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On a Duality in White Noise Analysis

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Abstract

In White Noise Analysis (WNA), various random quantities are analyzed as Hida distributions ([1]). They are given by generalized functions of "white noise" defined as the "derivative" of the Brownian motion. For these Hida distributions, two kinds of products, "normal product" and "convolution product", are defined, respectively, in the use of "S-transform" and "T-transform". In our work (T. Hasebe, I. Ojima and H.S., in preparation) from the algebraic viewpoint of "duality", we have found such a remarkable property about these notions that the products have no "zero devisors" among Hida distributions. As the fact is nothing but a WNA version of Titchmarsh's theorem, it is expected to play fundamental roles in developing the "operational calculus in WNA" along the line of Mikusiński's version for differential equations. After surveying the necessary setting-up for dealing with the products, transforms and duality between them, we will show this fact in the present notes.

First we take up the following Gel'fand triple

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}),$$

and construct the (essentially unique) measure μ which satisfies,

$$\int_{S'(\mathbb{R})} d\mu[X] e^{i\langle X,f\rangle} = e^{-\frac{1}{2}(f,f)},$$

by using Bochner-Minlos theorem. (Here, $S(\mathbb{R})$ and $S'(\mathbb{R})$ denote, respectively, the space of Schwartz test functions and of Schwaltz distributions.) $(S'(\mathbb{R}), \mu)$ is called white noise.

Then a "higher Gel'fand triple" can be constructed as follows:

$$(S) \subset (L^2) := L^2(S'(\mathbb{R}), \mu) \subset (S)^*,$$

where (S) and $(S)^*$ are the spaces called, respectively, "the space of white noise test functionals" and "the space of generalized white noise functionals". Elements of $(S)^*$ are called Hida distributions.

(S) has many nice properties: for example, every element F of (S) has partial derivatives in (S),

$$D_h F[X] := \lim_{\varepsilon \to 0} \frac{F[X+\varepsilon h] - F[X]}{\varepsilon},$$

which is well defined even for $h \in S'(\mathbb{R})$. In particular, D_{δ_t} are denoted by ∂_t and called Hida derivatives. (Here δ_t denotes Dirac delta function at t.)

Two important transforms are defined:

$$\mathcal{S} : \Phi[X] \mapsto \int_{S'(\mathbb{R})} d\mu[X]\Phi[X+f] =: (\mathcal{S}\Phi)(f),$$
$$\mathcal{T} : \Phi[X] \mapsto \int_{S'(\mathbb{R})} d\mu[X]\Phi[X]e^{i\langle X,f\rangle} =: (\mathcal{T}\Phi)(f),$$

which are called S-transform and T-transform, respectively. Their domains can be extended to $(S)^*$.

A characterization theorem for $(S)^*$ in terms of S- and T-transform is formulated by Potthoff and Streit ([2]):

Theorem 1 *The following three statements are equivalent:*

1. *F is a U-functional, i.e. $F : S(\mathbb{R}) \longrightarrow \mathbb{C}$ such that*
   (i) For all $\xi, \eta \in S(\mathbb{R})$, the mapping $\lambda \mapsto F(\eta + \lambda\xi)$, $\lambda \in \mathbb{R}$, has an entire analytic extension, which is denoted as $F(\eta + z\xi)$, $z \in \mathbb{C}$.
   (ii) There exists a $p \in \mathbb{N}_0$, so that the entire function $z \mapsto F(z\xi)$ is of order less than 2, uniformly on the unit ball in $S_p(\mathbb{R})$; i.e., there exist $p \in \mathbb{N}_0$ and $C > 0$ so that for all $\xi \in S(\mathbb{R})$ with $|\xi|_{2,p} \leq 1$ and all sufficiently large $r > 0$,

$$\sup_{z\in\mathbb{C},|z|\leq r} |f(z\xi)| \leq e^{Cr^2}.$$

2. *$F(\cdot)$ is the S-transform of a (unique) Hida distribution.*
3. *$F(\cdot)$ is the T-transform of a (unique) Hida distribution.*

Using the theorem, it can be proved that U-functionals form an algebra under the usual product. As both transforms are injective, we can define two types of products on $(S)^*$

$$a : b := \mathcal{S}^{-1}(\mathcal{S}a \cdot \mathcal{S}b),$$
$$a * b := \mathcal{T}^{-1}(\mathcal{T}a \cdot \mathcal{T}b),$$

where $:$ and $*$ are called "normal product" and"convolution". They have $\mathbf{1}$ and δ_0 (white noise delta-functional) as their units, respectively. They are intertwined by Kuo's Fourier transform $\hat{}$ $:= \mathcal{S}^{-1}\mathcal{T}$. It is essentially the unique transform from $(S)^*$ to itself satisfying the following properties

$$\hat{\mathbf{1}} = \delta_0, \qquad \hat{\delta_0} = \mathbf{1},$$
$$(\partial_t\Phi)\hat{} = iX(t)\hat{\Phi}, \qquad (X(t)\Phi)\hat{} = i\partial_t\hat{\Phi}.$$

From this characterization theorem, we can derive the fundamental consequence about these products

Proposition 2 *The products,* : *and* $*$, *have no zero devisor among Hida distributions.*

(As the proof will be given in our forthcoming paper [3], we omit it here.)

This is just the generalization of Titchmarsh's theorem which guarantees the absence of zero divisors in the convolution product among complex valued continuous functions on $[0, \infty)$. We recall here that Mikusiński formulated ([4]) Heaviside's operational calculus on the basis of Titchmarsh's theorem in the use of the usual convolution product. It is natural to expect that the above result will enable us to develop a kind of "operational calculus" in the context of WNA, which will be very instrumental for solving stochastic differential equations, for instance.

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