicated than the original vertex operator algebra V. Another reason is that only a few general theorems such as quantum Galois theory [8, 11] and the theory of g-twisted modules [9, 15] have been established so far (see also [7]).

Now, suppose V is well understood and g is given explicitly. We want to know (1) various properties of the vertex operator algebra V^g, and (2) the representation theory of V^g, namely, the classification of irreducible modules and the determination of fusion rules. There is a well known conjecture.

Conjecture 1.1 *Assume that V is a rational and C_2-cofinite vertex operator algebra. Let g be an automorphism of V of finite order. Then*

(1) *V^g is rational and C_2-cofinite.*

(2) *Any irreducible V^g-module will appear in some irreducible V-module or some irreducible g^i-twisted V-module, $1 \leq i \leq |g| - 1$.*

In this article we will briefly survey recently obtained results concerning orbifolds of some lattice vertex operator algebras by an automorphism of order 3. For details, please refer to [18].

2 Examples of orbifold

In this section we review some known examples of orbifold. Let $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a positive definite even lattice. Frenkel, Lepowsky and Meurman [10] constructed a vertex operator algebra V_L associated with L. The vertex operator algebra V_L is known to be rational and C_2-cofinite. Any isometry σ of the lattice L can be lifted to an automorphism $\hat{\sigma}$ of the vertex operator algebra V_L.

The most basic example of such an isometry is the -1 isometry $\theta : L \longrightarrow L; \alpha \mapsto -\alpha$. There is a canonical lift $\hat{\theta}$ of θ such that $\hat{\theta}^2 = 1$ (cf. [10, (6.4.13), (10.3.12)]). Let

$$V_L^{\pm} = \{v \in V_L \mid \hat{\theta}v = \pm v\}.$$

Then $V_L^+ = V_L^{\hat{\theta}}$ is a simple vertex operator algebra and V_L^- is an irreducible module for V_L^+. The orbifold V_L^+ of V_L by the involution $\hat{\theta}$ has been studied extensively (cf.

[1, 2, 3]). The rationality and the C_2-cofiniteness of V_L^+ were established. Moreover, the classification of irreducible modules and the determination of fusion rules were obtained. In particular, Conjecture 1.1 is true in this case.

In the case where the lattice L is the Leech lattice Λ, the vertex operator algebra V_Λ is holomorphic, that is, V_Λ is simple and rational, and furthermore V_Λ has a unique irreducible module. This is because Λ is a unimodular lattice. There is a unique irreducible $\hat{\theta}$-twisted V_Λ-module V_Λ^T. The involution $\hat{\theta}$ acts on V_Λ^T. Actually, the action of $\hat{\theta}$ on V_Λ^T is not canonical. Here we adopt the notation so that our action $\hat{\theta}$ on V_Λ^T is negative of the action described in [10]. We set

$$V_\Lambda^{T,\pm} = \{v \in V_\Lambda^T \mid \hat{\theta}v = \pm v\}.$$

Then $V_\Lambda^\pm$, $V_\Lambda^{T,\pm}$ form a complete set of representatives of equivalence classes of irreducible V_Λ^+-modules. The construction of the moonshine vertex operator algebra $V^\natural$ by Frenkel, Lepowsky and Meurman [10] was based on the orbifold V_Λ^+. In fact, $V^\natural$ was defined to be a direct sum of V_Λ^+ and its irreducible module $V_\Lambda^{T,-}$. It was shown that the vector space $V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,-}$ has a vertex operator algebra structure and its automorphism group $\mathrm{Aut}\, V^\natural$ is isomorphic to the Monster $\mathbb{M}$. One of the remarkable properties of $V^\natural$ is that the character $\mathrm{ch}\, V^\natural$ is related to the modular function $j(\tau)$.

Theorem 2.1 [10, Theorems 12.3.1, 12.3.4]
(1) $V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,-}$ *has a vertex operator algebra structure.*
(2) $\mathrm{ch}\, V^\natural = (j(\tau) - 744)q = 1 + 0 \cdot q + 196884q^2 + 21493760q^3 + \cdots$.
(3) $\mathrm{Aut}\, V^\natural \cong \mathbb{M}$.

The commutative non-associative algebra of the 196884 dimensional weight 2 space $V^\natural_{(2)}$, called the Griess algebra, plays a crucial role for the identification of $\mathrm{Aut}\, V^\natural$ with the Monster $\mathbb{M}$. In fact, the automorphism group of the Griess algebra is identical with the Monster.

