数理解析研究所講究録
第 1566 巻 2007 年 24-38

Modeling the Acoustic Behavior of Cancellous Bone

Robert P. Gilbert
Department of Mathematical Sciences, University of Delaware,
Newark, DE 19716, USA, gilbert@math.udel.edu

May 5, 2007

This research represents work done with James Buchanan, Ming Fang, Philippe Guyenne, Alex Panchenko and Ana Vasilic. It was supported in part by NSF grant INT 0438765 and by the Alexander v. Humboldt Senior Scientist Award at the Ruhr Universität Bochum. In this paper we report on our recent researches on the interrogation and modeling of cancellous bone.

1 Introduction

In [6, 5] Buchanan, Gilbert and Khashanah investigated the extent to which the most important parameters of the Biot model [1],[2]could be recovered by acoustic interrogation in an numerical experiment which simulated the physical experiment of Hosokawa and Otani [14], (See also of McKelvie and Palmer [15], Williams [17].) where a small specimen of cancellous bone was placed in a tank of water between an acoustic source and receiver. It was found that by using computer simulations we could estimate bone density to within a percent. Other bone parameters, such as bulk modulus, shear modulus, permeability, etc. were not so accurately determined.

The Biot model treats a poroelastic medium as an elastic frame with interstial pore fluid. Cancellous bone is anisotropic, however, as pointed out by Williams, if the acoustic waves passing through it travel in the trabecular direction an isotropic model may be acceptable. We will simulate a two dimensional version of the experiments described in McKelvie and Palmer and Hosokawa and Otani. The motion of the frame and fluid within the bone are tracked by position vectors $\mathbf{u} = [u, v]$ and $\mathbf{U} = [U, V]$. The constitutive equations used by Biot are those of a linear elastic material with terms added to account for the interaction of the frame and interstial fluid

$$\begin{aligned} \sigma_{xx} &= 2\mu e_{xx} + \lambda e + Q\epsilon, \\ \sigma_{yy} &= 2\mu e_{yy} + \lambda e + Q\epsilon, \\ \sigma_{xy} &= \mu e_{xy}, \sigma_{yx} = \mu e_{yx}, \\ \sigma &= Qe + R\epsilon \end{aligned} \tag{1.1}$$

where the solid and fluid dilatations are given by

$$e = \nabla \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \epsilon = \nabla \cdot \boldsymbol{U} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}. \tag{1.2}$$

The stress-strain relations are

$$e_{xx} = \frac{\partial u}{\partial x}, \; e_{xy} = e_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, e_{yy} = \frac{\partial v}{\partial y}. \tag{1.3}$$

The parameter μ, the complex frame shear modulus can be measured. The other parameters λ, R and Q occurring in the constitutive equations are calculated from the measured or estimated values of the parameters given in Table 4 using the formulas

$$\begin{aligned} \lambda &= K_b - \frac{2}{3}\mu + \frac{(K_r - K_b)^2 - 2\beta K_r(K_r - K_b) + \beta^2 K_r^2}{D - K_b} \\ R &= \frac{\beta^2 K_r^2}{D - K_b} \\ Q &= \frac{\beta K_r\left((1-\beta) K_r - K_b\right)}{D - K_b}. \end{aligned} \tag{1.4}$$

where

$$D = K_r(1 + \beta(K_r/K_f - 1)). \tag{1.5}$$

The bulk and shear moduli K_b and μ are often given imaginary parts to account for frame inelasticity. Equations (1.1), (1.2) and (1.3) and an argument based upon Lagrangian dynamics are shown in [1] to lead to the following equations of motion for the displacements $\boldsymbol{u}, \boldsymbol{U}$ and dilatations e, ϵ

$$\begin{aligned} \mu\nabla^2\boldsymbol{u} + \nabla[(\lambda + \mu)e + Q\epsilon] &= \frac{\partial^2}{\partial t^2}(\rho_{11}\boldsymbol{u} + \rho_{12}\boldsymbol{U}) + b\frac{\partial}{\partial t}(\boldsymbol{u} - \boldsymbol{U}) \\ \nabla[Qe + R\epsilon] &= \frac{\partial^2}{\partial t^2}(\rho_{12}\boldsymbol{u} + \rho_{22}\boldsymbol{U}) - b\frac{\partial}{\partial t}(\boldsymbol{u} - \boldsymbol{U}). \end{aligned} \tag{1.6}$$

