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Reproducing kernels and the Tikhonov regularization (再生核と Tikhonov 正則化法)

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Abstract

In this paper, some definite applications of the theory of reproducing kernels to the Tikhonov regularization representing the extremal functions in the regularization are introduced with typical examples.

1 Introduction

Let E be an arbitrary set, and let H_K be the reproducing kernel Hilbert space (RKHS) admitting a reproducing kernel $K(p,q)$ on E. For any Hilbert space $\mathcal{H}$ we consider a bounded linear operator L from H_K into $\mathcal{H}$. We shall consider the best approximate problem

$$\inf_{f\in H_K} \|Lf-\boldsymbol{b}\|_{\mathcal{H}} \tag{1}$$

for a vector $\boldsymbol{b}$ in $\mathcal{H}$. Then, we have

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Proposition 1.1 *([1,17]) For a vector* $\mathbf{b}$ *in* $\mathcal{H}$*, there exists a function* $\tilde{f}$ *in* H_K *such that*

$$\inf_{f \in H_K} \|Lf - \mathbf{b}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{b}\|_{\mathcal{H}} \tag{2}$$

if and only if, for the RKHS H_k *admitting the reproducing kernel defined by*

$$k(p,q) = (L^* LK(\cdot, q), L^* LK(\cdot, p))_{H_K}, \tag{3}$$

$$L^* \mathbf{b} \in H_k. \tag{4}$$

Furthermore, if the best approximation $\tilde{f}$ *satisfying (2) exists, then there exists a unique extremal function* $f_{\mathbf{b}}$ *with the minimum norm in* H_K*, and the function* $f_{\mathbf{b}}$ *is expressible in the form*

$$f_{\mathbf{b}}(p) = (L^* \mathbf{b}, L^* LK(\cdot, p))_{H_k} \quad on \quad E. \tag{5}$$

In Proposition 1.1, note that

$$(L^* \mathbf{b})(p) = (L^* \mathbf{b}, K(\cdot, p))_{H_K} = (\mathbf{b}, LK(\cdot, p))_{\mathcal{H}}; \tag{6}$$

that is, $L^* \mathbf{b}$ is expressible in terms of the known $\mathbf{b}, L, K(p,q)$ and $\mathcal{H}$. $f_{\mathbf{b}}$ in (5) is the Moore-Penrose generalized inverse solution $L^\dagger \mathbf{b}$ of the equation $Lf = \mathbf{b}$. Therefore, Proposition 1.1 gives a necessary and sufficient condition for the existence of the Moore-Penrose generalized inverse. Proposition 1.1 is rigid and is not practical in practical applications, because, practical data contain noise or errors and the criteria (4) is not suitable. So, we shall consider the Tikhonov regularization and we shall establish a good relation between the Tikhonov regularization and the theory of reproducing kernels. For the Tikhonov regularization, see, for example, [3,4].

2 Spectral theory

In order to discuss operator equations for general bounded linear operators L, following [3] we shall fix the well-established theory among spectral theory, the Moore-Penrose generalized inverse and the Tikhonov regularization. See [4] for the corresponding results for compact operators L.

Let $\{E_\lambda\}$ be a spectral family for the self-adjoint operator L^*L. If L^*L is continuously invertible, then

$$(L^*L)^{-1} = \int \frac{1}{\lambda} dE_\lambda.$$

In this case, the Moore-Penrose generalized inverse (5) can be represented by the Gaussian normal equation

$$f_{\mathbf{b}}(p) = \int \frac{1}{\lambda} dE_\lambda L^* \mathbf{b}. \tag{7}$$

If $\mathcal{R}(L)$ is non-closed and $\mathbf{b} \notin \mathcal{D}(L^\dagger)$, i.e. if the equation $Lf = \mathbf{b}$ is ill-posed, then the integral in (7) does not exist. Then, we shall define, for any fixed positive $\alpha > 0$

$$f_{\mathbf{b},\alpha}(p) = \int \frac{1}{\lambda + \alpha} dE_\lambda L^* \mathbf{b}. \tag{8}$$

By construction, the operator on the right-hand side of (8) acting on $\mathbf{b}$ is continuous, so that, for noisy data $\mathbf{b}^\delta$ with $\|\mathbf{b} - \mathbf{b}^\delta\|_{\mathcal{H}} \leq \delta$, we can bound the error between $f_{\mathbf{b},\alpha}$ and

$$f^\delta_{\mathbf{b},\alpha}(p) = \int \frac{1}{\lambda + \alpha} dE_\lambda L^* \mathbf{b}^\delta \tag{9}$$

as follows:

Proposition 2.1 *([3], pages 71-73) For any* $\mathbf{b} \in \mathcal{D}(L^\dagger)$,

$$\lim_{\alpha \to 0} \frac{1}{L^*L + \alpha I} L^* \mathbf{b} = \lim_{\alpha \to 0} f_{\mathbf{b},\alpha} = f_{\mathbf{b}}. \tag{10}$$

