数理解析研究所講究録
第 1565 巻 2007 年 205-216

TRANSLATION INVARIANT MODELS IN NONRELATIVISTIC QUANTUM ELECTRODYNAMICS

Fumio Hiroshima

Department of Mathematics, University of Kyushu
Fukuoka, Japan
hiroshima@math.kyushu-u.ac.jp

May 31, 2007

1 The Pauli-Fierz Hamiltonian

In this paper we discuss translation invariant nonrelativistic quantum electrodynamics by *functional integrations*. We assume that an electron is in low energy, its density of charge is smoothly localized. In particular, the ultraviolet divergence does not exist. Let us see some classical model. Let $E(t,x)$ and $B(t,x)$, $(t,x)\in\mathbb{R}\times\mathbb{R}^3$, be an electric field and a magnetic field respectively, and $q(t)$ the position of an electron at time $t\in\mathbb{R}$. The Maxwell equation with form factor φ is given by

$$\begin{aligned}\dot{B} &= -\nabla\times E,\\ \nabla\cdot B &= 0,\\ \dot{E} &= \nabla\times B - e\varphi(\cdot-q(t))\dot{q}(t),\\ \nabla\cdot E &= e\varphi(\cdot-q(t)).\end{aligned}$$

Let $(J,\rho)=(e\varphi(x-q(t))\dot{q}(t), e\varphi(x-q(t)))$. Then the Lagrangian density is given by

$$\mathcal{L}(t,x)=\frac{1}{2}m\dot{q}^2+\frac{1}{2}(E^2-B^2)+J\cdot A-\rho\phi, \tag{1.1}$$

where A and ϕ are a vector potential and a scalar potential related to E and B such as $E=-\dot{A}-\nabla\phi$ and $B=\nabla\times A$. Let $L=\int_{\mathbb{R}^3}\mathcal{L}(t,x)dx$. Then the conjugate momenta are given by

$$p(t):=\frac{\partial L}{\partial\dot{q}}=m\dot{q}(t)+e\int A(t,x)\varphi(x-q(t))dx,\quad \Pi(t,x):=\frac{\delta L}{\delta\dot{A}}=\dot{A}(t,x).$$

Then the Hamiltonian is given through the Legendre transformation as

$$H_{cl} = p\cdot\dot{q} + \int \dot{A}\Pi dx - L$$
$$= \frac{1}{2m}\left(p - e\int A(t,x)\varphi(x-q(t))dx\right)^2 + \frac{1}{2}\int\left\{\dot{A}(t,x)^2 + (\nabla\times A(t,x))^2\right\}dx + V_{cl}(q),$$

where V is a smeared external potential given by

$$V_{cl}(q) := \frac{1}{2}e^2\int \frac{\varphi(q-y)\varphi(q-y')}{4\pi|y-y'|}dydy'.$$

We quantized H_{cl} to define the Pauli-Fierz Hamiltonian.

Let us assume that the dimension of the state space is d and the photon is polarized to $d-1$ directions. Physically reasonable choice is $d=3$. Let $\mathcal{F}_{\rm b}$ be the Boson Fock space over $h_{\rm b} := \oplus^{d-1}L^2(\mathbb{R}^d)$, i.e., $\mathcal{F}_{\rm b} := \bigoplus_{n=0}^\infty[\otimes_s^n h_{\rm b}]$, where $\otimes_s^n h_{\rm b}$ denotes the n-fold symmetric tensor product of $h_{\rm b}$ with $\otimes_s^0 h_{\rm b} := \mathbb{C}$. $\Omega = \{1,0,0,...\}\in\mathcal{F}_{\rm b}$ is called the Fock vacuum. The annihilation operator and the creation operator on $\mathcal{F}_{\rm b}$ are denoted by $a(f)$ and $a^*(f)$, $f\in W$, respectively, and are defined by

$$(a^*(f)\Psi)^{(n)} := \sqrt{n}S_n(f\otimes\Psi^{(n-1)}) \tag{1.2}$$

and $a(f) := (a^*(\bar{f}))^*$, where S_n denotes the symmetrizer. Let $\mathcal{F}_{\rm b,fin}$ be the finite particle subspace of $\mathcal{F}_{\rm b}$. The annihilation operator and the creation operator leave $\mathcal{F}_{\rm b,fin}$ invariant and satisfy the canonical commutation relations on it:

$$[a(f),a^*(g)] = (\bar{f},g)1,\quad [a(f),a(g)]=0,\quad [a^*(f),a^*(g)]=0.$$

For $f=(f_1,...,f_{d-1})\in\oplus^{d-1}L^2(\mathbb{R}^d)$, we informally write $a^\sharp(f)$, where $a^\sharp$ stands for a or a^*, as $a^\sharp(f) = \sum_{j=1}^{d-1}\int a^\sharp(k,j)f_j(k)dk$. The quantized radiation field $A_\mu(x)$ with a form factor φ is defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}}\sum_{j=1}^{d-1}\int e_\mu(k,j)\left(\frac{\hat{\varphi}(k)}{\sqrt{\omega(k)}}a^*(k,j)e^{-ik\cdot x} + \frac{\hat{\varphi}(-k)}{\sqrt{\omega(k)}}a(k,j)e^{ik\cdot x}\right)dk.$$

Here $e(k,1),\cdots,e(k,d-1)$ denote generalized polarization vectors satisfying $k\cdot e(k,j) = 0$ and $e(k,i)\cdot e(k,j) = \delta_{ij}1$, $i,j=1,...,d-1$, and $\hat{\varphi}$ is the Fourier transform of form factor φ. Note that

$$\sum_{j=1}^{d-1}e_\alpha(k,j)e_\beta(k,j) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{|k|^2} := \delta^\perp_{\alpha\beta}(k),\quad \alpha,\beta=1,...,d.$$

Thus

$$(A_\mu(x)\Omega, A_\nu(x)\Omega)_{\mathcal{F}_{\rm b}} = \frac{1}{2}\int_{\mathbb{R}^d}\frac{|\hat{\varphi}(k)|^2}{\omega(k)}\delta^\perp_{\mu\nu}(k)dk$$

holds. Throughout this paper we use Assumption (A) below.