Symbol	Parameter
ρ_f	Density of the pore fluid
ρ_r	Density of frame material
K_b	Complex frame bulk modulus
μ	Complex frame shear modulus
K_f	Fluid bulk modulus
K_r	Frame material bulk modulus
β	Porosity
η	Viscosity of pore fluid
k	Permeability
α	Structure constant
a	Pore size parameter

Table 1: Parameters in the Biot model

Here ρ_{11} and ρ_{22} are density parameters for the solid and fluid, ρ_{12} is a density coupling parameter, and b is a dissipation parameter. These are calculated from the inputs of Table 4 using the formulas

$$\begin{aligned}\rho_{11} &= (1-\beta)\rho_r - \beta(\rho_f - m\beta)\\ \rho_{12} &= \beta(\rho_f - m\beta)\\ \rho_{22} &= m\beta^2\\ b &= \frac{F\left(a\sqrt{\omega\rho_f/\eta}\right)\beta^2\eta}{k}\end{aligned}$$

where

$$m = \frac{\alpha\rho_f}{\beta}$$

and the multiplicative factor $F(\zeta)$, which was introduced in [2] to correct for the invalidity of the assumption of Poiseuille flow at high frequencies, is given by

$$F(\zeta) = \frac{1}{4}\frac{\zeta T(\zeta)}{1-2T(\zeta)/i\zeta} \tag{1.7}$$

where T is defined in terms of Kelvin functions

$$T(\zeta) = \frac{\mathrm{ber}'(\zeta) + i\mathrm{bei}'(\zeta)}{\mathrm{ber}(\zeta) + i\mathrm{bei}(\zeta)}.$$

β	k	a	α	$\mathrm{Re}\, K_b$	$\mathrm{Re}\, \mu$
0.72	5×10^{-9}	4.71×10^{-4}	1.10	3.18×10^{9}	1.30×10^{9}
0.75	7×10^{-9}	8.00×10^{-4}	1.08	2.69×10^{9}	1.10×10^{9}
0.81	2×10^{-8}	1.20×10^{-3}	1.06	1.80×10^{9}	7.38×10^{8}
0.83	3×10^{-8}	1.35×10^{-3}	1.05	1.55×10^{9}	6.27×10^{8}
0.95	5×10^{-7}	2.20×10^{-3}	1.01	2.57×10^{8}	1.05×10^{8}

Table 2: Estimated values of some Biot parameters at different porosities taken from McKelvie and Palmer or Hosokawa and Otani.

The article of McKelvie and Palmer contains estimates of the Biot parameters of cancellous bone in the human *os calcis* (heel bone) for the normal ($\beta = 0.72$) and severely osteoporotic ($\beta = 0.95$) cases while the article of Hosokawa and Otani has estimates for bovine femoral bone for porosities of $\beta = 0.75, 0.81$ and 0.83. Table 2 contains estimates of these six Biot parameters for five bone specimens. In obtaining them we followed the estimation procedures used by Williams, McKelvie and Palmer, and Hosokawa and Otani. In generating test problems for a parameter recovery algorithm an estimate of the range of values a parameter might take is needed. Here is a discussion of how the values of the Biot parameters in Table 2 were calculated and our estimate of the range of values for each parameter:

For purposes of comparison we also computed the mean and standard deviation of all Phase 3 answers whose objective function value was within a factor of 2 of the lowest value and used these to find a 95% confidence interval for the mean. The result is shown in Table 3. Instances of underestimation, indicated by *, were more common, however only the underestimation of the error for the structure factor in Problem 83w was severe. On the other hand the overestimations of the error were less severe than with minimum/maximum/midpoint approach and on the whole better characterize the actual errors.

This suggested that perhaps the Biot model, which was the basis of our numerical experiment, was not sufficient to accurately model the acoustic response of cancellous bone. It is well known that the strength of bone depends heavily on its micro-structure [13]. Hence, it is imperative to develop new ultrasound methods for assessing the micro-structure *in vivo*.