Furthermore,

$$\|Lf_{\mathbf{b},\alpha} - Lf^\delta_{\mathbf{b},\alpha}\|_{\mathcal{H}} \leq \delta \tag{11}$$

and

$$\|f_{\mathbf{b},\alpha} - f^\delta_{\mathbf{b},\alpha}\|_{H_K} \leq \frac{\delta}{\sqrt{\alpha}}. \tag{12}$$

Proposition 2.2 *([3], pages 117-118) For any* $\mathbf{b} \in \mathcal{D}(L^{\dagger})$ *with* $\|\mathbf{b} - \mathbf{b}^{\delta}\|_{\mathcal{H}} \leq \delta$*, the function* $f^{\delta}_{\mathbf{b},\alpha}$ *defined by (9) is the unique minimizer of the Tikhonov functional*

$$\inf_{f \in H_K} \{\alpha \|f\|^2_{H_K} + \|\mathbf{b}^{\delta} - Lf\|^2_{\mathcal{H}}\}. \tag{13}$$

If $\alpha = \alpha(\delta)$ *is such that*

$$\lim_{\delta \to 0} \alpha(\delta) = 0$$

and

$$\lim_{\delta \to 0} \frac{\delta^2}{\alpha(\delta)} = 0,$$

then

$$\lim_{\delta \to 0} f^{\delta}_{\mathbf{b},\alpha} = f_{\mathbf{b}} = L^{\dagger}(\mathbf{b}). \tag{14}$$

Since practical data contain noise and errors, these results are very important.

3 Representation of the extremal functions in Tikhonov regularization

Our main purpose here is to give an effective representation of the extremal functions $f_{\mathbf{b},\alpha}$ or $f^{\delta}_{\mathbf{b},\alpha}$ in the Tikhonov regularization, since the representation by spectral theory is abstract, in many practical problems.

We set, for any fixed positive $\alpha > 0$

$$K_L(\cdot, p; \alpha) = \frac{1}{L^*L + \alpha I} K(\cdot, p).$$

Then, by introducing the inner product,

$$(f, g)_{H_K(L;\alpha)} = \alpha(f, g)_{H_K} + (Lf, Lg)_{\mathcal{H}}, \tag{15}$$

we shall construct the Hilbert space $H_K(L; \alpha)$ comprising functions of H_K. This space, of course, admits a reproducing kernel. Furthermore, we obtain, directly

Proposition 3.1 *([19]) The extremal function $f_{\mathbf{b},\alpha}(p)$ in the Tikhonov regularization*

$$\inf_{f\in H_K}\{\alpha\|f\|^2_{H_K}+\|\mathbf{b}-Lf\|^2_{\mathcal{H}}\} \tag{16}$$

is represented in terms of the kernel $K_L(p,q;\alpha)$ as follows:

$$f_{\mathbf{b},\alpha}(p)=(\mathbf{b},LK_L(\cdot,p;\alpha))_{\mathcal{H}} \tag{17}$$

where the kernel $K_L(p,q;\alpha)$ is the reproducing kernel for the Hilbert space $H_K(L;\alpha)$ and it is determined as the unique solution $\tilde{K}(p,q;\alpha)$ of the equation:

$$\tilde{K}(p,q;\alpha)+\frac{1}{\alpha}(L\tilde{K}_q,LK_p)_{\mathcal{H}}=\frac{1}{\alpha}K(p,q) \tag{18}$$

with

$$\tilde{K}_q=\tilde{K}(\cdot,q;\alpha)\in H_K \quad \text{for} \quad q\in E, \tag{19}$$

and

$$K_p=K(\cdot,p)\in H_K \quad \text{for} \quad p\in E.$$

In (17), when $\boldsymbol{b}$ contains errors or noise, we need its error estimate. For this, we can obtain the general result:

Theorem 3.1 *([14], [5]). In* (17), *we obtain the estimate*

$$|f_{\mathbf{b},\alpha}(p)|\le\frac{1}{\sqrt{\alpha}}\sqrt{K(p,p)}\|\mathbf{b}\|_{\mathcal{H}}.$$

For many concrete applications of these general theorems, see, for example, [5, 8-16,18-26].

4 Discretization

In several concrete examples, we consider as the reproducing kernel Hilbert space H_K the Sobolev Hilbert spaces on the whole spaces which admit concrete reproducing kernels and as the Hilbert space $\mathcal{H}$ the Hilbert spaces L_2 on the whole spaces. Then the related reproducing kernels $K_L(p,q;\alpha)$ and the extremal functions $f_{\boldsymbol{b},\alpha}$ can be determined concretely in terms of the Fourier integrals from the general equation (18). See, [8-11,13,19-21]. Here, we shall propose a new algorithm to solve numerically the equation (18) which is, in general, an integral equation of Fredholm of the second kind. Our algorithm will give a new type discretization whose effectivity was proved by examples ([8]), since to solve the equation (18) is decisively important to obtain the concrete representation (17).

We take a complete orthonormal system $\{e_j\}_{j=1}^{\infty}$ of the Hilbert space $\mathcal{H}$.