(A) *Form factor $\hat{\varphi}$ satisfies $\sqrt{\omega}\hat{\varphi}, \hat{\varphi}/\omega \in L^2(\mathbb{R}^d)$ and $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k) = \hat{\varphi}(k)$.*

$A_\mu(x)$ is essentially self-adjoint on $\mathcal{F}_{\mathrm{b,fin}}$, and its unique self-adjoint extension is denoted by the same symbol. Next we define the second quantization. Let $\mathcal{C}(\mathcal{K} \to \mathcal{L})$ be the set of contraction operators from $\mathcal{K}$ to $\mathcal{L}$. The second quantization Γ is the functor:

$$\Gamma : \mathcal{C}(L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)) \to \mathcal{C}(\mathcal{F}_\mathrm{b} \to \mathcal{F}_\mathrm{b})$$

given by

$$\Gamma(T) := \bigoplus_{n=0}^{\infty} \otimes^n (\oplus^{d-1} T).$$

For a self-adjoint operator h on $L^2(\mathbb{R}^d)$, $\{\Gamma(e^{ith})\}_{t\in\mathbb{R}}$ is a strongly continuous one-parameter unitary group on $\mathcal{F}_\mathrm{b}$. Then there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathcal{F}_\mathrm{b}$ such that $\Gamma(e^{ith}) = e^{itd\Gamma(h)}$. The number operator is defined by $N := d\Gamma(1)$. Let $\omega(k) = |k|$ be the multiplication operator on $L^2(\mathbb{R}^d)$. Define the free Hamiltonian H_{rad} on $\mathcal{F}_\mathrm{b}$ by

$$H_{\mathrm{rad}} := d\Gamma(\omega). \tag{1.3}$$

The Hilbert space $\mathcal{H}$ of state vectors for the total system under consideration is given by

$$\mathcal{H} := L^2(\mathbb{R}^d) \otimes \mathcal{F}_\mathrm{b}. \tag{1.4}$$

Under the identification $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F}_\mathrm{b} dx$, we define the self-adjoint operator A on $\mathcal{H}$ by $A_\mu := \int_{\mathbb{R}^d}^{\oplus} A_\mu(x)dx$. The total Hamiltonian H, the so-called Pauli-Fierz Hamiltonian, is described by

$$H := \frac{1}{2}(-i\nabla \otimes 1 - eA)^2 + V \otimes 1 + 1 \otimes H_{\mathrm{rad}}, \tag{1.5}$$

where $e \in \mathbb{R}$ is a coupling constant. The proposition below is established in [H00b, H02].

Proposition 1.1 *Assume that V is relatively bounded with respect to $-\Delta$ with a relative bound strictly smaller than one. Then H is self-adjoint on $D(H_0)$ and essentially self-adjoint on any core of self-adjoint operator $-(1/2)\Delta \otimes 1 + 1 \otimes H_{\mathrm{rad}}$, and bounded from below,*

Define the field momentum by $P_{\mathrm{f}\mu} := d\Gamma(k_\mu)$ and the total momentum

$$P_\mu^{\mathrm{T}} := \overline{-i\nabla_\mu \otimes 1 + 1 \otimes P_{\mathrm{f}\mu}}, \tag{1.6}$$

where $\overline{X}$ denotes the closure of closable operator X. Now we set $V = 0$. Then it is seen that H is *translation invariant*;

$$e^{isP_\mu^{\mathrm{T}}} H e^{-isP_\mu^{\mathrm{T}}} = H, \quad s \in \mathbb{R}, \quad \mu = 1, ..., d.$$

Then we can decompose H on $\sigma(P_\mu^{\mathrm{T}}) = \mathbb{R}$. Define

$$H(P) := \frac{1}{2}(P - P_\mathrm{f} - eA(0))^2 + H_{\mathrm{rad}}, \quad P \in \mathbb{R}^d. \tag{1.7}$$

Note that $H(P)$ is a well defined symmetric operator on $D(H_{\mathrm{rad}}) \cap D(P_\mathrm{f}^2)$ by assumption **(A)**. The next proposition is established in [H06, LMS06].

Proposition 1.2 *$H(P)$ is self-adjoint on $D(H_{\rm rad}) \cap (\cap_{\mu=1}^d D(P_{{\rm f}\mu}^2))$ and it follows that*

$$\int_{\mathbb{R}^d}^{\oplus} H(P)dP \cong H. \tag{1.8}$$

So $H(P)$ is our main object and $P \in \mathbb{R}^d$ is called the total momentum. We want to investigate spectral properties of $H(P)$ by making use of functional integrations.

2 Functional integral representations

Let $(b(t))_{t\geq 0} = (b_1(t), \cdots, b_d(t))_{t\geq 0}$ be the d-dimensional Brownian motion starting at 0 on a probability space $(W, \mathcal{B}, db)$. Set $X_s := x + b(s)$, $x \in \mathbb{R}^d$, and $dX := dx \otimes db$.