Apart from the above mentioned examples, only a few more examples of orbifold have been studied in detail. In fact, some orbifolds of special type of lattice vertex operator algebras by an automorphism of order 3 can be found in [17, 18] (see also [5]). We remark that even for an automorphism g of order 2, there is no general results concerning Conjecture 1.1.

3 Main results

Let p be an odd prime such that $p-1$ divides 24, that is $p = 3, 5, 7$, or 13. Then there is a fixed-point-free isometry τ of the Leech lattice Λ of order p. It is expected that an analogous construction of the moonshine vertex operator algebra $V^\natural$ may be possible by using a lift $\hat{\tau} \in \mathrm{Aut}\, V_\Lambda$ of τ in place of the canonical lift $\hat{\theta}$ of the -1 isometry θ (cf. [10, Introduction]).

Now we consider the case $p = 3$. Thus τ is a fixed-point-free isometry of Λ of order 3. The first step should be the study of the orbifold $V_\Lambda^{\hat{\tau}} = \{v \in V_\Lambda \mid \hat{\tau}v = v\}$ of V_Λ by $\hat{\tau}$. By [7] V_Λ has a unique irreducible $\hat{\tau}^i$-twisted module for $i = 1, 2$. Let $V_\Lambda^{T_i}$ be the irreducible $\hat{\tau}^i$-twisted V_Λ-module obtained by the method of [6, 14]. Set $U(\varepsilon) = \{u \in U \mid \hat{\tau}u = \xi^\varepsilon u\}$ for $U = V_\Lambda, V_\Lambda^{T_1}, V_\Lambda^{T_2}$ and $\varepsilon = 0, 1, 2$, where $\xi = \exp(2\pi\sqrt{-1}/3)$. Thus $V_\Lambda(0) = V_\Lambda^{\hat{\tau}}$. Our main theorem is as follows.

Theorem 3.1 [18]

(1) *$V_\Lambda^{\hat{\tau}}$ is rational and C_2-cofinite.*

(2) *There are exactly nine equivalence classes of irreducible $V_\Lambda^{\hat{\tau}}$-modules, which are represented by $V_\Lambda(\varepsilon)$, $V_\Lambda^{T_1}(\varepsilon)$, $V_\Lambda^{T_2}(\varepsilon)$, $\varepsilon = 0, 1, 2$.*

We will sketch the proof of the main theorem. We start with a $\sqrt{2}A_2$ lattice L. Thus $L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ with $\langle \beta_i, \beta_i \rangle = 4$ and $\langle \beta_1, \beta_2 \rangle = -2$. Let $\beta_0 = -(\beta_1 + \beta_2)$. Then $\langle \beta_i, \beta_i \rangle = 4$ and $\langle \beta_i, \beta_j \rangle = -2$ if $i \neq j$ for $i, j \in \{0, 1, 2\}$. Consider an isometry τ of L induced by the permutation

$$\tau : \beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1.$$

Note that τ is fixed-point-free and of order 3 on L.

Let $L^\perp = \{\alpha \in \mathbb{Q} \otimes L \mid \langle \alpha, L \rangle \subset \mathbb{Z}\}$ be the dual lattice of L. We extend τ to an isometry of $L^\perp$. There are twelve cosets of L in $L^\perp$. In fact, $L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and the twelve cosets are parameterized by $\mathcal{K}$ and $\mathbb{Z}_3$, where $\mathcal{K} = \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is Klein's four-group. For each $x \in \mathcal{K}$ we assign $\beta(x) \in L^\perp$ by $\beta(0) = 0$, $\beta(a) = \beta_2/2$, $\beta(b) = \beta_0/2$, and $\beta(c) = \beta_1/2$. Set

$$L^{(x,j)} = \beta(x) + \frac{j}{3}(-\beta_1 + \beta_2) + L.$$

Then $L^{(x,j)}$, $x \in \mathcal{K}$, $j \in \mathbb{Z}_3$ are the twelve cosets of L in $L^\perp$.