For fixed $\{\lambda_j\}_{j=1}^{\infty}(\lambda_j > 0)$, we consider the general extremal problem for (16)

$$\inf_{f\in H_K}\left\{\alpha\|f\|_{H_K}^2 + \sum_{j=1}^{\infty}\lambda_j|(\boldsymbol{b}-Lf,e_j)_{\mathcal{H}}|^2\right\}. \tag{20}$$

That is,

$$\|\boldsymbol{b}-Lf\|_{\mathcal{H}}^2$$

is replaced by

$$\sum_{j=1}^{\infty}\lambda_j|(\boldsymbol{b},e_j)_{\mathcal{H}}-(Lf,e_j)_{\mathcal{H}}|^2.$$

Then, we shall give an algorithm constructing the reproducing kernel $K_{\alpha,\lambda_j}(p,q)$ of the Hilbert space $H_{K_{\alpha,\lambda_j}}$ with the norm square

$$\alpha\|f\|_{H_K}^2 + \sum_{j=1}^{\infty}\lambda_j|(Lf,e_j)_{\mathcal{H}}|^2. \tag{21}$$

Here, of course, we assume that (21) converges for $\{\lambda_j\}_{j=1}^{\infty}(\lambda_j > 0)$. However, in a practical application, of course, we consider only finite terms in (21) and by finite terms we can give a good approximation of (21).

We shall start with the first step. The reproducing kernel $K^{(1)}(p,q)$ of the Hilbert space with the norm square

$$\alpha\|f\|_{H_K}^2 + \sum_{j=1}^{1} \lambda_j |(Lf, e_j)_{\mathcal{H}}|^2 \tag{22}$$

is given by

$$K^{(1)}(p,q) = K^{(0)}(p,q) - \frac{\lambda_1 (e_1, LK_p^{(0)})_{\mathcal{H}} (LK_q^{(0)}, e_1)_{\mathcal{H}}}{1 + \lambda_1 (L(e_1, LK_q^{(0)})_{\mathcal{H}}, e_1)_{\mathcal{H}}}, \tag{23}$$

for

$$K^{(0)}(p,q) = \frac{1}{\alpha} K(p,q).$$

For the second step, the reproducing kernel $K^{(2)}(p,q)$ of the Hilbert space with the norm square

$$\alpha\|f\|_{H_K}^2 + \sum_{j=1}^{2} \lambda_j |(Lf, e_j)_{\mathcal{H}}|^2 \tag{24}$$

is given by

$$K^{(2)}(p,q) = K^{(1)}(p,q) - \frac{\lambda_2 (e_2, LK_p^{(1)})_{\mathcal{H}} (LK_q^{(1)}, e_2)_{\mathcal{H}}}{1 + \lambda_2 (L(e_2, LK_q^{(1)})_{\mathcal{H}}, e_2)_{\mathcal{H}}}, \tag{25}$$

by using the reproducing kernel $K^{(1)}(p,q)$. In this way, we can obtain the desired representation of $K_{\alpha,\lambda_j}(p,q) = K^{(\infty)}(p,q)$. Then, we obtain

Proposition 4.1 *For any $\mathbf{b} \in \mathcal{H}$, the extremal function $f_{\alpha,\lambda,\mathbf{b}}$ in the extremal problem (20) is given by*

$$f_{\alpha,\lambda,\mathbf{b}}(p) = \sum_{j=1}^{\infty} \lambda_j (\mathbf{b}, e_j)_{\mathcal{H}} (e_j, LK_{\alpha,\lambda_j}(\cdot, p))_{\mathcal{H}}, \tag{26}$$

where we assume that (21) converges on E.

We consider a general extremal problem in (20) by considering a general weight $\{\lambda_j\}$. This means that for a larger λ_{j_0}, the speed of the convergence

$$(Lf, e_{j_0})_{\mathcal{H}} \to (\mathbf{b}, e_{j_0})_{\mathcal{H}}$$

is higher. This technique is a very important for practical applications. For examples, see [10].

5 Error estimate

In the representation of (26), when the data $(\boldsymbol{b}, e_j)_{\mathcal{H}}$ contain errors or noise, we need its error estimate. For this we obtain the good result, which is corresponding to Proposition 2.2:

Theorem 5.1 *In (26), we obtain the estimate*

$$|f_{\alpha,\lambda,\boldsymbol{b}}(p)|$$

$$\leq \frac{1}{\sqrt{\alpha}} \left(\sum_{j=1}^{\infty} (\lambda_j |(\boldsymbol{b}, e_j)_{\mathcal{H}}|^2) \right)^{1/2} \sqrt{K(p,p)}. \tag{27}$$