Problem		β	k	a	α	$\mathrm{Re}\, K_b$	$\mathrm{Re}\, \mu$
71w	Error	0.12%	1.83%	5.41%	0.55%	0.71%	3.44%
	Est. Err	0.22%	8.34%	9.49%	1.10%	11.27%	3.61%
75w	Error	0.00%	21.20%	22.86%	0.34%	9.19%	6.34%
	Est. Err	0.19%	30.41%	38.40%	1.19%	12.24%	15.15%
79w	Error	0.14%	1.17%	4.10%	0.32%	7.67%	1.64%
	Est. Err	0.12%*	36.14%	44.77%	0.99%	9.00%	1.44%*
83w	Error	0.94%	23.30%	25.94%	2.78%	0.02%	0.02%
	Est. Err	1.27%	17.71%*	17.87%*	0.87%*	0.99%	0.66%
87w	Error	0.03%	2.57%	4.15%	0.33%	6.86%	7.10%
	Est. Err	0.30%	32.76%	27.61%	0.90%	39.87%	13.19%
91w	Error	0.56%	23.52%	17.41%	1.24%	21.12%	34.73%
	Est. Err	0.99%	17.14%*	16.44%*	0.86%*	26.49%	22.22%*
Average							
Worst							

Table 3: Phase 3 percentage errors when using mean values for Problems 71w, ..., 91w. Estimated errors are calculated from 95 percent confidence intervals.

2 Two-scale Convergence

Using the method of homogenization, we described the microstructure of the composite material, bone plus blood-marrow, in terms of a cell problem where all ingredients exist in equilibrium [8]. The two-phases of material are assumed to have the following constitutive equations

$$\sigma^\epsilon = \theta^\epsilon \sigma^{f,\epsilon} + (1 - \theta^\epsilon)\sigma^{s,\epsilon}, \tag{2.1}$$

The viscoelastic behavior of the trabeculae is modelled by a Kelvin-Voigt constitutive equation

$$\sigma_{ij}^{s,\epsilon} = (A^s + i\omega B^s)_{ijkl} e(u^\epsilon)_{kl}. \tag{2.2}$$

Here ω is the wave frequency and $e(u^\epsilon)$ is the strain tensor defined by

$$e(u^\epsilon)_{ij} = \frac{1}{2}(\partial_i u_j^\epsilon + \partial_j u_i^\epsilon) \quad i, j = 1, 2, 3.$$

The constants A_{ijkl}^s are the elasticity coefficients of the solid are assumed to have the classical symmetry and positivity conditions. The constants B_{ijkl}^s

describe viscosity of the solid, with the classical symmetry and positivity conditions.

The marrow was modelled as a slightly compressible viscous barotropic fluid with the constitutive equations

$$\sigma_{ij}^{f,\varepsilon} = (A^f + i\omega B^f)_{ijkl} e(u^\varepsilon)_{kl}. \tag{2.3}$$

In (2.3),

$$A_{ijkl}^f = c^2 \rho_f \delta_{ij}\delta_{kl}, \quad B_{ijkl}^f = 2\eta\delta_{ik}\delta_{jl} + \xi\delta_{ij}\delta_{kl}. \tag{2.4}$$

Here, c is the sound speed, $\rho_f > 0$ is a constant density of the marrow at rest, η, ξ are constant viscosities which are subject to the following conditions:

$$\eta > 0, \qquad \frac{\xi}{\eta} > -\frac{2}{3}.$$

From (2.4) one can obtain more explicit constitutive equations

$$\sigma^{f,\varepsilon} := c^2 \rho_f \nabla \cdot u^\varepsilon I + 2i\omega\eta\, e(u^\varepsilon) + i\omega\xi\nabla \cdot u^\varepsilon I. \tag{2.5}$$

The equations of motion for the trabeculae (solid part) are given by

$$-\omega^2 \rho_s u^\varepsilon \; - \; \operatorname{div}\,(\sigma^{s,\varepsilon}) \; = \; F\rho_s \qquad \text{in } \Omega_s^\varepsilon, \tag{2.6}$$

Here the trabeculae stress is defined in (2.2), and $\rho_s > 0$ is the constant density of the trabeculae at rest.

In the marrow part,

$$-\omega^2 \rho_f\, u^\varepsilon \; - \; \operatorname{div}\,(\sigma^{f,\varepsilon}) \; = \; F\rho_f \qquad \text{in } \Omega_f^\varepsilon, \tag{2.7}$$

The transition conditions between fluid and solid parts are given by the continuity of displacement

$$[u^\varepsilon] = 0 \text{ on } \Gamma_\varepsilon, \tag{2.8}$$

where $[\cdot]$ indicates the jump across the boundary of $\Gamma_\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$, and the continuity of the traction

$$\sigma^{s,\varepsilon} \cdot \nu = \sigma^{f,\varepsilon} \cdot \nu \text{ on } \Gamma_\varepsilon. \tag{2.9}$$