A $\mathcal{K}$ code of length ℓ is simply an additive subgroup of $\mathcal{K}^\ell$. For $x, y \in \mathcal{K}$, define

$$x \cdot y = \begin{cases} 1 & \text{if } x \neq y, x \neq 0, y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_\ell) \in \mathcal{K}^\ell$, let $\langle \lambda, \mu \rangle_\mathcal{K} = \sum_{i=1}^\ell \lambda_i \cdot \mu_i \in \mathbb{Z}_2$. For a $\mathcal{K}$-code C of length ℓ, we define its dual code by

$$C^\perp = \{\lambda \in \mathcal{K}^\ell \mid \langle \lambda, \mu \rangle_\mathcal{K} = 0 \text{ for all } \mu \in C\}.$$

A $\mathcal{K}$-code C is said to be self-orthogonal if $C \subset C^\perp$ and self-dual if $C = C^\perp$. For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{K}^\ell$, its support is defined to be $\mathrm{supp}_\mathcal{K}(\lambda) = \{i \mid \lambda_i \neq 0\}$. The cardinality of $\mathrm{supp}_\mathcal{K}(\lambda)$ is called the weight of λ. We denote the weight of λ by $\mathrm{wt}_\mathcal{K}(\lambda)$. A $\mathcal{K}$-code C is said to be even if $\mathrm{wt}_\mathcal{K}(\lambda)$ is even for any $\lambda \in C$.

Define an action of τ on $\mathcal{K}$ by $\tau(0) = 0$, $\tau(a) = b$, $\tau(b) = c$, and $\tau(c) = a$. This action of τ on $\mathcal{K}$ is compatible with the isometry τ. Indeed, we have $\tau(L^{(x,j)}) = L^{(\tau(x),j)}$. We extend the action of τ to $\mathcal{K}^\ell$ componentwise so that $\tau\lambda = (\tau(\lambda_1), \ldots, \tau(\lambda_\ell))$.

Lemma 3.2 [13, Lemma 2.8] *Let C be a $\mathcal{K}$-code of length ℓ.*

(1) *If C is even, then C is self-orthogonal.*

(2) *If C is τ-invariant, then C is even if and only if C is self-orthogonal.*

A $\mathbb{Z}_3$-code of length ℓ is a subspace of the vector space $\mathbb{Z}_3^\ell$. For $\gamma = (\gamma_1, \ldots, \gamma_\ell)$, $\delta = (\delta_1, \ldots, \delta_\ell) \in \mathbb{Z}_3^\ell$, we consider the ordinary inner product $\langle \gamma, \delta \rangle_{\mathbb{Z}_3} = \sum_{i=1}^{\ell} \gamma_i \delta_i \in \mathbb{Z}_3$. The dual code $D^\perp$ of a $\mathbb{Z}_3$-code D is defined to be

$$D^\perp = \{\gamma \in \mathbb{Z}_3^\ell \mid \langle \gamma, \delta \rangle_{\mathbb{Z}_3} = 0 \text{ for all } \delta \in D\}.$$

Then D is said to be self-orthogonal if $D \subset D^\perp$ and self-dual if $D = D^\perp$.

For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{K}^\ell$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathbb{Z}_3^\ell$, let

$$L_{(\lambda,\gamma)} = L^{(\lambda_1,\gamma_1)} \oplus \cdots \oplus L^{(\lambda_\ell,\gamma_\ell)} \subset (L^\perp)^{\oplus \ell},$$

where $(L^\perp)^{\oplus\ell}$ denotes an orthogonal sum of ℓ copies of $L^\perp$. We extend the isometry τ of $L^\perp$ to an isometry of $(L^\perp)^{\oplus\ell}$ componentwise. For $P \subset \mathcal{K}^\ell$ and $Q \subset \mathbb{Z}_3^\ell$, set

$$L_{P\times Q} = \bigcup_{\lambda\in P, \gamma\in Q} L_{(\lambda,\gamma)}.$$

If C is a $\mathcal{K}$-code of length ℓ and D is a $\mathbb{Z}_3$-code of the same length, then $L_{C\times D}$ becomes an additive subgroup of $(L^\perp)^{\oplus\ell}$. However, $L_{C\times D}$ is not an integral lattice in general. Let $(L_{C\times D})^\perp = \{\alpha \in (\mathbb{Q} \otimes_{\mathbb{Z}} L)^{\oplus\ell} \mid \langle \alpha, L_{C\times D}\rangle \subset \mathbb{Z}\}$. The following two lemmas are easily verified.

Lemma 3.3 $(L_{C\times D})^\perp = L_{C^\perp\times D^\perp}$.

Lemma 3.4 (1) *If C is even and D is self-orthogonal, then $L_{C\times D}$ is an even lattice.*

(2) *If both C and D are self-dual, then $L_{C\times D}$ is a unimodular lattice.*

Suppose C is a τ-invariant even $\mathcal{K}$-code of length ℓ and D is a self-orthogonal $\mathbb{Z}_3$-code of the same length. Then $L_{C\times D}$ is a positive definite even lattice by Lemma 3.4. Moreover, τ induces an isometry of $L_{C\times D}$, for we are assuming that C is τ-invariant. Note that τ is fixed-point-free on $L_{C\times D}$.