2.1 Functional integral representations for e^{-tH}

Let $\mathcal{A}_0(f)$ be a Gaussian random process on a probability space (Q_0, Σ_0, μ_0) indexed by real $f = (f_1, ..., f_d) \in \overset{d}{\oplus} L^2(\mathbb{R}^d)$ with mean zero and covariance given by

$$\int_{Q_0} \mathcal{A}_0(f)\mathcal{A}_0(g)d\mu_0 = q_0(f,g), \tag{2.1}$$

where

$$q_0(f,g) := \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_{\mathbb{R}^d} \delta_{\alpha\beta}^{\perp}(k)\overline{\hat{f}_\alpha(k)}\hat{g}_\beta(k)dk.$$

The existence of probability space (Q_0, Σ_0, μ_0) and Gaussian random variable $\mathcal{A}_0(f)$ are known by the Minlos theorem. In a similar way, we can construct two other Gaussian random variables. Let $\mathcal{A}_1(f)$ indexed by real $f \in \overset{d}{\oplus} L^2(\mathbb{R}^{d+1})$ and $\mathcal{A}_2(f)$ by real $f \in \overset{d}{\oplus} L^2(\mathbb{R}^{d+2})$ be Gaussian random processes on probability spaces (Q_1, Σ_1, μ_1) and (Q_2, Σ_2, μ_2), respectively, with mean zero and covariances given by

$$\int_{Q_1} \mathcal{A}_1(f)\mathcal{A}_1(g)d\mu_1 = q_1(f,g), \quad \int_{Q_2} \mathcal{A}_2(f)\mathcal{A}_2(g)d\mu_2 = q_2(f,g), \tag{2.2}$$

where

$$q_1(f,g) := \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_{\mathbb{R}^{d+1}} \delta_{\alpha\beta}^{\perp}(k)\overline{\hat{f}_\alpha(k,k_0)}\hat{g}_\beta(k,k_0)dkdk_0,$$

$$q_2(f,g) := \frac{1}{2}\sum_{\alpha,\beta=1}^d \int_{\mathbb{R}^{d+1+1}} \delta_{\alpha\beta}^{\perp}(k)\overline{\hat{f}_\alpha(k,k_0,k_1)}\hat{g}_\beta(k,k_0,k_1)dkdk_0dk_1.$$

From now on $q = 0, 1, 2$. We extend it for $f = f_{\rm R} + if_{\rm I}$ with $f_{\rm R} = (f + \bar{f})/2$ and $f_{\rm I} = (f - \bar{f})/(2i)$ as $\mathcal{A}_q(f) = \mathcal{A}_q(f_{\rm R}) + i\mathcal{A}_q(f_{\rm I})$. The n-particle subspace $L_n^2(Q_q)$ of $L^2(Q_q)$ is defined by

$$L_n^2(Q_q) = \overline{\text{L.H.}\{: \mathcal{A}_q(f_1)\cdots\mathcal{A}_q(f_n) : |f_j \in L^2(\mathbb{R}^{d+q}), j = 1, ..., n\}}.$$

Here $:X:$ denotes the Wick product of X. The identity $L^2(Q_q) = \oplus_{n=0}^{\infty} L_n^2(Q_q)$ is known as the Wiener-Itô decomposition. We also define the second quantization on $L^2(Q_q)$. Let $\Gamma_{qq'} : \mathcal{C}(L^2(\mathbb{R}^{d+q}) \to L^2(\mathbb{R}^{d+q'})) \to \mathcal{C}(L^2(Q_q) \to L^2(Q_{q'}))$ be defined by

$$\Gamma_{qq'}T1 = 1, \quad \Gamma_q(T) : \mathcal{A}_q(f_1) \cdots \mathcal{A}_q(f_n) :=: \mathcal{A}_{q'}([T]_d f_1) \cdots \mathcal{A}_{q'}([T]_d f_n) : .$$

Set $\Gamma_{qq} = \Gamma_q$ for simplicity. In particular since $\{\Gamma_q(e^{ith})\}_{t\in\mathbb{R}}$ with a self-adjoint operator h on $L^2(\mathbb{R}^d)$ is a strongly continuous one-parameter unitary group, there exists a self-adjoint operator $d\Gamma_q(h)$ on $L^2(Q_q)$ such that $\Gamma_q(e^{ith}) = e^{itd\Gamma_q(h)}$, $t \in \mathbb{R}$. We set $N_q := d\Gamma_q(1)$. Let h be a multiplication operator in $L^2(\mathbb{R}^d)$. We define the families of isometries,

$$L^2(\mathbb{R}^d) \xrightarrow{j_s} L^2(\mathbb{R}^{d+1}) \xrightarrow{\xi_t=\xi_t(h)} L^2(\mathbb{R}^{d+2}), \quad s,t \in \mathbb{R}, \tag{2.3}$$

by

$$\widehat{j_s f}(k,k_0) := \frac{e^{-isk_0}}{\sqrt{\pi}} \left(\frac{\omega(k)}{\omega(k)^2 + |k_0|^2}\right)^{1/2} \hat{f}(k), \quad (k,k_0) \in \mathbb{R}^d \times \mathbb{R}, \tag{2.4}$$

$$\widehat{\xi_t f}(k,k_0,k_1) := \frac{e^{-itk_1}}{\sqrt{\pi}} \left(\frac{h(k)}{h(k)^2 + |k_1|^2}\right)^{1/2} \hat{f}(k,k_0), \quad (k,k_0,k_1) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}.$$