At the exterior boundary we imposed zero Dirichlet condition:

$$u^\varepsilon = 0 \text{ on } \partial\Omega. \tag{2.10}$$

This led to a weak formulation of the slightly compressible problem as

$$-\omega^2 \int_\Omega \rho^\varepsilon\, u^\varepsilon(x)\bar{\phi}(x) + \int_\Omega \theta^\varepsilon(A^f + i\omega B^f)e(u^\varepsilon) : e(\bar{\phi}) + \tag{2.11}$$
$$\int_\Omega (1-\theta^\varepsilon)(A^s + i\omega B^s)e(u^\varepsilon) : e(\bar{\phi}) \;=\; \int_\Omega F\rho^\varepsilon\bar{\phi},\ \forall\phi \in H_0^1(\Omega)^n,$$

where $\bar{\cdot}$ denotes the complex conjugate. In [8] we constructed the cell problem by assuming that u^1 is representable in the form

$$u^1(x,y) = \boldsymbol{N}^{pq}(y)e_x(u^0)_{pq}(x) + \boldsymbol{M}^{pq}(y,\omega)e_x(u^0)_{pq}(x), \tag{2.12}$$

where the summation convention is assumed. Here the $u^1(x,y)$ are vectors and, therefore, the matrices $\boldsymbol{N}^{pq}$ and the $\boldsymbol{M}^{pq}$ have vector components, i.e. the right hand side is a linear combination of these vectors with scalar coefficients $(e_x(u^0))_{pq}$.

The strong form of the variation formulation requires that we find $\boldsymbol{N}^{pq}$ such that

$$\operatorname{div}(K_N(\mathcal{E}^{pq} + e_y(\boldsymbol{N}^{pq}))) = 0 \quad \text{in } \mathcal{Y}, \tag{2.13}$$

$$B^f(\mathcal{E}^{pq} + e_y(\boldsymbol{N}^{pq}))\nu = A^s(\mathcal{E}^{pq} + e_y(\boldsymbol{N}^{pq}))\nu, \quad \text{on } \partial\mathcal{Y}_f \cap \partial\mathcal{Y}_s,$$

$$[\boldsymbol{N}^{pq}] = 0, \qquad \text{on } \partial\mathcal{Y}_f \cap \partial\mathcal{Y}_s.$$

Here

$$K_N = i\omega\theta B^f + (1-\theta)A^s. \tag{2.14}$$

Similarly, the strong form of the variation equation for $\boldsymbol{M}^{pq}$ is to find a solution of

$$\begin{aligned}&\operatorname{div}(K_M(\mathcal{E}^{pq} + e_y(\boldsymbol{M}^{pq})))\\ &= -\operatorname{div}(K_M e_y(\boldsymbol{N}^{pq})) \quad \text{in } \mathcal{Y},\end{aligned} \tag{2.15}$$

$$\begin{aligned}&(A^f + i\omega B^f)(\mathcal{E}^{pq} + e_y(\boldsymbol{M}^{pq}) + e_y(\boldsymbol{N}^{pq}))\nu\\ &= (A^s + i\omega B^s)(\mathcal{E}^{pq} + e_y(\boldsymbol{M}^{pq}) + e_y(\boldsymbol{N}^{pq}))\nu\\ &\text{on } \partial\mathcal{Y}_f \cap \partial\mathcal{Y}_s,\end{aligned}$$

$$[\boldsymbol{M}^{pq}] = 0, \qquad \text{on } \partial\mathcal{Y}_f \cap \partial\mathcal{Y}_s,$$

where

$$K_M = \theta(A^f + i\omega B^f) + (1-\theta)(A^s + i\omega B^s). \tag{2.16}$$

3 Isotropic Case

For isotropic trabeculae we may write out explicitly the equations for the vectors components of the $\mathbf{N}^{pq}$

$$\mathbf{N}^{pq} = [N_1^{pq}, N_2^{pq}].$$

They are seen to satisfy the system of partial differential equations [8, 9]

$$(\lambda + 2\mu)\frac{\partial^2 N_k^{pq}}{\partial y_k \partial y_l} + 2\mu \bigtriangleup \mathbf{N}_l^{pq} = 0 \text{ in } \mathcal{Y}_s$$

$$(\eta + 2\xi)\frac{\partial^2 N_k^{pq}}{\partial y_k \partial y_l} + 2\eta \bigtriangleup \mathbf{N}_l^{pq} = 0 \text{ in } \mathcal{Y}_f. \tag{3.1}$$