6 Discrete point data case

As a very general algorithm, we shall consider the discrete point data case such that: In (16), we shall consider the corresponding problem:

$$\inf_{f \in H_K} \left\{ \alpha \|f\|_{H_K}^2 + \sum_{j=1}^{\infty} \lambda_j |f(p_j) - b_j|^2 \right\}, \tag{28}$$

for fixed discrete points $\{p_j\}_j$ of the set E and for given values $\{b_j\}_j$. Then, the corresponding kernels for (23) and (25) are given similarly

$$K^{(1)}(p, q; \{p_1\}) = K^{(0)}(p, q) - \frac{\lambda_1 K^{(0)}(p, p_1) K^{(0)}(p_1, q)}{1 + \lambda_1 K^{(0)}(p_1, p_1)}, \tag{29}$$

and

$$K^{(2)}(p, q; \{p_1, p_2\}) = K^{(1)}(p, q; \{p_1\}) - \frac{\lambda_2 K^{(1)}(p, p_2; \{p_1\}) K^{(1)}(q, p_2; \{p_1\})}{1 + \lambda_2 K^{(1)}(p_2, p_2; \{p_1\})}. \tag{30}$$

In this way, we obtain the reproducing kernel $K_{\alpha,\lambda_j}(p, q; \{p_j\})$ and the corresponding results:

Theorem 6.1 *For any $\{b_j\}$, the extremal function $f_{\alpha,\lambda,\{b_j\}}$ in the extremal problem (28) is given by*

$$f_{\alpha,\lambda,\{b_j\}}(p) = \sum_{j=1}^{\infty} \lambda_j b_j K_{\alpha,\lambda_j}(\cdot, p; \{p_j\}), \tag{31}$$

where we assume that (31) converges on E. Furthermore, we obtain the estimate

$$|f_{\alpha,\lambda,\{b_j\}}(p)|$$
$$\leq \frac{1}{\sqrt{\alpha}} \left(\sum_{j=1}^{\infty} (\lambda_j |b_j|^2) \right)^{1/2} \sqrt{K(p,p)}. \tag{32}$$

The most prototype application of the general theory of this paper is a simple construction of the Moore-Penrose generalized inverse for any matrix:

A Construction of a Natural Inverse of Any Matrix by Using the Theory of Reproducing Kernels by K. Iwamura, T. Matsuura and S. Saitoh (PAJMS Vol. 1 no: 2 (December 2005)).

7 A typical example for the inversion of the heat conduction

We shall give simple approximate real inversion formulas for the Gaussian convolution (the Weierstrass transform)

$$u_F(x,t) = (L_t F)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} F(\xi) \exp\left\{ -\frac{|\xi - x|^2}{4t} \right\} d\xi \tag{33}$$

for the functions of $L_2(\mathbf{R}^n)$. This integral transform which represents the solution $u(x,t)$ of the heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{on} \quad \mathbf{R}^n \times \{t > 0\}$$

satisfying the initial condition

$$u(x,0) = F(x) \quad \text{on} \quad \mathbf{R}^n,$$

is very fundamental and has many applications to mathematical sciences.

Over twenty years ago, in the one dimensional case $n = 1$, the author [17] gave a surprise characterization of the image $u_F(x,t)$ of (33) for $L_2(\mathbf{R}) =$

$L_2(\mathbf{R}, dx)$ functions in terms of an analytic function and established a very simple complex inversion formula. The paper created a new method and many applications to general integral transforms in the framework of Hilbert spaces and analytic extension formulas. See, for example [17] and [27], and their many references. However, in particular, its real inversion formulas are very involved, for example, recall that:

For a bounded and continuous function $F(x)$ and for $t = 1$, for the differential operator $D = \frac{d}{dx}$

$$e^{-D^2}[(L_1F)(x)] = F(x) \quad \text{pointwisely} \quad \text{on} \quad \mathbf{R}$$

([29]). So, one might think that its real inversion formulas will be essentially involved for catching "analyticity" in terms of the data on the real line as in the real inversion formulas of the Laplace transform. See also [6,7] for recent related articles.

Indeed, this inverse problem is very famous as a typical ill-posed problem that is very difficult.

In those papers [22,13], however we were able to obtain simple and practical approximate real inversion formulas by the method in Section 6 using the Sobolev reproducing Hilbert spaces. Furthermore, we illustrated their numerical experiments by using computers and we can realize that we were able to obtain practical real inversion formulas.

In [14], we applied the Paley-Wiener spaces as the reproducing kernel Hilbert spaces in the above theory and we got an improved numerical inversion.

At first we shall fix notations and basic results in the Paley-Wiener spaces and at the same time we shall show the basic relation of the sampling theory and the theory of reproducing kernels.