Next, define the families of operators J_s and $\Xi_t = \Xi_t(h)$, $s,t \in \mathbb{R}$;

$$L^2(Q_0) \xrightarrow{J_s} L^2(Q_1) \xrightarrow{\Xi_t} L^2(Q_2)$$

by

$$J_s = \Gamma_{01}(j_s), \quad \Xi_t = \Gamma_{12}(\xi_t). \tag{2.5}$$

Define $\mathcal{A}_{q,\mu}(f) = \mathcal{A}_q(\oplus_{\ell=1}^d \delta_{\ell\mu} f)$. We see that $d\Gamma_0(-i\nabla) \cong P_f$ and $d\Gamma_0(\omega(-i\nabla)) \cong H_{\rm rad}$. We can see that $\mathcal{H} \cong \int_{\mathbb{R}^d}^{\oplus} L^2(Q_0)dx$ i.e., $F \in \mathcal{H}$ can be regarded as an $L^2(Q_0)$-valued L^2-function on $\mathbb{R}^d$. Note that in the Fock representation the test function $\hat{f}$ of $A_\mu(\hat{f})$ is taken in the momentum representation, but in the Schrödinger representation, f of $\mathcal{A}_{0,\mu}(f)$ in the position representation. We can see that

$$H \cong \frac{1}{2}(-i\nabla \otimes 1 - e\mathcal{A}_0^{\tilde{\varphi}})^2 + V \otimes 1 + 1 \otimes H_{\rm rad},$$

where $\tilde{\varphi} := (\hat{\varphi}/\sqrt{\omega})^\vee$. By the Feynman-Kac formula and the fact $J_0^* J_t = e^{-tH_{\rm rad}}$ we can see that

$$(F, e^{-t(-(1/2)\Delta + V + H_{\rm rad})}G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s)ds} (J_0 F(X_0), J_t G(X_t))_{L^2(Q_1)} dX.$$

Adding the minimal perturbation: $-i\nabla_\mu \otimes 1 \to -i\nabla_\mu \otimes 1 - e\mathcal{A}_0^{\tilde{\varphi}}$, we have the functional integral representation below [H97].

$$(F, e^{-tH}G)_{\mathcal{H}} = \int_{\mathbb{R}^d \times W} e^{-\int_0^t V(X_s)ds} (J_0 F(X_0), e^{-ie\mathcal{A}_1(\mathcal{K}_1^{[0,t]}(x))} J_t G(X_t))_{L^2(Q_1)} dX, \tag{2.6}$$

where $\mathcal{K}_1^{[0,t]}(x) := \oplus_{\mu=1}^d \int_0^t j_s \tilde{\varphi}(\cdot - X_s) db_\mu(s) \in \oplus^d L^2(\mathbb{R}^{d+1})$.

2.2 Functional integral representations for $e^{-tH(P)}$

We now construct the functional integral representation of $(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}$. We use the identification $\mathcal{F}_b \cong L^2(Q_0)$ without notices. For $\Psi \in L^2(Q_0)$, we set $\Psi_t := J_t e^{-iP_f \cdot b(t)}\Psi$, $t \geq 0$.

Theorem 2.1 *Let* $\Psi, \Phi \in \mathcal{F}_b$. *Then*

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b} = \int_W (\Psi_0, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP\cdot b(t)} db, \tag{2.7}$$

where $\mathcal{K}_1^{[0,t]}(0) := \oplus_{\mu=1}^d \int_0^t j_s \tilde{\varphi}(\cdot - b(s)) db_\mu(s)$.

Proof: We show an outline of the proof. See [H06] for detail. Set $F_s = \rho_s \otimes \Psi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{b,\mathrm{fin}}$ and $G_r = \rho_r \otimes \Phi \in L^2(\mathbb{R}^d) \otimes \mathcal{F}_{b,\mathrm{fin}}$, where ρ_s is the heat kernel:

$$\rho_s(x) = (2\pi s)^{-d/2} e^{-|x|^2/(2s)}, \quad s > 0. \tag{2.8}$$

By the fact that $H = U^{-1}\left(\int_{\mathbb{R}^d}^{\oplus} H(P)dP\right)U$ and $Ue^{-i\xi\cdot P^{\mathrm{T}}}U^{-1} = \int_{\mathbb{R}^d}^{\oplus} e^{-i\xi\cdot P}dP$, we have

$$(F_s, e^{-tH}e^{-i\xi\cdot P^{\mathrm{T}}}G_r)_{\mathcal{H}} = \int_{\mathbb{R}^d} dP((UF_s)(P), e^{-tH(P)}e^{-i\xi\cdot P}(UG_r)(P))_{\mathcal{F}_b}, \quad \xi \in \mathbb{R}^d.$$

Here $(UF_s)(P) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\cdot P}e^{ix\cdot P_f}\rho_s(x)\Psi dx$. Note that

$$\lim_{s\to 0}(UF_s)(P) = \frac{1}{\sqrt{(2\pi)^d}}\Psi \tag{2.9}$$

strongly in $\mathcal{F}_b$ for each $P \in \mathbb{R}^d$. Hence we have by the Lebesgue dominated convergence theorem,

$$\lim_{s\to 0}(F_s, e^{-tH}e^{-i\xi\cdot P^{\mathrm{T}}}G_r)_{\mathcal{F}_b} = \frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} dP(\Psi, e^{-tH(P)}e^{-i\xi\cdot P}(UG_r)(P))_{\mathcal{F}_b}. \tag{2.10}$$

On the other hand we see that by (2.6)

$$\lim_{s\to 0}(F_s, e^{-tH}e^{-i\xi\cdot P^{\mathrm{T}}}G_r)_{\mathcal{H}} = \int_W \rho_r(b(t)-\xi)(J_0\Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_f}\Phi)_{L^2(Q_1)} db. \tag{2.11}$$

Here we used that $\int_W db\rho_r(b_t + x - \xi)(J_0\Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(x))}J_t e^{-i\xi P_f}\Phi)$ is continuous at $x = 0$ and $e^{-i\xi\cdot \tilde{P}_T}(\rho(X_t)\otimes\Phi) = \rho(X_t - \xi)\otimes e^{-i\xi\cdot P_f}\Phi$. Then we obtained by (2.10) and (2.11) that

$$\begin{aligned}&\frac{1}{\sqrt{(2\pi)^d}}\int_{\mathbb{R}^d} e^{-i\xi\cdot P}(\Psi, e^{-tH(P)}(UG_r)(P))_{\mathcal{F}_b}dP \\ &= \int_W \rho_r(b(t)-\xi)(J_0\Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))}J_t e^{-i\xi\cdot P_f}\Phi)_{L^2(Q_1)} db.\end{aligned} \tag{2.12}$$