We may express these as matrix equations

$$(\lambda + 2\mu)\mathbf{H}\mathbf{N}^{pq} + 2\mu \bigtriangleup N^{pq} = 0 \quad \text{in } \mathcal{Y}_s, \tag{3.2}$$

and

$$(\eta + 2\xi)\mathbf{H}N^{pq} + 2\eta \bigtriangleup \mathbf{N}^{pq} = 0 \quad \text{in } \mathcal{Y}_f. \tag{3.3}$$

where $\mathbf{H}$ is the *Hessian* operator, i.e.

$$\mathbf{H} := \begin{pmatrix} \frac{\partial^2}{\partial y_1^2} & \frac{\partial^2}{\partial y_1 \partial y_2} \\ \frac{\partial^2}{\partial y_1 \partial y_2} & \frac{\partial^2}{\partial y_2^2} \end{pmatrix}.$$

To determine the matrix solutions M^{pq}, we introduce $Q^{pq} = M^{pq} + N^{pq}$, solve the problems for Q^{pq} and then obtain the solutions for M^{pq}. We obtain the following equation holding in $\mathcal{Y}_f$ [9]

$$i\omega\eta \bigtriangleup \mathbf{Q}^{pq} + \left(c^2\rho_f + i\omega\xi + i\omega\eta\right)\mathbf{H}\mathbf{Q}^{pq} = 0 \quad \text{in } \mathcal{Y}_f \tag{3.4}$$

We obtain the weak formulation of the effective equation

$$\int_\Omega \rho F \cdot \bar{\varphi} = -\omega^2 \int_\Omega \rho u^0 \cdot \bar{\varphi} + \int_\Omega \left[A^\star + i\omega B^\star + C^\star(\omega)\right] e(u^0) : e(\bar{\varphi}). \tag{3.5}$$

This leads to

THEOREM 1 *Let u^ε be the unique solution in $H_0^1(\Omega)$ of*

$$\begin{array}{ll} \mathcal{L}_\varepsilon u^\varepsilon = F\rho^\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{array} \tag{3.6}$$

where $\mathcal{L}_\varepsilon$ denotes the second order partial differential operator

$$\mathcal{L}_\varepsilon u^\varepsilon = -\text{ div } \left(((1-\theta^\varepsilon)\sigma^{s,\varepsilon} + \theta^\varepsilon\sigma^{f,\varepsilon})e(u^\varepsilon)\right) - \omega^2\rho^\varepsilon u^\varepsilon,$$

$\theta^\varepsilon \equiv \theta\left(\frac{x}{\varepsilon}\right)$ is the characteristic function of the marrow part Ω_f^ε and

$$\rho^\varepsilon = \theta^\varepsilon\rho_f + \rho_s(1-\theta^\varepsilon).$$

Then, there exist a subsequence $\{u^\varepsilon\}$, not relabeled, such that $\{u^\varepsilon\}$ converges weakly in $H_0^1(\Omega)$ to a limit $u^0 \in H_0^1(\Omega)$, and ρ^ε converges weak-$\star$ in $L^\infty(\Omega)$ to ρ. The pair u^0, ρ is a weak solution of the homogenized equation

$$\begin{array}{ll} \mathcal{L}u = F\rho & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \tag{3.7}$$

where $\mathcal{L}$ denotes the homogenized operator such that

$$\mathcal{L}u = -\text{ div }_x\left\{A^\star e(u^0) + i\omega B^\star e(u^0) + C^\star(\omega)e(u^0)\right\} - \omega^2\rho u$$

The effective constant tensors $A^\star, B^\star$ are defined, in (??), (??), respectively. The effective frequency dependent tensor $C^\star(\omega)$ is defined in (??). The vectors $\mathbf{N}^{pq}, \mathbf{M}^{pq}$ that appear in (??)-(??) are solutions of the auxiliary cell problems (??), (??), respectively.

4 Numerical Experiments

Using the physical values given in the table and the computed value for λ we were able to compute the coefficients in the effective equations from the cell problem solutions N_{ij} qnd M_{ij} $i.j = 1.2$ [9]. We show two of these coefficients below. It is important to realize that these coefficients are in themselves only used to compute the effective *constant* coefficients appearing in the effective equations (3.7).