There are various examples of $L_{C\times D}$. In the case $\ell = 12$, it is known that the Leech lattice Λ can be expressed in the form $L_{C\times D}$ for a τ-invariant self-dual $\mathcal{K}$-code and a self-dual $\mathbb{Z}_3$-code (cf. [12]). From now on we adopt the expression of $\Lambda = L_{C\times D}$ and fix such C and D.

We consider a sequence

$$L_{\{\mathbf{0}\}\times\{\mathbf{0}\}} = L^{\oplus 12} \subset L_{\{\mathbf{0}\}\times D} \subset \Lambda = L_{C\times D}$$

of sublattices, where $\mathbf{0}$ denotes the zero codeword. Correspondingly, we have a sequence

$$V_{L_{\{\mathbf{0}\}\times\{\mathbf{0}\}}} = V_L^{\otimes 12} \subset V_{L_{\{\mathbf{0}\}\times D}} \subset V_\Lambda = V_{L_{C\times D}}$$

of vertex operator subalgebras.

There is a natural lift $\hat{\tau} \in \mathrm{Aut}\, V_L$ of the isometry τ of L of order 3. We can extend it to an automorphism of $V_{L_{C\times D}}$ of order 3 whose restriction to $V_{L_{\{0\}\times\{0\}}} = V_L^{\otimes 12}$ is $(\hat{\tau},\ldots,\hat{\tau})$. For simplicity of notation, we denote the automorphism of $V_{L_{C\times D}}$ obtained in this way by the same symbol $\hat{\tau}$. Our main concern is the orbifold $V_{L_{C\times D}}^{\hat{\tau}}$ of $V_\Lambda = V_{L_{C\times D}}$ by $\hat{\tau}$. For this, we consider subalgebras which appear in the sequence

$$(V_L^{\hat{\tau}})^{\otimes 12} \subset V_{L_{\{0\}\times\{0\}}}^{\hat{\tau}} \subset V_{L_{\{0\}\times D}}^{\hat{\tau}} \subset V_{L_{C\times D}}^{\hat{\tau}}.$$

Indeed, we can analyze any module for $V_{L_{\{0\}\times\{0\}}}^{\hat{\tau}}$, $V_{L_{\{0\}\times D}}^{\hat{\tau}}$, and $V_{L_{C\times D}}^{\hat{\tau}}$ as a module for $(V_L^{\hat{\tau}})^{\otimes 12}$. In this process the knowledge about the vertex operator algebra $V_L^{\hat{\tau}}$ is indispensable. We quote some properties of $V_L^{\hat{\tau}}$ from [16, 17].

(1) $V_L^{\hat{\tau}}$ is rational and C_2-cofinite.

(2) $V_L^{\hat{\tau}}$ has exactly 30 inequivalent classes of irreducible modules. Their representatives can be described explicitly. Among them, twelve are contained in irreducible V_L-modules, while nine appear in irreducible $\hat{\tau}$-twisted V_L-modules and the remaining nine appear in irreducible $\hat{\tau}^2$-twisted V_L-modules.

(3) Fusion rules are known partially. Some of the irreducible $V_L^{\hat{\tau}}$-modules are simple currents, but some are not simple currents.

Let U be an irreducible $V_{L_{C\times D}}^{\hat{\tau}}$-module. Our argument is divided into three steps.

Step1: Since $V_L^{\hat{\tau}}$ is rational, $(V_L^{\hat{\tau}})^{\otimes 12}$ is also rational. Thus U is a direct sum of irreducible $(V_L^{\hat{\tau}})^{\otimes 12}$-modules.

Step2: An irreducible $(V_L^{\hat{\tau}})^{\otimes 12}$-module is a tensor product of some 12 irreducible $V_L^{\hat{\tau}}$-modules. Thus every irreducible direct summand in U of Step1 can be described as a tensor product of irreducible $V_L^{\hat{\tau}}$-modules.

Step3: Fusion rules among the irreducible $V_L^{\hat{\tau}}$-modules impose certain restrictions on the irreducible direct summands in U. Using these conditions we determine U.