Since

$$\int_{\mathbb{R}^d} \|e^{-tH(P)}UG_r(P)\|^2_{\mathcal{F}_b} dP \le \int_{\mathbb{R}^d} \|UG_r(P)\|^2_{\mathcal{F}_b} dP = \|G_r\|^2_{\mathcal{H}} < \infty,$$

we have $(\Psi, e^{-tH(\cdot)}(UG_r)(\cdot))_{\mathcal{F}_b} \in L^2(\mathbb{R}^d)$ for $r \neq 0$. Then taking the inverse Fourier transform of both sides of (2.12) with respect to P, we have

$$\begin{aligned}&(\Psi, e^{-tH(P)}(UG_r)(P))_{\mathcal{F}_b}\\ &= \frac{1}{\sqrt{(2\pi)^d}} \int_W db \int_{\mathbb{R}^d} d\xi e^{iP\cdot\xi} \rho_r(b(t)-\xi)(J_0\Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-i\xi\cdot P_f}\Phi)_{L^2(Q_1)} \quad (2.13)\end{aligned}$$

for almost every $P \in \mathbb{R}^d$. Both sides (2.13) are continuous in P, then (2.13) is true for all $P \in \mathbb{R}^d$. Taking $r \to 0$ on both sides of (2.13), we have by the Lebesgue dominated convergence theorem and (2.9) that

$$(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b} = \int_W (J_0\Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_f\cdot b(t)}\Phi)_{L^2(Q_1)} e^{iP\cdot b(t)} db = (2.7).$$

Thus the theorem follows for $\Psi, \Phi \in \mathcal{F}_{b,\text{fin}}$. Let $\Psi, \Phi \in \mathcal{F}_b$, and $\Psi_n, \Phi_n \in \mathcal{F}_{b,\text{fin}}$ such that $\Psi_n \to \Psi$ and $\Phi_n \to \Phi$ strongly as $n \to \infty$. Since

$$|(J_0\Psi_n, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_f\cdot b(t)}\Phi_n)_{L^2(Q_1)}| \le \|\Psi_n\|_{\mathcal{F}_b}\|\Phi_n\|_{\mathcal{F}_b} \le c$$

with some constant c independent of n, we have by the Lebesgue dominated convergence theorem

$$\begin{aligned}&\lim_{n\to\infty} \int_W (J_0\Psi_n, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_f\cdot b(t)}\Phi_n)_{L^2(Q_1)} e^{iP\cdot b(t)} db\\ &= \int_W (J_0\Psi, e^{-ieA_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_f\cdot b(t)}\Phi)_{L^2(Q_1)} e^{iP\cdot b(t)} db,\end{aligned}$$

and it is immediate that $\lim_{n\to\infty}(\Psi_n, e^{-tH(P)}\Phi_n)_{\mathcal{F}_b} = (\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}$. Hence (2.7) is proven. **qed**

2.3 Applications

Let $L^2_{\text{fin}}(Q_q) := \bigcup_{N=0}^{\infty} [\oplus_{n=0}^N L^2_n(Q_q)]$ and T a self-adjoint operator on $L^2(\mathbb{R}^{d+q})$. Let us define the operator $\Pi_{q,\mu}(Tf)$ on $L^2_{\text{fin}}(Q_q)$ by

$$\Pi_{q,\mu}(Tf) := i[d\Gamma_q(T), \mathcal{A}_{q,\mu}(f)]$$

for $f \in D(T)$. In the case f is real-valued, $\Pi_{q,\mu}(Tf)$ is a symmetric operator. The self-adjoint extension of $\Pi_{q,\mu}(f)$ with real f is denoted by the same symbol.

Let $\mathcal{K}_+ := \{\Psi \in L^2(Q_0) | \Psi \ge 0\}$ and $\mathcal{K}_+^0 := \{\Psi \in \mathcal{K}_+ | \Psi > 0\}$. It is well known that $e^{iP_f\cdot v}\mathcal{K}_+ \subset \mathcal{K}_+$ for $v \in \mathbb{R}^d$. Fundamental fact is that for real $f \in L^2(\mathbb{R}^{d+1})$,

$$J_0^* e^{i\Pi_{1,\mu}(f)} J_t[\mathcal{K}_+ \setminus \{0\}] \subset \mathcal{K}_+^0, \quad (2.14)$$

i.e., $J_0^* e^{i\Pi_{1,\mu}(f)} J_t$ is positivity improving. See [H00a]. We define $\vartheta := \exp\left(i\frac{\pi}{2}N\right)$.

Theorem 2.2 *$\vartheta e^{-tH(0)}\vartheta^{-1}$ is positivity improving.*

Proof: Let $\Psi, \Phi \in \mathcal{K}_+ \setminus \{0\}$. It is seen that

$$(\Psi, \vartheta e^{-tH(0)}\vartheta^{-1}\Phi)_{\mathcal{F}_b} = \int_W (\Psi, J_0^* e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_f\cdot b(t)}\Phi)_{L^2(Q_0)} db. \tag{2.15}$$

Here we used the facts that $J_t e^{-iP_f\cdot b(t)} e^{-i(\pi/2)N} = e^{-i(\pi/2)\tilde{N}} J_t e^{-iP_f\cdot b(t)}$ and

$$e^{i(\pi/2)\tilde{N}} e^{-ieA_1(f)} e^{-i(\pi/2)\tilde{N}} = e^{-ie\Pi_1(f)},$$

where $\tilde{N} = d\Gamma_1(1)$. Since $J_0^* e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))} J_t e^{-iP_f\cdot b(t)}$ is positivity improving for each $b \in W$, specifically the integrand in (2.15) is strictly positive for each $b \in W$. Hence the right-hand side of (2.15) is strictly positive, which implies that $\vartheta e^{-tH(0)}\vartheta^{-1}\mathcal{K}_+ \setminus \{0\} \subset \mathcal{K}_+^0$. Thus the theorem follows. **qed**

Immediate corollaries are as follows.