Symbol	Parameter
$\rho_f = 10^3$	Density of the pore fluid
$\rho_r =$	Density of frame material
$K_b = 2.76 \times 10^9$	Complex frame bulk modulus
$\mu = 1.15 \times 10^9$	Frame shear modulus
$K_f = 2 \times 10^9$	Fluid bulk modulus
$K_r = 2 \times 10^10$	Frame material bulk modulus
$\beta = 0.76$	Porosity
$\eta = 1.5$	Viscosity of pore fluid
ξ =0	
$\omega = \pi \times 10^6$	Sound frequency
c =1483	Speed of sound

Table 4: Parameters used in our model

The system of equations (3.7) for the parameters chosen becomes

$$3.86\frac{\partial^2 u}{\partial x^2} + 0.0867\frac{\partial^2 u}{\partial x \partial y} + 0.1018\frac{\partial^2 v}{\partial y^2} + 2.9109\frac{\partial^2 v}{\partial x \partial y} + 0.9468\frac{\partial^2 u}{\partial y^2} + 0.00007\frac{\partial^2 v}{\partial x^2} - u$$

$$0.08678\frac{\partial^2 u}{\partial x^2} + 2,8366\frac{\partial^2 u}{\partial x \partial y} + 3.7888\frac{\partial^2 v}{\partial y^2} + 0.08651\frac{\partial^2 v}{\partial x \partial y} + 0.00005\frac{\partial^2 u}{\partial y^2} + 0.9468\frac{\partial^2 v}{\partial x^2} - v$$

By computing the characteristic equation of the symbol of this pseudodifferential operator the roots are seen to be

$$\frac{\xi}{\eta} = 0.0177 \pm 0.951i, -0.0401 \pm 1.040i.$$

and hence the system is elliptic, as expected.

5 The Slightly Compressible Monophasic Polymer

Work is in progress [11] on a cancellous bone model where the interstitial fluid is taken to be a non-Newtonian fluid. As before we assume an elastic matrix

$$\rho_s \frac{\partial^2 u^\varepsilon}{\partial t^2} - \mathrm{div}(Ae(\mathbf{u}^\varepsilon)) = F\rho_s; \qquad (5.1)$$

whereas, in the fluid part, $\Omega_s^\varepsilon \times]0,T[.$, we have a Stokes system describing the motion

$$\rho_f \frac{\partial^2 \mathbf{u}^\varepsilon}{\partial t^2} - \text{div}\,\left(\sigma^{f,\varepsilon}\right) = F\rho_f \tag{5.2}$$

in $\Omega_f^\varepsilon \times]0,T[.$; whereas,

$$\sigma^{f,\varepsilon} := -p^\varepsilon I + 2\mu\eta_p\left(e\left[\mathbf{v}^\varepsilon\right]\right) \tag{5.3}$$

$$\text{div}\;\mathbf{v} = 0, \text{ in } \Omega_f^\varepsilon \times]0,\mathrm{T}[, \tag{5.4}$$

where $\mathbf{v} = \dot{\mathbf{u}}$ is the fluid velocity and p^ε is the fluid pressure. For the slightly compressible case which occurs in acoustics we replace the pressure by $c^2\rho\text{div}\mathbf{u}$. A quantitative acoustic model must take into account that cancellous bone is mostly blood and marrow. This blood marrow mixture is a polymer, which suggests that we model its viscosity as a non Newtonian (shear dependent) fluid. There are two widely used shear-dependent viscosity laws in practice. The first is the power law, or Ostwald-de Waele model

$$\eta_p\left(e(\dot{\mathbf{u}})\right) = \eta_p\left(e(\dot{\mathbf{u}})\right) := \mu|e(\dot{\mathbf{u}})|^{r-2}, \quad 1 < r < 2, \mu > 0, \tag{5.5}$$

and the Careau law, which takes into account that polymers show a finite nonzero constant Newtonian viscosity at very low shear rates [?],

$$\begin{aligned} \eta_p\left(e(\dot{\mathbf{u}})\right) = \eta_p\left(e(\dot{\mathbf{u}})\right) &:= \left(\eta_0 - \eta_\infty\right)\left(1 + \lambda|e(\dot{\mathbf{u}})|^2\right)^{\frac{r}{2}-1} + \eta_\infty \\ & 1 < r < 2, \quad \eta_0 > \eta_\infty \geq 0, \quad \lambda > 0. \end{aligned} \tag{5.6}$$

We shall assume that η_p obeys one of these lawes with exponent r in what follows.