Actually, we first classify irreducible modules for $V_{L_{\{0\}\times\{0\}}}^{\hat{\tau}}$ and $V_{L_{\{0\}\times D}}^{\hat{\tau}}$. When we discuss irreducible modules for these two vertex operator algebras, only simple current irreducible $V_L^{\hat{\tau}}$-modules are involved and the argument is relatively easy. However, for irreducible $V_{L_{C\times D}}^{\hat{\tau}}$-modules we need to deal with non-simple current extensions. In fact, this is the most difficult part in the proof of Theorem 3.1.

4 Further discussions

Recall the construction of $V^\natural$ by Frenkel, Lepowsky and Meurman [10]. The irreducible $\hat{\theta}$-twisted V_Λ-module V_Λ^T is a direct sum $V_\Lambda^{T,+} \oplus V_\Lambda^{T,-}$ of two irreducible V_Λ^+-modules $V_\Lambda^{T,+}$ and $V_\Lambda^{T,-}$. The weights of $V_\Lambda^{T,-}$ are integers, while those of $V_\Lambda^{T,+}$ are half integers. For the construction of $V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,-}$, the irreducible V_Λ^+-module $V_\Lambda^{T,-}$ of integral weights is used.

In our case the irreducible $\hat{\tau}^i$-twisted V_Λ-module $V_\Lambda^{T_i}$, $i = 1, 2$ is a direct sum of three irreducible V_Λ^+-modules $V_\Lambda^{T_i}(\varepsilon)$, $\varepsilon = 0, 1, 2$;

$$V_\Lambda^{T_i} = V_\Lambda^{T_i}(0) \oplus V_\Lambda^{T_i}(1) \oplus V_\Lambda^{T_i}(2), \quad i = 1, 2.$$

Among the six irreducible $V_\Lambda^{\hat{\tau}}$-modules $V_\Lambda^{T_i}(\varepsilon)$, $i = 1, 2$, $\varepsilon = 0, 1, 2$, only $V_\Lambda^{T_1}(1)$ and $V_\Lambda^{T_2}(2)$ have integral weights. Thus if one expect a similar construction as in [10], a prospective candidate should be a direct sum of $V_\Lambda(0)$, $V_\Lambda^{T_1}(1)$, and $V_\Lambda^{T_2}(2)$. In this context we have two conjectures.

Conjecture 4.1 *All the nine irreducible $V_\Lambda^{\hat{\tau}}$-modules $V_\Lambda(\varepsilon)$, $V_\Lambda^{T_1}(\varepsilon)$, $V_\Lambda^{T_2}(\varepsilon)$, $\varepsilon = 0, 1, 2$ are simple currents.*

Conjecture 4.2 *Let $W = V_\Lambda(0) \oplus V_\Lambda^{T_1}(1) \oplus V_\Lambda^{T_2}(2)$. Then W has a vertex operator structure and it is isomorphic to $V^\natural$.*

We look at the weight 2 subspace. Recall that the weight 2 subspace $V^\natural_{(2)}$ of $V^\natural$ is of 196884 dimension. As a module for the Monster $\mathbb{M}$, $V^\natural_{(2)}$ is divided into a direct sum of two irreducible modules, one corresponds to the principal character χ_1 and the other corresponds to the irreducible character χ_2 of degree 196883 in the ATLAS notation (cf. [4]). By abuse of notation we identify χ_i with its representation space, so that we may write $V^\natural_{(2)} = \chi_1 \oplus \chi_2$ symbolically.

The conjugacy classes of elements of order at most 3 in the Monster are as follows (cf. [4]).

$$\begin{array}{rl} \text{the unity}: & 1A, \\ \text{elements of order } 2: & 2A,\ 2B, \\ \text{elements of order } 3: & 3A,\ 3B,\ 3C. \end{array}$$

Now, we calculate the action of $\hat{\theta}$ on $V^\natural_{(2)}$. Since $V^\natural = V_\Lambda^+ \oplus V_\Lambda^{T,-}$, we have

$$\begin{array}{rcccc} V^\natural_{(2)} & = & (V_\Lambda^+)_{(2)} & \oplus & (V_\Lambda^{T,-})_{(2)}, \\ \hat{\theta}: & & 1 & & -1. \end{array}$$

Since $\dim(V_\Lambda^+)_{(2)} = 98580$ and $\dim(V_\Lambda^{T,-})_{(2)} = 98304$, the trace of the action of $\hat{\theta}$ on $V^\natural_{(2)}$ is

$$\mathrm{tr}_{V^\natural_{(2)}} \hat{\theta} = 98580 - 98304 = 276.$$

On the other hand, ATLAS [4] tells us the character values of $\chi_1 + \chi_2$ on the conjugacy classes $2A$ and $2B$. They are

$$\chi_1(2A) + \chi_2(2A) = 4372,$$
$$\chi_1(2B) + \chi_2(2B) = 276.$$

Hence we know that $\hat{\theta}$ corresponds to a $2B$ element of the Monster.