Corollary 2.3 *The ground state $\varphi_g(0)$ of $H(0)$ is unique up to multiple constants, if it exists, and it can be taken as $\vartheta\varphi_g(0) > 0$ in the Schrödinger representation.*

Corollary 2.4 *It follows that*

$$|(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}| \le (|\Psi|, e^{-t(\frac{1}{2}P_f^2 + H_{rad})}|\Phi|)_{L^2(Q_0)}, \tag{2.16}$$

$$|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_b}| \le (|\Psi|, \vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}. \tag{2.17}$$

Proof: When L is positivity preserving, we have $|L\Psi| \le L|\Psi|$. Furthermore,

$$|(\Psi, e^{-tH(P)}\Phi)_{\mathcal{F}_b}| \le \int_W (J_0|\Psi|, J_t e^{-iP_f\cdot b(t)}|\Phi|)_{L^2(Q_1)} db = (|\Psi|, e^{-t(\frac{1}{2}P_f^2 + H_{rad})}|\Phi|)_{L^2(Q_0)}$$

where we used that $b(t)$ is Gaussian with $\int |b_\mu(t)|^2 db = 1/2$. Thus (2.16) follows. We have

$$(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_b} = \int_W (\Psi_0, e^{-ie\Pi_1(\mathcal{K}_1^{[0,t]}(0))}\Phi_t)_{L^2(Q_1)} e^{iP\cdot b(t)} db. \tag{2.18}$$

Then

$$|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Phi)_{\mathcal{F}_b}| \le (|\Psi|, \vartheta e^{-tH(0)}\vartheta^{-1}|\Phi|)_{L^2(Q_0)}.$$

Hence (2.17) follows. **qed**

Let $E(P, e^2) = \inf \sigma(H(P))$.

Corollary 2.5 *(1) $0 = E(0,0) \le E(0,e^2) \le E(P,e^2)$, (2) Assume that the ground state $\varphi_g(0)$ of $H(0)$ exists for $e \in [0, e_0)$ with some $e_0 > 0$. Then $E(0,e^2)$ is concave, continuous and monotonously increasing function on e^2, (3) $E(0,e^2) \le \inf \sigma(H)$.*

Proof: (2.17) implies $|(\Psi, \vartheta e^{-tH(P)}\vartheta^{-1}\Psi)_{\mathcal{F}_b}| \le e^{-tE(0,e^2)}\|\Psi\|^2_{\mathcal{F}_b}$. Since ϑ is unitary, (1) follows. Let $\varphi_g(0)$ be the ground state of $H(0)$. Thus by Corollary 2.3, $(1, \varphi_g(0))_{L^2(Q_0)} \ne 0$. Hence

$$E(0,e^2) = \lim_{t\to\infty} -\frac{1}{t}\log(\Omega, e^{-tH(0)}\Omega)_{\mathcal{F}_b} = \lim_{t\to\infty} -\frac{1}{t}\log\int_W e^{-\frac{e^2}{2}q_0(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))}db.$$

Since $e^{-\frac{e^2}{2}q_0(\mathcal{K}_1^{[0,t]}(0),\mathcal{K}_1^{[0,t]}(0))}$ is log convex on e^2, $E(0,e^2)$ is concave. Then $E(0,e^2)$ is continuous on $(0,e_0)$. Since $E(0,e^2)$ is also continuous at $e^2=0$ by the fact that $H(0)$ converges as $e^2\to 0$ in the uniform resolvent sense, $E(0,e^2)$ is continuous on $[0,e_0)$. Then $E(0,e^2)$ can be expressed as $E(0,e^2) = \int_0^{e^2}\phi(t)dt$ with some positive function ϕ. Thus $E(0,e^2)$ is monotonously increasing on e^2. Then (2) is obtained. We have

$$(F, (1\otimes\vartheta)e^{-tH}(1\otimes\vartheta^{-1})G)_{\mathcal{H}} = \int_{\mathbb{R}^d} dP(F(P), \vartheta e^{-tH(P)}\vartheta^{-1}G(P)))_{\mathcal{F}_b}.$$

Then by (2.17) it is seen that

$$|(F, (1\otimes\vartheta)e^{-tH}(1\otimes\vartheta^{-1})F)_{\mathcal{H}}| \le e^{-tE(0,e^2)}\int_{\mathbb{R}^d} dP\|F(P)\|^2_{\mathcal{F}_b} = e^{-tE(0,e^2)}\|F\|^2_{\mathcal{H}}.$$

Thus (3) follows. **qed**

3 The n point Euclidean Green functions

The functional integral representations derived in the previous section can be extended to the n point Euclidean Green functions.

Theorem 3.1 *Let $K = d\Gamma(h)$ with a multiplication operator h in $L^2(\mathbb{R}^d)$. We assume that $\Phi_0, \Phi_m \in \mathcal{F}_b$ and $\Phi_j \in \mathcal{F}_b^\infty$ for $j = 1, ..., m-1$ with $\Phi_j = \Phi_j(A(f_1^j), \cdots, A(f_{n_j}^j))$. Then for $P_0, \cdots, P_{m-1} \in \mathbb{R}^d$,*

$$\begin{aligned}&(\Phi_0, \prod_{j=1}^m e^{-(s_j-s_{j-1})K}e^{-(t_j-t_{j-1})H(P_{j-1})}\Phi_j)_{\mathcal{F}_b}\\ &= \int_W(\hat\Phi_0, e^{-ie\mathcal{A}_2(\mathcal{K}_2(0))}\prod_{j=1}^m\hat\Phi_j)_{L^2(Q_2)}e^{+i\sum_{j=1}^m(b(t_j)-b(t_{j-1}))P_{j-1}}db,\end{aligned} \tag{3.1}$$

where $\mathcal{K}_2(0) := \oplus_{\mu=1}^d \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \xi_{s_j} j_s \tilde{\hat\varphi}(\cdot - b(s))db_\mu(s)$ *and*

$$\hat\Phi_j := \Xi_{s_j}J_{t_j}e^{-iP_f\cdot b(t_j)}\Phi_j = \Phi_j\left(\mathcal{A}_2\left(\xi_{s_j}j_{t_j}f_1^j(\cdot - b(t_j))\right), \cdots, \mathcal{A}_2\left(\xi_{s_j}j_{t_j}f_{n_j}^j(\cdot - b(t_j))\right)\right).$$