The fluid stress is then given by

$$\sigma^{f,\varepsilon} := \left[-c^2\rho\text{div}\mathbf{u}^\varepsilon I + 2\mu\eta_p\left(e\left[\mathbf{v}\right]\right)\right] e(\dot{\mathbf{u}}).$$

The transition conditions between fluid and solid parts are given by the continuity of displacement

$$[\mathbf{u}^\varepsilon] = 0 \text{ on } \Gamma_\varepsilon \times]0,T[, \tag{5.7}$$

where $[\cdot]$ indicates the jump across the boundary, and the continuity of the contact force

$$\sigma^{s,\varepsilon} \cdot \nu = \sigma^{f,\varepsilon} \cdot \nu \text{ on } \Gamma \times]0,T[. \tag{5.8}$$

At the exterior boundary we assume periodicity, namely that the

$$\{\mathbf{u}^\varepsilon, p^\varepsilon\} \text{ are } L-\text{periodic} . \tag{5.9}$$

To simplify our discussion we assume that there is no flow or deformation at $t = 0$, that is

$$\begin{aligned} \mathbf{u}^\varepsilon(x,0) &= 0, \\ \dot{\mathbf{u}}^\varepsilon(x,0) &= 0 \quad \text{in } \Omega. \end{aligned} \tag{5.10}$$

We may reformulate the system as a variational problem, namely find $\mathbf{u}^\varepsilon \in C^1([0,T]; L^2_{\text{per}}(\Omega)^n) \cap C([0,T]; H^1_{\text{per}}(\Omega)^n)$ such that

$$\int_\Omega \rho^\varepsilon \dot{\mathbf{v}}^\varepsilon(t)\varphi + \int_{\Omega^\varepsilon_f} 2\mu\eta_p\,(e(\mathbf{v}^\varepsilon(t)))\, e(\mathbf{v}) : e(\varphi) \tag{5.11}$$

$$+ \int_{\Omega^\varepsilon_s} AD(\mathbf{u}^\varepsilon(t)) : e(\varphi) - \int_{\Omega^\varepsilon_f} c^2 \rho \,\mathrm{div}\, \mathbf{u}^\varepsilon \,\mathrm{div}\, \varphi \;=\; \int_\Omega F\rho_\varepsilon\varphi,$$

$$\forall \varphi \in H^1_{\text{per}}(\Omega)^n, \qquad \text{in} \quad \mathcal{D}'(]0,T[)$$

where

$$\mathbf{u}^\varepsilon(0) = 0, \text{ and } \mathbf{v}^\varepsilon(0) = 0 \tag{5.12}$$

$$\mathrm{div}\, \mathbf{u} = 0 \text{ in } \Omega^\varepsilon_f \times]0,T[, \tag{5.13}$$

$$\rho_\varepsilon = \chi_{\Omega^\varepsilon_f}\rho_f + \chi_{\Omega^\varepsilon_s}\rho_s, \tag{5.14}$$

and χ_A is the characteristic function of A. It is trivial to establish [11] the following a priori estimates starting with equations using the $C^1([0,T]; H^1_{per}(\Omega)^n)$ short-time regularity and setting $\varphi = \frac{\partial u^\varepsilon}{\partial t}$ as the test function in (5.11).

$$\|\dot{\mathbf{u}}^\varepsilon\|_{L^\infty]0,T[;L^2(\Omega)^n)} \leq C\|F\|_{L^2(]0,T[\times\Omega)^n} = C(F), \tag{5.15}$$

$$\|e(\mathbf{u}^\varepsilon)\|_{L^\infty(]0,T[;L^2(\Omega^\varepsilon_s)^n)(\Omega^\varepsilon_s)^{n^2})} \leq C(F), \tag{5.16}$$

$$\|e\,(\dot{\mathbf{u}}^\varepsilon)\,\|_{L^r(]0,T[;L^r((\Omega^\varepsilon_f)^{n^2})} \leq C(F). \tag{5.17}$$

On the other hand by writing $e(\mathbf{u}^\varepsilon(t)) = \int_0^t e\left(\frac{\partial \mathbf{u}^\varepsilon}{\partial \eta}(\eta)\right) d\eta$ with Hölder's inequality we have

$$\|e(\mathbf{u}^\varepsilon(t))\|_{L^r(\Omega^\varepsilon_f)^n} \leq t^{1/q}\|e\left(\frac{\partial \mathbf{u}^\varepsilon}{\partial \eta}\right)\|_{L^r(]0,T[\times\Omega^\varepsilon_f)^{n^2}}$$

from which

$$\|e(\mathbf{u}^\varepsilon)\|_{L^\infty(0,T;L^r(\Omega)^{n^2})} \leq \bar{C}(F). \tag{5.18}$$