Next, we examine the action of $\hat{\tau}$ on the weight 2 subspace of $W = V_\Lambda(0) \oplus V_\Lambda^{T_1}(1) \oplus V_\Lambda^{T_2}(2)$. We have

$$\begin{array}{cccccc} W_{(2)} = & V_\Lambda(0)_{(2)} & \oplus & V_\Lambda^{T_1}(1)_{(2)} & \oplus & V_\Lambda^{T_2}(2)_{(2)}, \\ \hat{\tau}: & 1 & & \xi & & \xi^2. \end{array}$$

Since $\dim V_\Lambda(0)_{(2)} = 65664$, $\dim V_\Lambda^{T_1}(1)_{(2)} = 65610$, and $\dim V_\Lambda^{T_2}(2)_{(2)} = 65610$, the trace of the action of $\hat{\tau}$ on $W_{(2)}$ is

$$\mathrm{tr}_{W_{(2)}} \hat{\tau} = 65664 - 65610 = 54.$$

Moreover,

$$\begin{aligned} \chi_1(3A) + \chi_2(3A) &= 783, \\ \chi_1(3B) + \chi_2(3B) &= 54, \\ \chi_1(3C) + \chi_2(3C) &= 0 \end{aligned}$$

by [4]. Hence $\hat{\tau}$ should correspond to a $3B$ element if Conjecture 4.2 is true.

References

[1] T. Abe, G. Buhl and C. Dong, Rationality, regularity, and C_2-cofiniteness, *Trans. Amer. Math. Soc.* **356** (2004), 3391–3402.

[2] T. Abe and C. Dong, Classification of irreducible modules for the vertex operator algebra V_L^+: General case, *J. Algebra* **273** (2004), 657–685.

[3] T. Abe, C. Dong and H.S. Li, Fusion rules for the vertex operator algebra $M(1)$ and V_L^+, *Comm. Math. Phys.* **253** (2005), 171–219.

[4] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An atlas of finite groups*, Clarendon Press, Oxford, 1985.

[5] C. Dong, C.H. Lam, K. Tanabe, H. Yamada and K. Yokoyama, $\mathbb{Z}_3$ symmetry and W_3 algebra in lattice vertex operator algebras, *Pacific J. Math.* **215** (2004), 245–296.

[6] C. Dong and J. Lepowsky. The algebraic structure of relative twisted vertex operators, *J. Pure and Applied Algebra* **110** (1996), 259–295.

[7] C. Dong, H.S. Li and G. Mason, Modular-invariance of trace functions in orbifold theory and generalized moonshine, *Comm. Math. Phys.* **214** (2000), 1–56.

[8] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.* **86** (1997), 305–321.

[9] C. Dong and G. Yamskulna, Vertex operator algebras, generalized doubles and dual pairs, *Math. Z.* **241** (2002), 397–423.

[10] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. **134**, Academic Press, 1988.

[11] A. Hanaki, M. Miyamoto and D. Tambara, Quantum Galois theory for finite groups, *Duke Math. J.* **97** (1999), 541–544.

[12] K. Kitazume, C.H. Lam and H. Yamada, Decomposition of the moonshine vertex operator algebra as Virasoro modules, *J. Algebra* **226** (2000), 893–919.

[13] C.H. Lam and H. Yamada, $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes and vertex operator algebras, *J. Algebra* **224** (2000), 268–291.

[14] J. Lepowsky, Calculus of twisted vertex operators, *Proc. Natl. Acad. Sci. USA* **82** (1985), 8295–8299.

[15] M. Miyamoto and K. Tanabe, Uniform product of $A_{g,n}(V)$ for an orbifold model V and G-twisted Zhu algebra, *J. Algebra* **274** (2004), 80–96.

[16] K. Tanabe, On intertwining operators and finite automorphism groups of vertex operator algebras, *J. Algebra* **287** (2005), 174–198.

[17] K. Tanabe and H. Yamada, The fixed point subalgebra of a lattice vertex operator algebra by an automorphism of order three, to appear in *Pacific J. Math.*

[18] K. Tanabe and H. Yamada, Fixed point subalgebras of lattice vertex operator algebras by an automorphism of order three, preprint, arXiv:math.QA/0610013.