Proof: See [H06] for detail.

We shall show some applications of Theorem 3.1, by which we can construct a sequence of measures on W converging to $(\varphi_g(P), T\varphi_g(P))_{\mathcal{F}_b}$ for some bounded operator T. In particular $T = e^{-\beta N}$ and $T = e^{-iA(f)}$ are taken as examples. It is known that $H(P)$ has a unique ground state $\varphi_g(P)$ and $(\varphi_g(P), \Omega)_{\mathcal{F}_b} \neq 0$ for sufficiently small e.

Corollary 3.2 *We suppose that $H(P)$ has the unique ground state $\varphi_g(P)$ and it satisfies $(\varphi_g(P), \Omega)_{\mathcal{F}_b} \neq 0$. Then for $\beta > 0$,*

$$(\varphi_g(P), e^{-\beta N}\varphi_g(P)) = \lim_{t\to\infty} \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP\cdot b(2t)} d\mu_{2t},$$

where $D(t) := q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0))$ and μ_{2t} is a measure on W given by

$$d\mu_{2t} := \frac{1}{Z} e^{-(e^2/2)q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0))} db$$

with normalizing constant Z such that $\int_W e^{iP\cdot b(2t)} d\mu_{2t} = 1$.

Proof: We define the family of isometries $\xi_s = \xi_s(1)$, $s \in \mathbb{R}$, by (2.3). By Theorem 3.1 we have

$$\begin{aligned}(e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)_{\mathcal{F}_b} &= \int_W db e^{iP\cdot b(2t)} (1, e^{-ieA_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))} 1)_{L^2(Q_2)} \\ &= \int_W db e^{iP\cdot b(2t)} e^{-(e^2/2)q_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0))}.\end{aligned}$$

Noticing that $q_2(\xi_s f, \xi_t g) = e^{-|s-t|} q_1(f, g)$, we have

$$q_2(\xi_0 \mathcal{K}_1^{[0,t]}(0) + \xi_\beta \mathcal{K}_1^{[t,2t]}(0)) = q_1(\mathcal{K}_1^{[0,2t]}(0), \mathcal{K}_1^{[0,2t]}(0)) - (1 - e^{-\beta}) q_1(\mathcal{K}_1^{[0,t]}(0), \mathcal{K}_1^{[t,2t]}(0)).$$

Then

$$\frac{(e^{-tH(P)}\Omega, e^{-\beta N} e^{-tH(P)}\Omega)}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)} = \int_W e^{(e^2/2)(1-e^{-\beta})D(t)} e^{iP\cdot b(2t)} d\mu_{2t}. \tag{3.2}$$

The corollary follows from (3.2) and

$$s - \lim_{t\to\infty} \frac{e^{-tH(P)}\Omega}{\|e^{-tH(P)}\Omega\|_{\mathcal{F}_b}} = \frac{(\varphi_g(P), \Omega)_{\mathcal{F}_b}}{|(\varphi_g(P), \Omega)_{\mathcal{F}_b}|} . \varphi_g(P)$$

qed

Corollary 3.3 *Assume the same assumptions as in Corollary 3.2. Then*

$$(\varphi_g(P), e^{-iA(f)}\varphi_g(P))_{\mathcal{F}_b} = \lim_{t\to\infty} \int_W e^{-eq_1(\mathcal{K}_1^{[0,2t]}(0), f^t) - \frac{1}{2}q_0(f,f)} e^{iP\cdot b(2t)} d\mu_{2t}, \tag{3.3}$$

where $f^t := \oplus_{\mu=1}^d j_t f_\alpha(\cdot - b(t))$.

Proof: We have by Theorem 3.1

$$\begin{aligned}
(\varphi_{\rm g}(P), e^{-iA(f)}\varphi_{\rm g}(P))_{\mathcal{F}_{\rm b}} &= \lim_{t\to\infty} \frac{(e^{-tH(P)}\Omega, e^{-iA(f)}e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}}}{(e^{-tH(P)}\Omega, e^{-tH(P)}\Omega)_{\mathcal{F}_{\rm b}}} \\
&= \lim_{t\to\infty} \frac{1}{Z}\int_W dbe^{iP\cdot b(2t)}(1, e^{-i(e\mathcal{A}_1(\mathcal{K}_1^{[0,2t]}(0))+\mathcal{A}_1(j_tf))}1)_{L^2(Q_1)} \\
&= \lim_{t\to\infty} \frac{1}{Z}\int_W dbe^{iP\cdot b(2t)}e^{-\frac{1}{2}q_1(e\mathcal{K}_1^{[0,2t]}(0)+f^t)}.
\end{aligned}$$