Moreover, if $F \in H^k(0,T;L^2(\Omega)^n)$ then

$$\|\frac{\partial^{k+1}\mathbf{u}^\varepsilon}{\partial t^{k+1}}\|_{L^\infty(0,T;L^2(\Omega)^n)} \leq C\|F\|_{H^k(]0,T[\times\Omega)^n} = \tilde{C}(F) \tag{5.19}$$

Likewise

$$\|e\left(\frac{\partial^k\mathbf{u}^\varepsilon}{\partial t^k}\right)\|_{L^\infty(0,T;L^2(\Omega)^n)} \leq \tilde{C}(F). \tag{5.20}$$

We show that uniqueness and existence follow by using the results of [3], [4] and following the procedure pf [10].

References

[1] M.A. Biot. Theory of propogation of elastic waves in a fluid-saturated porous solid. i. lower frequency range. *J. Acoust. Soc Am.*, 28(2):168–178, 1956.

[2] M.A. Biot. Theory of propogation of elastic waves in a fluid-saturated porous solid. ii. higher frequency range. *J. Acoust. Soc. Am.*, 28(2):179–191, 1956.

[3] A. Bourgeat and A. Mikeli*Homogenization of a polymer flow through a porous medium*, Nonlinear Analysis **26** 1221-1253.

[4] A. Bourgeat, O. Gipouloux and E. Marusi;Paloka *Filtration law for polymer flow through a porous media* Multiscale. Mod. and Simulation **1** (2003) 432-457.

[5] J. L. Buchanan, R. P. Gilbert, and K. Khashanah. Determination of the parameters of cancellous bone using low frequency acoustic measurements. *J. of Computational Acoustics*, 12(2), 2004.

[6] J.L. Buchanan, R.P. Gilbert, and K. Khashanah. Recovery of the poroelastic parameters of cancellous bone using low frequency acoustic interrogation. In A. Wirgin, editor, *Acoustics, Mechanics, and the Related Topics of Mathematical Analysis*, pages 41–47. World Scientific, 2002.

[7] S. Chaffai, F. Padilla, G. Berger and P. Languier, In vitro measurement of the frequency dependent attenuation in cancellous bone between 0.2 and 2 MHz, *J. Acoust, Soc. Amer.*, 108:1281-1289, 2000.

[8] Fang, M., Gilbert, R.P, Panchenko, A. and Vasilic, A.: *Homogenizing the Time Harmonic Acoustics of Bone: The Monophasic case*, (in press) Math. Comput. Modelling (2007).

[9] Fang, M., Gilbert, R P., Guyenne, P. and A. and Vasilic, A.: *Homogenizing the Time Harmonic Acoustics of Bone with Some Numerical Experiments*, preprint (2007).

[10] R. P. Gilbert and A. Mileli*Homogenizing the acoustic properties of the seabed: Part I* Nonlinear Analysis **40** 185-212.

[11] Gilbert, R.P, Panchenko, A. and Vasilic, A.: *Homogenizing the acoustics of cancellous bone with an interstitial non-Newtonian fluid* , (preprint) (2007)

[12] R. Hodgskinson, C. F. Njeh, J.D. Currey, and C. M. Langton, title=The ability of ultrasound velocity to predict the stiffness of cancellous bone in vitro, *Bone*, 21:183-190, 1997.

[13] T. Hildebrand and P. Rüegsegger: *A new method for the model independent assessment of thickness in three-dimensional images*, J. of Microsc., **185**, 67-75 (1997)

[14] A. Hosokawa and T. Otani. Ultrasonic wave propagation in bovine cancellous bone. *J. Acouist. Soc. Am.*, 101:558–562, 1997.

[15] M. L. McKelvie and S. B. Palmer. The interaction of ultrasound with cancellous bone. *Phys. Med. Biol.*, 10:1331–1340, 1991.

[16] E. F. Morgon and O. C. Yeh and W. C. Chang and T. M. Keaveny, Nonlinear behavior of trabecular bone at small strains, *J. Biomechanical Engineering*,1-9, 2001.

[17] J.L. Williams. Prediction of some experimental results by biot's theory. *J. Acoust. Soc. Am.*, 91:1106–1112, 1992.

[18] B. Yavari and A. Bedford. Comparison of numerical calculations of two biot coefficients with analytical solutions. *J. Acoust. Soc. Am.*, 90:985–990, 1991.