We shall consider the integral transform, for $L_2(\mathbf{R}^n, (-\pi/h, +\pi/h)^n), (h > 0)$ functions g

$$f(z) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \chi_h(t) g(t) e^{-iz\cdot t} dt. \tag{34}$$

Here, $z = (z_1, z_2, ..., z_n), t = (t_1, t_2, ..., t_n), dt = dt_1 \cdot dt_2 \cdots dt_n, z \cdot t = z_1 t_1 + \cdots + z_n t_n$ and

$$\chi_h(t) = \Pi_{\nu=1}^n \chi(t_\nu, (-\pi/h, +\pi/h)),$$

the characteristic function χ of $(-\pi/h, +\pi/h)$. In order to identify the image space, we form the reproducing kernel

$$K_h(z,\overline{u}) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \chi_h(t)e^{-iz\cdot t}\overline{e^{-iu\cdot t}}dt \tag{35}$$
$$= \Pi_\nu^n \frac{1}{\pi(z_\nu - \overline{u}_\nu)} \sin\frac{\pi}{h}(z_\nu - \overline{u}_\nu).$$

The image space of (34) is called the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$ $(:= W_h)$ comprised of all analytic functions of exponential type satisfying, for each ν, for some constant C_ν and as $z_\nu \to \infty$

$$|f(z_1, ..., z_\nu, z_{\nu+1}, ..., z_n)| \leq C_\nu \exp\left(\frac{\pi|z_\nu|}{h}\right)$$

and

$$\int_{\mathbf{R}^n} |f(x)|^2 dx < \infty.$$

From the identity, for multi-index $j = (j_1, j_2, ..., j_n) \in \mathcal{Z}^n$

$$K_h(jh, j'h) = \Pi_{\nu=1}^n \frac{1}{h}\delta(j_\nu, j'_\nu)$$

(the Kronecker's δ), for each ν, since $\delta(j_\nu, j'_\nu)$ is the reproducing kernel for the Hilbert space ℓ^2, from the Parseval's identity we have the isometric identities in (34)

$$\begin{aligned} \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} |g(t)|^2 dt &= h^n \sum_j |f(jh)|^2 \\ &= \int_{\mathbf{R}^n} |f(x)|^2 dx. \end{aligned}$$

That is, the reproducing kernel Hilbert space H_{K_h} with $K_h(z,\overline{u})$ is characterized as a space comprising the Paley-Wiener space W_h and with the norms above in the both senses of discrete and continuous versions. Here we used the well-known result that $\{jh\}_j$ is a uniqueness set for the Paley-Wiener space W_h; that is, $f(jh) = 0$ for all j implies $f \equiv 0$. Then, the reproducing property of $K_h(z,\overline{u})$ states that

$$f(x) = (f(\cdot), K_h(\cdot, x))_{H_{K_h}} = h^n \sum_j f(jh)K_h(jh, x)$$

$$= \int_{\mathbf{R}^n} f(\xi) K_h(\xi, x) d\xi,$$

in particular, on the real space x. This representation is the sampling theorem which represents the whole data $f(x)$ in terms of the discrete data $\{f(jh)\}_j$. For a general theory of sampling and error estimates for some finite points $\{hj\}_j$, see [17].

Following our general theory, we can obtain the concrete results:

Proposition 7.1 *([14]) For any function $g \in L_2(\mathbf{R}^n)$ and for any $\lambda > 0$, the best approximate function $F^*_{t,\lambda,h,g}$ in the sense*

$$\inf_{F \in H_{K_h}} \left\{ \lambda \|F\|^2_{H_{K_h}} + \|g - u_F(\cdot, t)\|^2_{L_2(\mathbf{R}^n)} \right\}$$

$$= \lambda \|F^*_{t,\lambda,h,g}\|^2_{H_{K_h}} + \|g - u_{F^*_{t,\lambda,h,g}}(\cdot, t)\|^2_{L_2(\mathbf{R}^n)} \tag{36}$$

exists uniquely and $F^*_{t,\lambda,h,g}$ is represented by

$$F^*_{t,\lambda,h,g}(x) = \int_{\mathbf{R}^n} g(\xi) Q_{t,\lambda,h}(\xi - x) d\xi \tag{37}$$

for

$$Q_{t,\lambda,h}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{\chi_h(p) e^{-ip\cdot(\xi - x)} dp}{\lambda e^{|p|^2 t} + e^{-|p|^2 t}}.$$

If, for $F \in H_{K_h}$ we consider the output $u_F(x,t)$ and we take $u_F(\xi, t)$ as g, then we have the result: as $\lambda \to 0$

$$F^*_{t,\lambda,h,g} \to F, \tag{38}$$

uniformly.

Here we note the fact that for the Sobolev space case, for $\lambda = 0$ the corresponding representation (37) does not exist ([22],[13]), meanwhile for the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$ case of (37), for $\lambda = 0$ the representation (37) is still valid; that is, in Proposition 7.1, the result is valid for even $\lambda = 0$. Hence, we can consider the results for $\lambda = 0$ in the spirit of Tikhonov regularization in which we are interested in a small λ or λ tending to zero. That is, when we use the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$, we need not to consider the Tikhonov regularization. Then,

$$(L_t F^*_{t,0,h,g})(x) = (g(\cdot), K_h(\cdot, x))_{L_2(\mathbf{R}^n)}$$

as we see from (35). Since the output is the orthogonal projection of g onto the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$, we can estimate the difference of the output of our inverse $F^*_{t,0,h,g}$ and g, clearly as

$$\|L_t F^*_{t,0,h,g} - g\|_{L_2(\mathbf{R}^n)}$$

which is the distance from g onto the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$. Of course, $F^*_{t,0,h,g}$ is the Moore-Penrose generalized inverse of the operator equation, for any $g \in L_2(\mathbf{R}^n)$ and $F \in W\left(\frac{\pi}{h}\right)$,

$$L_t F = g.$$

For the Paley-Wiener space $W\left(\frac{\pi}{h}\right)$, we need not use Tikhonov regularization and we can look for the Moore-Penrose generalized inverse $F^*_{t,0,h,g}$ by using the theory of reproducing kernels ([17], pp. 178-180). However, we had better to calculate the extremal functions $F^*_{t,\lambda,h,g}$ in the Tikhonov regularization and to set $\lambda = 0$, because the structure of the Moore-Penrose generalized inverses is involved.