Note that $q_1(f^t, f^t) = q_0(f, f)$. Then the corollary follows. **qed**

Remark 3.4 *It is informally written as*

$$\begin{aligned}
&q_1(\mathcal{K}_1^{[S,T]}(0), \mathcal{K}_1^{[S',T']}(0)) \\
&= \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\int_S^T db_\alpha(s)\int_{S'}^{T'} db_\beta(r)\int_{\mathbb{R}^d}\delta^\perp_{\alpha\beta}(k)\frac{|\hat\varphi(k)|^2}{\omega(k)}e^{-|s-r|\omega(k)}e^{-ik(b(s)-b(r))}dk.
\end{aligned}$$

and

$$q_1(\mathcal{K}_1^{[0,2t]}(0), f^t) = \frac{1}{2}\sum_{\alpha,\beta=1}^{d}\int_0^{2t} db_\alpha(s)\int_{\mathbb{R}^d}\delta^\perp_{\alpha\beta}(k)\frac{\overline{\hat\varphi(k)}}{\sqrt{\omega(k)}}\hat f_\beta(k)e^{ik\cdot(b(s)-b(t))}e^{-|s-t|\omega(k)}dk.$$

4 The Pauli-Fierz Hamiltonian with spin 1/2

Let us include the spin of the electron. Let $d = 3$ and $\sigma_1, \sigma_2, \sigma_3$ be the 2×2 Pauli matrices given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli-Fierz Hamiltonian with spin 1/2 is defined by

$$H_\sigma(P) = \frac{1}{2}(P - P_{\rm f} - eA(0))^2 + H_{\rm rad} - \frac{e}{2}\sum_{\mu=1}^{3}\sigma_\mu B_\mu(0),$$

where $B(0) = {\rm rot}A(x)$. Although $H_\sigma(P)$ acts on $\mathbb{C}^2\otimes\mathcal{F}_{\rm b}$, it can be reduced to the self-adjoint operator on $L^2(\mathbb{Z}/2\mathbb{Z}; Q_0)$. The functional integral representation of $e^{-tH_\sigma(P)}$ can be also constructed by making use of $3+1$ dimensional Lévy process $(b(t), N_t)$ with values in $\mathbb{R}^3\times(\mathbb{N}\cup\{0\})$, where N_t denotes the Poisson process on a measure space $(S, \Sigma, P_{\rm P})$ with $\mathbb{E}_{P_{\rm P}}[N_t = N] = e^{-t}t^N/N!$. For $\sigma\in\mathbb{Z}/2\mathbb{Z}$ we define $\sigma_t = \sigma(-1)^{N_t}$. Let $\mathcal{B}_q(x) = {\rm rot}\mathcal{A}_q(x)$. The net result is

Theorem 4.1 *Let $\Phi, \Psi \in L^2(\mathbb{Z}/2\mathbb{Z}; Q_0)$. Then*

$$(\Phi, e^{-tH_\sigma(P)}\Psi) = \lim_{\epsilon\to 0} e^t \sum_{\sigma\in\mathbb{Z}/2\mathbb{Z}} \int_{W\times S} db \otimes dP_{\mathrm{P}} \left[e^{iP\cdot b(t)} \int_{Q_1} d\mu_1 \overline{J_0\Phi(\sigma)} e^{X_t^\epsilon} J_t e^{-iP_f\cdot b(t)} \Psi(\sigma_t) \right], \tag{4.1}$$

where

$$X_t = -ie\sum_{\mu=1}^{3}\int_0^t \mathcal{A}_{1,\mu}(\lambda(\cdot - b(s)))db_s^\mu - \int_0^t (-\frac{e}{2})\sigma_s \mathcal{B}_{1,3}(j_s\lambda(\cdot - b(s)))ds$$
$$+ \int_0^{t+} \log\left(-H_{\mathrm{od}}(b(s), -\sigma_{s-}, s) - \epsilon\psi_\epsilon(H_{\mathrm{od}}(b(s), -\sigma_{s-}, s))\right) dN_s$$

and

$$H_{\mathrm{od}}(x, -\sigma, s) = \frac{e}{2}(\mathcal{B}_{1,1}(j_s\lambda(\cdot - b(s)) - i\sigma\mathcal{B}_{1,2}(j_s\lambda(\cdot - b(s)))$$

with the indicator function $\psi_\epsilon(x) = \begin{cases} 1, & |x| < \epsilon/2, \\ 0, & |x| \geq \epsilon/2. \end{cases}$

Proof: See [HL07] for detail. **qed**

Acknowledgment This work was partially done in "Open Quantum System" at the Ervin Schrödinger Institute (ESI) in March 2005, and "Feynman-Kac Formulas and their Applications" at the Wolfgang Pauli Institute (WPI) of Universität Vienna in March 2006. The author wishes to thank institutes for their kind hospitality. This work is financially supported by Grant-in-Aid for Science Research (C) 17540181 from JSPS.

References

[H97] F. Hiroshima, Functional integral representations of quantum electrodynamics, *Rev. Math. Phys.* **9** (1997), 489–530.

[H00a] F. Hiroshima, Ground states of a model in nonrelativistic quantum electrodynamics II, *J. Math. Phys.* **41** (2000), 661–674.

[H00b] F. Hiroshima, Essential self-adjointness of translation-invariant quantum field models for arbitrary coupling constants, *Commun. Math. Phys.* **211** (2000), 585–613.

[H02] F. Hiroshima, Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants, *Ann. Henri Poincaré*, **3** (2002), 171–201.

[H05] F. Hiroshima, Multiplicity of ground states in quantum field models:applications of asymptotic fields, *J. Funct. Anal.* **224** (2005) 431–470

[H06] F. Hiroshima, Fiber Hamiltonian in non-relativistic quantum electrodynamics, preprint 2006.

[HL07] F. Hirohisma and J. Lörinczi, Functional integral representation of the Pauli-Fierz Hamiltonian with spin 1/2, preprint 2007.

[LMS06] M. Loss, T. Miyao and H. Spohn, Lowest energy states in nonrelativistic QED: Atoms and ions in motion, *J. Funct. Anal.* **243** (2006), 353-393.