We consider the heat conduction for the RKHS H_K, however, our inverstion formula in the sense (36) will show that for a very general function containing the delta function, our inversion formula is valid, because we are considering the approximate inversion by the functions H_K.

8 Numerical Real Inversion Formulas of the Laplace Transform

We shall give a very natural and numerical real inversion formula of the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt} F(t) dt, \quad p > 0 \tag{39}$$

for functions F of some natural function space. This integral transform is, of course, very fundamental in mathematical science. The inversion of the

Laplace transform is, in general, given by a complex form, however, we are interested in and are requested to obtain its real inversion in many practical problems. However, the real inversion will be very involved and one might think that its real inversion will be essentially involved, because we must catch "analyticity" from the real or discrete data. Note that the image functions of the Laplace transform are analytic on some half complex plane. For complexity of the real inversion formula of the Laplace transform, we recall, for example, the following formulas:

$$\lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} f^{(n)}\left(\frac{n}{t}\right) = F(t)$$

and

$$\lim_{n\to\infty} \Pi_{k=1}^{n}\left(1+\frac{t}{k}\frac{d}{dt}\right)\left[\frac{n}{t}f\left(\frac{n}{t}\right)\right] = F(t),$$

([28,29]). See also the great references [30-31]. The problem may be related to analytic extension problems, see [6,7] and [17].

8.1 A Natural Situation for Real Inversion Formulas

In order to apply our general theory to the real inversion formula of the Lapace transform, we shall recall the "natural situation" based on [18].

We shall introduce the simple reproducing kernel Hilbert space (RKHS) H_K comprised of absolutely continuous functions F on the positive real line $\mathbf{R}^+$ with finite norms

$$\left\{\int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt\right\}^{1/2}$$

and satisfying $F(0) = 0$. This Hilbert space admits the reproducing kernel $K(t,t')$

$$K(t,t') = \int_0^{\min(t,t')} \xi e^{-\xi} d\xi \tag{40}$$

(see [17], pages 55-56). Then we see that

$$\int_0^\infty |(\mathcal{L}F)(p)p|^2 dp \le \frac{1}{2}\|F\|^2_{H_K}; \tag{41}$$

that is, the linear operator on H_K

$$(\mathcal{L}F)(p)p$$

into $L_2(\mathbf{R}^+, dp) = L_2(\mathbf{R}^+)$ is bounded ([18]). For the reproducing kernel Hilbert spaces H_K satisfying (341), we can find some general spaces ([18]). Therefore, from the general theory, we obtain

Proposition 8.1 *([18]). For any $g \in L_2(\mathbf{R}^+)$ and for any $\alpha > 0$, the best approximation $F^*_{\alpha,g}$ in the sense*

$$\inf_{F \in H_K} \left\{ \alpha \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F)(p)p - g\|^2_{L_2(\mathbf{R}^+)} \right\}$$

$$= \alpha \int_0^\infty |F^{*\prime}_{\alpha,g}(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F^*_{\alpha,g})(p)p - g\|^2_{L_2(\mathbf{R}^+)} \tag{42}$$

exists uniquely and we obtain the representation

$$F^*_{\alpha,g}(t) = \int_0^\infty g(\xi) \left(\mathcal{L}K_\alpha(\cdot, t)\right)(\xi) \xi d\xi. \tag{43}$$

Here, $K_\alpha(\cdot, t)$ is determined by the functional equation

$$K_\alpha(t, t') = \frac{1}{\alpha} K(t, t') - \frac{1}{\alpha} ((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}K_t)(p)p)_{L_2(\mathbf{R}^+)} \tag{44}$$

for

$$K_{\alpha,t'} = K_\alpha(\cdot, t')$$

and

$$K_t = K(\cdot, t)$$

We shall look for the approximate inversion $F^*_{\alpha,g}(t)$ by using (43). For this purpose, we take the Laplace transform of (44) in t and change the variables t and t' as in

$$(\mathcal{L}K_\alpha(\cdot, t))(\xi)$$

$$= \frac{1}{\alpha} (\mathcal{L}K(\cdot, t'))(\xi) - \frac{1}{\alpha} ((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}(\mathcal{L}K_t)(p)p))(\xi))_{L_2(\mathbf{R}^+)}. \tag{45}$$

Note that

$$K(t,t') = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for} \quad t \le t' \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for} \quad t \ge t'. \end{cases}$$

$$(\mathcal{L}K(\cdot,t'))(p)$$
$$= e^{-t'p}e^{-t'}\Big[\frac{-t'}{p(p+1)} + \frac{-1}{p(p+1)^2}\Big] + \frac{1}{p(p+1)^2}. \tag{46}$$

$$\int_0^\infty e^{-qt'}(\mathcal{L}K(\cdot,t'))(p)dt' = \frac{1}{pq(p+q+1)^2}. \tag{47}$$

Therefore, by setting

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t),$$

which is needed in (3.11), we obtain the Fredholm integral equation of the second type

$$\alpha H_\alpha(\xi,t) + \int_0^\infty H_\alpha(p,t)\frac{1}{(p+\xi+1)^2}dp$$
$$= -\frac{e^{-t\xi}e^{-t}}{\xi+1}\left(t + \frac{1}{\xi+1}\right) + \frac{1}{(\xi+1)^2}. \tag{48}$$

By solving this integral equation, we were able to obtain reasonable numerical real inversion formulas in [16].

9 Inversion formulas for linear physical systems using reproducing kernels

Inverse problems in mathematics which are expected to be applied to practical problems will, sometimes, have weak points in the viewpoint that the background theories are not faithful for practical and physical problems. For example, equations are the representations of some ideal models and are not those of faithful models in the real physical world. Sometimes, boundary conditions for the equations are involved in physical units and sometimes their physical realizations and observations are very difficult. Here, we shall give a new inversion formula for a linear system based on physical experimental data and by using reproducing kernels and the Tikhonov regularization.

In particular, we will not assume any analytical assumption on the linear system, but we use physical experimental data for obtaining an approximate inversion formula for the linear system L.

9.1 Approach looking for the inversion

Physically or by computers we can observe only discrete data, so, as a very general algorithm, we shall consider the discrete point data case such that: In (16), we shall consider the corresponding problem:

$$\inf_{f\in H_K}\left\{\alpha\|f\|^2_{H_K}+\sum_{j=1}^{N}|(Lf)(P_j)-d_j|^2\right\}, \tag{49}$$

for fixed discrete points $\{P_j\}_j$ of the set E and for given values $\mathbf{d}=\{d_j\}_j$; that is, $\mathcal{H}$ is the usual Euclidean space $\mathbf{R}^N$.

In order to use the representation (17), we need $LK_L(\cdot,p;\alpha))$ and it is determined by (18). In (18), we operate L as functions in p and we have

$$\alpha L_p\tilde{K}(p,q;\alpha)+L_p(L\tilde{K}_q,LK_p)_{\mathcal{H}}=L_pK(p,q). \tag{50}$$

Here, when we can take $\alpha=0$ in the sense of numerical, we can take, of course, $\alpha=0$ in those arguments.

However, in order to use our method, we must realize some physical objects as the N data $\mathbf{d}=\{d_j\}_j$, $N\times N$ values $L_pK(p,q)$ and $N\times N$ values L_pLK_p of real values; that is, f and $\mathbf{d}=\{d_j\}_j$ are numerical representations of some physical objects in the system $Lf=\mathbf{d}$.

Since the reproducing kernel Hilbert space H_K is the function space approximating the solution of the operator equation $Lf=\mathbf{d}$, we can take many simple reproducing kernel Hilbert spaces as in ([17]), however, from the present situation, the reproducing kernel $K(p,q)$ must be realized as the physical object for the present system.

9.2 Physical viewpoints

We see in our inversion formula (17), we use a concrete reproducing kernel $K(p,q)$ through (18), but we do not use any Hilbert space structure of the reproducing kernel $K(p,q)$. By the theory of reproducing kernels, for any positive matrix there exists a uniuely determined reproducing kernel Hilbert space; that is, recall the fundamental fact:

We consider any positive matrix $K(p,q)$ on a fixed set E; that is, for an abstract set E and for a complex-valued function $K(p,q)$ on $E \times E$, it satisfies that for any finite points $\{p_j\}$ of E and for any complex numbers $\{C_j\}$,

$$\sum_j \sum_{j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geq 0. \tag{51}$$

Then, by the fundamental theorem by Moore–Aronszajn, we have:

Proposition 9.1 *([17]) For any positive matrix $K(p,q)$ on E, there exists a uniquely determined functional Hilbert space H_K(RKHSH_K) comprising functions $\{f\}$ on E and admitting the reproducing kernel $K(p,q)$ satisfying and characterized by*

$$K(\cdot, q) \in H_K \text{ for any } q \in E \tag{52}$$

and, for any $q \in E$ and for any $f \in H_K$

$$f(q) = (f(\cdot), K(\cdot, q))_{H_K}. \tag{53}$$

Furthermore, in our inversion formula (17), in (16), we are looking for approximations of the inversion in the function space H_K, so, in general, the space H_K is a sufficient large class of functions in the sense that we can approximate the inverse by the functions in H_K. For example, for any characteristic function on any interval, we can approximate it by the Sobolev Hilbert space of 1 dimensional uniformly. This will mean that for the input, we can consider a suitable positive matrix satisfying (51), here, by a suitable positive matrix, we mean that the positive matrix may be realized as the physical data and it will also depend on its physical system.

In connection with these points of view, for example, for the 2 dimensional Sobolev space, we shall use the more simple reproducing kernel

$$K(x_1, x_2, y_1, y_2) = \frac{1}{4} \exp(-|x_1 - y_1|) \exp(-|x_2 - y_2|), \tag{54}$$

which is the usual product of the 1 dimensional Sobolev reproducing kernels and its reproducing kernel Hilbert space is the tensor product of the two

Hilbert spaces of the one dimensional Sobolev Hilbert space (see [17] for this structure).

We shall introduce several simple reproducing kernels on the whole real line space. Note here that for multidimensional spaces, we can consider the products as in (54). Furthermore, the restriction of a reproducing kernel to any subset is again a reproducing kernel. The sum and the usual product of two reproducing kernels on a same set are again reproducing kernels. For these elementary facts, see, ([17]). On the whole real space $\mathbf{R}$, the followings are reproducing kernels:

(1) Any positive semidefinite matrix.
(2) $\delta(x-y)$ ($\delta(0)=1$ and $\delta(x)=0$ for $x \neq 0$).
(3) For any $\alpha > 0$, $\exp(-\alpha|x-y|)$.
(4) $\exp(\alpha xy)$ $\quad$ ($\alpha > 0$).
(5) $\exp(-\alpha(x-y)^2)$ $\quad$ ($\alpha > 0$).
(6) $\exp(-|x-y|)(1+|x-y|)$.
(7) $\min(x,y)$.
(8) For any $\alpha > 0$, $\frac{\sin(\alpha(x-y))}{x-y}$.

On the half space $\{x > 0\}$, the followings are reproducing kernels:

(1) $\frac{1}{(x+y)^{2q}}$ $\quad$ ($q \geq \frac{1}{2}$).
(2) $\frac{1}{(x^2+y^2)^{2q}}$ $\quad$ ($q \geq \frac{1}{2}$).
(3) $\exp\{\min(x,y)\}$.
(4) $\min\left\{\frac{x(1-x)^N}{1-x}, \frac{y(1-y)^N}{1-y}\right\}$ $\quad$ ($N \geq 1$, integer).

On the interval $\{-1 < x < 1\}$, the followings are reproducing kernels:

(1) $\frac{1}{(1-xy)^{2q}}$ $\quad$ ($q \geq \frac{1}{2}$).
(2) $\log\frac{1}{1-xy}$.
(3) $\frac{1}{xy}\log\frac{1}{1-xy}$.
(4) $\frac{1}{[\cosh\alpha(x-y)]^{2q}}$ $\quad$ ($q \geq \frac{1}{2}, \alpha > 0$).

Furthermore, note that any reproducing kernel $K(p,q)$ on an arbitrary set E for a separable reproducing kernel Hilbert space is represented in the form, for some functions $\{\varphi_j(p)\}$ on E

$$K(p,q) = \sum_j \varphi_j(p)\overline{\varphi_j(q)},$$

that converges absolutely on $E \times E$. Conversely, any function $K(p,q)$ which is represented in this way for arbitrary complex-valued functions $\{\varphi_j(p)\}$ on E is a reproducing kernel.

9.3 Exact algorithm

We shall state the exact algorithm looking for the extremal function $f_{\mathbf{d},\alpha}(p)$ in (17), clearly in the setting (50).

1) We set

$$X(P,q) = (L_p\tilde{K}(p,q;\alpha))(P),$$
$$k(P,q) = (L_pK(p,q))(P) \tag{55}$$

and

$$\kappa(P,Q) = (L_qL_pK(p,q))(P,Q). \tag{56}$$

2) As the solution of the regular linear equations (50)

$$\alpha X(P_j,q) + \sum_{j'=1}^{N} X(P_{j'},q)\kappa(P_j,P_{j'}) = k(P_j,q); j = 1,2,...,N, \tag{57}$$

we determine $X(P_j,q)$. Then we obtain the approximate inverse

$$f_{\mathbf{d},\alpha}(p) = \sum_{j=1}^{N} d_jX(P_j,p). \tag{58}$$

Therefore, for some concrete problem for its inversion, we need the experimental data (55) and (56) of the two types in 1) and the procedure 2) is a mathematical problem.

By Theorem 3.1, we note that in (58), the following estimate holds:

$$|f_{\mathbf{d},\alpha}(p)| \leq \frac{1}{\sqrt{\alpha}}\sqrt{K(p,p)}\left(\sum_{j=1}^{N} d_j^2\right)^{1/2}.$$

The simplest and the most typical case of the above algorithm is that the system L is any type matrix of type m and n (without loss of generality we assume that $n \geq m$), and the positive matrix is the identity matrix of size n. Even this case, it seems that the approximate inversion formula (58) is new.

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